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Analysis and control of a model describing the effects of hard water consumption on kidney disease

Abstract

Kidney problems caused by hard water arise from the prolonged exposure to high concentrations of calcium and magnesium, which can lead to stone formation, mineral imbalances, and reduced kidney function. While the minerals themselves are not inherently harmful, their accumulation in the renal system poses significant long-term risks, particularly for vulnerable populations. The issue is further complicated by regional variations in water composition and lifestyle factors influencing hydration and diet. Addressing hard water's impact on kidney health requires an integrated approach that combines awareness, proper water treatment, and balanced nutrition. As water is an essential resource and a fundamental component of human physiology, maintaining its quality is vital for the protection of kidney function and the preservation of overall health. It is important to know how the hard water affects kidney health. The effect of hard water on kidney health is very nonlinear. In this study, bifurcation analysis and multi-objective nonlinear model predictive control are performed on a model that describes the effects of hard water consumption on kidney diseases. Bifurcation analysis is a powerful mathematical tool used to deal with the nonlinear dynamics of any process. Several factors must be considered, and multiple objectives must be met simultaneously. The MATLAB program MATCONT was used to perform the bifurcation analysis. The MNLMPC calculations were performed using the optimization language PYOMO in conjunction with the state-of-the-art global optimization solvers IPOPT and BARON. The bifurcation analysis revealed the existence of a branch point. The MNLMC converged to the utopia solution. The branch point (which causes multiple steady-state solutions from a singular point) is very beneficial because it enables the multiobjective nonlinear model predictive control calculations to converge to the Utopia point (the best possible solution) in the model.

Keywords: bifurcation, optimization, control, kidney, hard-water

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Introduction

Background

Hard water, characterized by high concentrations of dissolved minerals such as calcium, magnesium, and in some cases iron, is a common issue in many regions of the world. While it is not generally harmful in small quantities, chronic exposure to hard water through drinking and cooking can have various implications for human health, particularly for the kidneys. The kidneys are vital organs responsible for filtering waste, balancing electrolytes, and maintaining the body's fluid equilibrium. Over time, constant exposure to mineral-heavy water can place additional strain on these organs, leading to an increased risk of kidney stones, impaired kidney function, and other related health problems. The relationship between hard water and kidney health is complex, influenced by both the chemical composition of the water and individual physiological factors such as hydration habits, diet, and genetic predispositions.

While the evidence linking hard water to kidney problems is substantial, it is also nuanced. Some studies suggest that moderate hardness may not pose significant risks and might even offer protective effects against certain cardiovascular conditions due to magnesium's beneficial role. The real danger lies in the extremes—either excessively hard water or water contaminated with both high mineral and heavy metal content. In many regions, people mitigate these risks by using water softening systems or reverse osmosis filtration, which effectively remove calcium and magnesium ions. However, excessive softening can strip water of essential minerals, leading to its own set of health implications, such as sodium overload in sodium-based softening systems. Therefore, the goal should be balance: ensuring that water hardness stays within safe and beneficial limits while preventing mineral accumulation in the body.

From a public health perspective, the issue of hard water and kidney health requires both preventive education and technological intervention. Regular water quality testing is crucial to determine hardness levels and potential contamination. People living in areas with extremely hard water should be advised to use filtration systems that maintain mineral balance while reducing total hardness. Additionally, dietary guidance can play an important role in minimizing the risks associated with hard water. Reducing dietary oxalate, maintaining sufficient hydration, and moderating salt and protein intake can help offset the impact of excessive calcium or magnesium intake from drinking water. Healthcare providers should also consider local water quality when evaluating recurrent kidney stone patients, as environmental factors often play a larger role than recognized.

It is very important to understand the nonlinearity and use rigorous control measures to minimize the adverse effects of hard water on kidney health and the main focus of this paper is to perform rigorous bifurcation analysis and multiobjective nonlinear model predictive calculations on a model describing the effects of hard water on kidney health.

Sengupta[1] discussed the potential health impacts of hard water. Pizzorno[2,3] discussed the various aspects of the kidney dysfunction epidemic. Abraham et al.[4] researched the chronic kidney disease hotspots in developing countries in South Asia. Carraro et al.[5] developed an epidemiological model for proliferative kidney disease in salmonid populations. Wasana et al.[6] showed the effects of fluoride, heavy metals, and hardness in drinking water on kidney tissues. Tambaru et al.[7] showed the effects of hard water consumption on kidney function using insights from mathematical modeling. Almarashi et al.[8] developed a new mathematical model for diagnosing chronic diseases (kidney failure) using ann. Walk et al.[9] modelled the effects of multiple myeloma on kidney function. Ndii et al.[10] discussed an optimal control strategy for the effects of hard water consumption on kidney-related diseases.

Model equations

The variables sv, iv, irv, rv, and wv represent the normalized values of the susceptible population, population with kidney dysfunction, population with kidney disease, the recovered population and the population consuming hard water. β, γ represent the rate at which people consume the hard water and the rate at which they develop kidney disease. The recovery rate of Individuals with kidney dysfunction is p1u1, where p1 is the recovery probability due to treatment and u1 is the treatment/control rate. The recovery rate of Individuals with kidney diseases is p2u1, where p2 is the recovery probability due to treatment and u2 is the treatment/control rate. The self-recovery rate is τ. b represents the increased level of the hardness of water. The model equations are

$$\begin{aligned}
&\lambda = \frac{wv}{(1+wv)} \\
&d(sv) = \mu - \beta(\lambda)sv - \mu(sv) \\
&d(iv) = \beta(\lambda)sv - \gamma(iv) - \mu(iv) - p1(u1)iv \\
&d(irv) = \gamma(iv) - tau(irv) - \mu(irv) - p2(u2)irv \\
&d(rv) = \tau(irv) + p1(u1)iv + p2(u2)irv - \mu(rv) \\
&d(wv) = b(wv)(1-wv) - u3(wv)
\end{aligned} \quad (1)$$

The base parameter values are

μ =1/65; β =0.1; γ =1/5; τ =1/5; p1=0.5; p2=0.5; b=0.05; u1=0;u2=0;u3=0;

Bifurcation analysis

The MATLAB software MATCONT is used to perform the bifurcation calculations. Bifurcation analysis deals with multiple steady-states and limit cycles. Multiple steady states occur because of the existence of branch and limit points. Hopf bifurcation points cause limit cycles. A commonly used MATLAB program that locates limit points, branch points, and Hopf bifurcation points is MATCONT.[11,12] This program detects Limit points(LP), branch points(BP), and Hopf bifurcation points(H) for an ODE system

$$\frac{dx}{dt} = f(x, \alpha) \quad (2)$$

$x \in R^n$ Let the bifurcation parameter be α. Since the gradient is orthogonal to the tangent vector,

The tangent plane at any point $w = [w_1, w_2, w_3, w_4, ...w_{n+1}]$ must satisfy

$$Aw = 0 \quad (3)$$

Where A is

$$A = [\partial f / \partial x \quad | \partial f / \partial \alpha] \quad (4)$$

Where $\partial f / \partial x$ is the Jacobian matrix. For both limit and branch points, the Jacobian matrix $J = [\partial f / \partial x]$ must be singular.

For a limit point, there is only one tangent at the point of singularity. At this singular point, there is a single non-zero vector, y, where Jy=0. This vector is of dimension n. Since there is only one tangent the vector

$y = (y_1, y_2, y_3, y_4, ...y_n)$ must align with $\hat{w} = (w_1, w_2, w_3, w_4, ...w_n)$. Since

$$J\hat{w} = Aw = 0 \quad (5)$$

the n+1[th] component of the tangent vector w_{n+1} = 0 at a limit point (LP).

For a branch point, there must exist two tangents at the singularity. Let the two tangents be z and w. This implies that

$$\begin{aligned} Az &= 0 \\ Aw &= 0 \end{aligned} \quad (6)$$

Consider a vector v that is orthogonal to one of the tangents (say w). v can be expressed as a linear combination of z and w ($v = \alpha z + \beta w$). Since $Az = Aw = 0$; $Av = 0$ and since w and v are orthogonal, $w^T v = 0$. Hence $Bv = \begin{bmatrix} A \\ w^T \end{bmatrix} v = 0$ which implies that B is singular.

Hence, for a branch point (BP) the matrix $B = \begin{bmatrix} A \\ w^T \end{bmatrix}$ must be singular.

At a Hopf bifurcation point,

$$\det(2 f_x(x, \alpha) @ I_n) = 0 \quad (7)$$

@ indicates the bialternate product while I_n is the n-square identity matrix. Hopf bifurcations cause limit cycles and should be eliminated because limit cycles make optimization and control tasks very difficult. More details can be found in Kuznetsov[13,14] and Govaerts.[15]

Multiobjective nonlinear model predictive control (MNLMPC)

The rigorous multiobjective nonlinear model predictive control (MNLMPC) method developed by Flores Tlacuahuaz et al.[16] was used. Consider a problem where the variables $\sum_{t_i=0}^{t_i=t_f} q_j(t_i)$ (j=1, 2..n) have to be optimized simultaneously for a dynamic problem

$$\frac{dx}{dt} = F(x, u) \quad (8)$$

t_f Being the final time value, and n the total number of objective variables and u the control parameter. The single objective optimal control problem is solved individually optimizing each of the variables $\sum_{t_i=0}^{t_i=t_f} q_j(t_i)$ The optimization of $\sum_{t_i=0}^{t_i=t_f} q_j(t_i)$ will lead to the values q_j^*. Then, the multiobjective optimal control (MOOC) problem that will be solved is

$$\min\left(\sum_{j=1}^{n}\left(\sum_{t_i=0}^{t_i=t_f} q_j(t_i) - q_j^*\right)\right)^2 \quad (9)$$

$$subject\ to\ \frac{dx}{dt} = F(x, u);$$

This will provide the values of u at various times. The first obtained control value of u is implemented and the rest are discarded. This procedure is repeated until the implemented and the first obtained control values are the same or if the Utopia point where ($\sum_{t_i=0}^{t_i=t_f} q_j(t_i) = q_j^*$ for all j) is obtained.

Pyomo[17] is used for these calculations. Here, the differential equations are converted to a Nonlinear Program (NLP) using the orthogonal collocation method The NLP is solved using IPOPT[18] and confirmed as a global solution with BARON.[19]

The steps of the algorithm are as follows

Optimize $\sum_{t_i=0}^{t_i=t_f} q_j(t_i)$ and obtain q_j^*.

Minimize $(\sum_{j=1}^{n}(\sum_{t_i=0}^{t_i=t_f} q_j(t_i) - q_j^*))^2$ and get the control values at various times.

Implement the first obtained control values

Repeat steps 1 to 3 until there is an insignificant difference between the implemented and the first obtained value of the control variables or if the Utopia point is achieved. The Utopia point is when

$\sum_{t_i=0}^{t_i=t_f} q_j(t_i) = q_j^*$ for all j.

Sridhar[20] demonstrated that when the bifurcation analysis revealed the presence of limit and branch points tthe MNLMPC calculations to converge to the Utopia solution. The Utopia solution (the most beneficial solution) is the one where all objectives are met to the fullest degree. For this, the singularity condition, caused by the presence of the limit or branch points was imposed on the co-state equation.[21] If the minimization of q_1 lead to the value q_1^* and the minimization of q_2 lead to the value q_2^* The MNLPMC calculations will minimize the function $(q_1 - q_1^*)^2 + (q_2 - q_2^*)^2$. The multiobjective optimal control problem is

$$\min \quad (q_1 - q_1^*)^2 + (q_2 - q_2^*)^2 \quad \textit{subject to} \quad \frac{dx}{dt} = F(x,u) \quad (10)$$

Differentiating the objective function results in

$$\frac{d}{dx_i}((q_1 - q_1^*)^2 + (q_2 - q_2^*)^2) = 2(q_1 - q_1^*)\frac{d}{dx_i}(q_1 - q_1^*) + 2(q_2 - q_2^*)\frac{d}{dx_i}(q_2 - q_2^*)$$

* MERGEFORMAT (11)

The Utopia point requires that both $(q_1 - q_1^*)$ and $(q_2 - q_2^*)$ are zero. Hence

$$\frac{d}{dx_i}((q_1 - q_1^*)^2 + (q_2 - q_2^*)^2) = 0 \qquad (12)$$

The optimal control co-state equation (Upreti; 2013)[43] is

$$\frac{d}{dt}(\lambda_i) = -\frac{d}{dx_i}((q_1 - q_1^*)^2 + (q_2 - q_2^*)^2) - f_x\lambda_i; \quad \lambda_i(t_f) = 0 \quad (13)$$

λ_i is the Lagrangian multiplier. t_f is the final time. The first term in this equation is 0 and hence

$$\frac{d}{dt}(\lambda_i) = -f_x\lambda_i; \lambda_i(t_f) = 0 \qquad \text{* MERGEFORMAT (14)}$$

At a limit or a branch point, for the set of ODE $\frac{dx}{dt} = f(x,u)$ f_x is singular. Hence there are two different vectors-values for $[\lambda_i]$ where $\frac{d}{dt}(\lambda_i) > 0$ and $\frac{d}{dt}(\lambda_i) < 0$. In between there is a vector $[\lambda_i]$ where $\frac{d}{dt}(\lambda_i) = 0$. This coupled with the boundary condition $\lambda_i(t_f) = 0$ will lead to $[\lambda_i] = 0$ This makes the problem an unconstrained optimization problem, and the optimal solution is the Utopia solution.

Results

When u3 was used as a bifurcation parameter a branch point (BP) was located at (sv, iv, irv, rv, wv, u3) values of (1, 0, 0, 0, 0, 0.050000) (Figure 1a) .

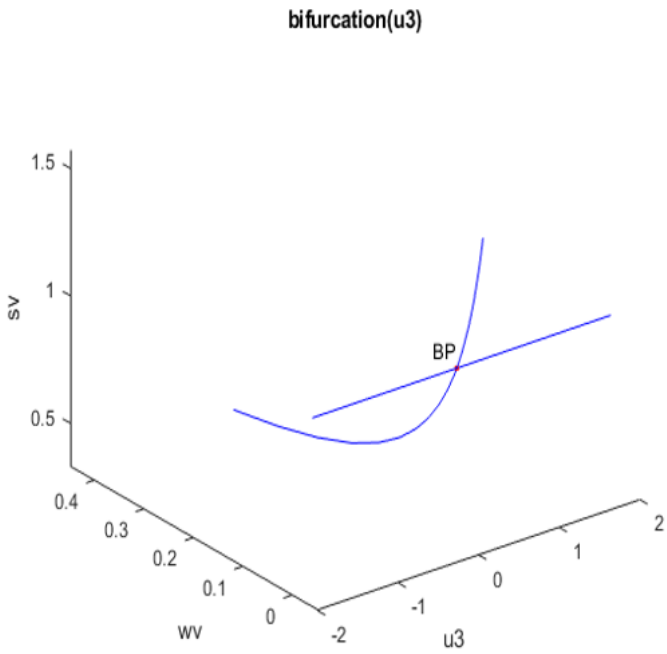

Figure 1 a Bifurcation with u3 as bifurcation parameter.

For the MNLMPC, u1, u u3 are the control parameters, and $\sum_{t_i=0}^{t_i=t_f} iv(t_i), \sum_{t_i=0}^{t_i=t_f} irv(t_i), \sum_{t_i=0}^{t_i=t_f} wv(t_i)$ were minimized individually, and each led to a value of 0. The overall optimal control problem will involve the minimization of

$$(\sum_{t_i=0}^{t_i=t_f} iv(t_i) - 0)^2 + (\sum_{t_i=0}^{t_i=t_f} irv(t_i) - 0)^2 + (\sum_{t_i=0}^{t_i=t_f} wv(t_i) - 0)^2 \text{ was}$$

minimized subject to the equations governing the model.

This led to a value of zero (the Utopia point). The MNLMPC values of the control variables, u1, u2, and u3 were 0.2646, 0.3384, and 0.019285. The MNLMPC profiles are shown in Figure 2a-2d. The control profiles of u1, u2, and u3 exhibited noise (Figure **2c**) and this was remedied using the Savitzky-Golay filter to produce the smooth profiles u1sg, u2sg, and u3sg (Figure **2d**).

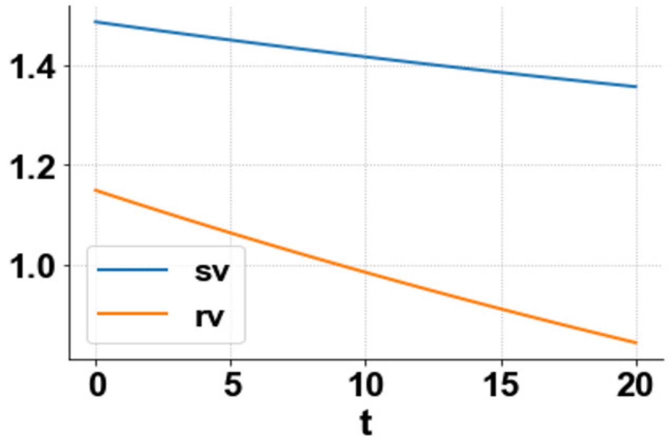

Figure 2a MNLMPC sv, rv profiles.

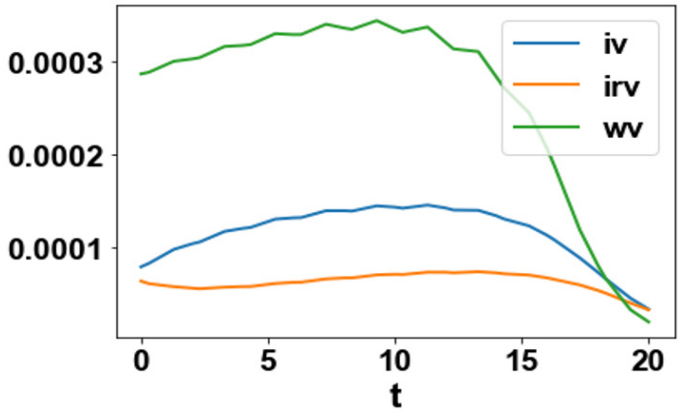

Figure 2b MNLMPC iv, irv, wv profiles.

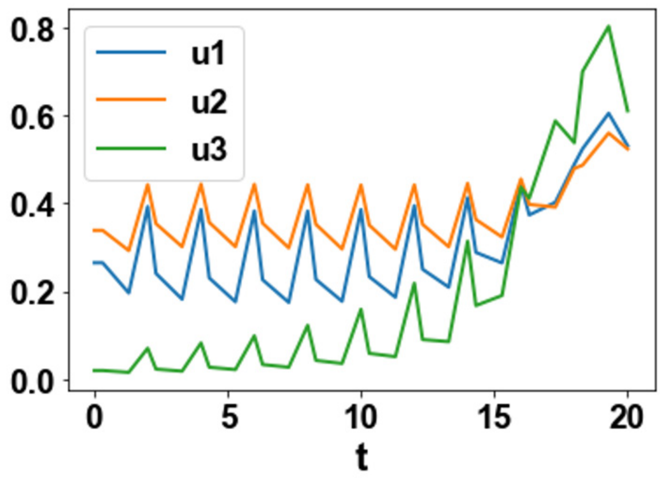

Figure 2c MNLMPC u1, u2, u3 profiles.

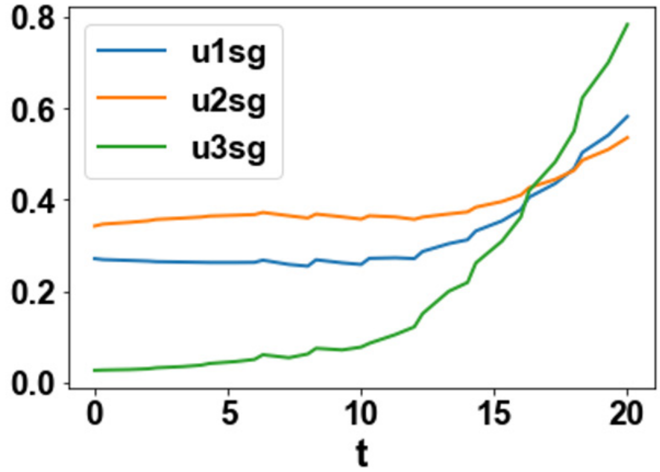

Figure 2d MNLMPC u1sg, u2sg, u3sg profiles.

Discussion of results

Theorem

If one of the functions in a dynamic system is separable into two distinct functions, a branch point singularity will occur in the system.

Proof

Consider a system of equations

$$\frac{dx}{dt} = f(x,\alpha) \qquad (2)$$

$x \in R^n$. Defining the matrix A as

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} & \frac{\partial f_1}{\partial x_4} \cdots\cdots & \frac{\partial f_1}{\partial x_n} & \frac{\partial f_1}{\partial \alpha} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} & \frac{\partial f_2}{\partial x_4} \cdots\cdots & \frac{\partial f_2}{\partial x_n} & \frac{\partial f_2}{\partial \alpha} \\ \cdots\cdots & & & & & \\ \cdots\cdots & & & & & \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \frac{\partial f_n}{\partial x_3} & \frac{\partial f_n}{\partial x_4} \cdots\cdots & \frac{\partial f_n}{\partial x_n} & \frac{\partial f_n}{\partial \alpha} \end{bmatrix} \qquad (3)$$

α is the bifurcation parameter. The matrix A can be written in a compact form as

$$A = [\frac{\partial f_p}{\partial x_q}.| \quad \frac{\partial f_p}{\partial \alpha}] \qquad (4)$$

The tangent at any point x; ($z = [z_1, z_2, z_3, z_4, \ldots. z_{n+1}]$) must satisfy

$$Az = 0 \qquad (5)$$

The matrix $\{\frac{\partial f_p}{\partial x_q}\}$ must be singular at both limit and branch points..

The n+1 th component of the tangent vector $z_{n+1} = 0$ at a limit point

(LP) and for a branch point (BP) the matrix $B = \begin{bmatrix} A \\ z^T \end{bmatrix}$

must be singular. Any tangent at a point y that is defined by $z = [z_1, z_2, z_3, z_4, ...z_{n+1}]$) must satisfy

$$Az = 0 \quad (6)$$

For a branch point, there must exist two tangents at the singularity. Let the two tangents be z and w. This implies that

$$\begin{aligned} Az &= 0 \\ Aw &= 0 \end{aligned} \quad (7)$$

Consider a vector v that is orthogonal to one of the tangents (say z). v can be expressed as a linear combination of z and w ($v = \alpha z + \beta w$). Since $Az = Aw = 0$; $Av = 0$ and since z and v are orthogonal,

$z^T v = 0$. Hence $Bv = \begin{bmatrix} A \\ z^T \end{bmatrix} v = 0$ which implies that B is singular where $B = \begin{bmatrix} A \\ z^T \end{bmatrix}$

Let any of the functions f_i are separable into 2 functions ϕ_1, ϕ_2 as

$$f_i = \phi_1 \phi_2 \quad (8)$$

At steady-state $f_i(x, \alpha) = 0$ and this will imply that either $\phi_1 = 0$ or $\phi_2 = 0$ or both ϕ_1 and ϕ_2 must be 0. This implies that two branches $\phi_1 = 0$ and $\phi_2 = 0$ will meet at a point where both ϕ_1 and ϕ_2 are 0.

At this point, the matrix B will be singular as a row in this matrix would be

$$[\frac{\partial f_i}{\partial x_k} | \frac{\partial f_i}{\partial \alpha}] \quad (9)$$

However,

$$[\frac{\partial f_i}{\partial x_k} = \phi_1(=0)\frac{\partial \phi_2}{\partial x_k} + \phi_2(=0)\frac{\partial \phi_1}{\partial x_k} = 0(\forall k = 1.,,n) \quad (10)$$

$$\frac{\partial f_i}{\partial \alpha} = \phi_1(=0)\frac{\partial \phi_2}{\partial \alpha} + \phi_2(=0)\frac{\partial \phi_1}{\partial \alpha}] = 0$$

This implies that every element in the row $[\frac{\partial f_i}{\partial x_k} | \frac{\partial f_i}{\partial \alpha}]$ would be 0, and hence the matrix B would be singular. The singularity in B implies that there exists a branch point.

The branch point occurs at (sv, iv, irv, rv, wv, u3) values (1, 0, 0, 0, 0, 0.050000). Here, the two distinct functions can be obtained from the fifth ODE in the model

$$\frac{d(wv)}{dt} = b(wv)(1 - wv) - u3(wv) \quad (11)$$

The two distinct equations are

$$\begin{aligned} wv &= 0 \\ b(1 - wv) - u3 &= 0 \end{aligned} \quad (12)$$

With wv = 0, u3 =0.05, b=0.05 both the distinct equations are satisfied validating the theorem.

The presence of the branch point causes the MNLMPC calculations to attain the Utopia solution, validating the analysis of Sridhar.[20]

Conclusion

Bifurcation analysis and multiobjective nonlinear control (MNLMPC) studies on a model describing the effects of hard water consumption on kidney diseases. The bifurcation analysis revealed the existence of a branch point. The branch point (which cause multiple steady-state solutions from a singular point) is very beneficial from a numerical standpoint because it enables the multiobjective nonlinear model predictive control calculations to converge to the Utopia point (the best possible solution) in the model and from a a clinical standpoint because all objectives are met to the fullest degree A combination of bifurcation analysis and multiobjective Nonlinear Model Predictive Control(MNLMPC) on a model describing the effects of hard water consumption on kidney diseases is the main contribution of this paper.

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Conflict of interest

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