Review Article

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On the Hermite-Hadamard inequality. Some methodological remarks

Abstract

In this paper, we present some historical notes and methodological observations about the classic Hermite-Hadamard Inequality, which allows us to obtain new versions of this inequality.

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Introduction

Convex functions play, in contemporary Mathematics, a very prominent role, first of all, because they are especially easy to minimize (for example, any minimum of a convex function is a global minimum). For this reason, there is a very rich theory for solving convex optimization problems that has many practical applications (e.g. circuit design, controller design, modelling, etc.). On the other hand, this concept is very useful for the development of many branches of Mathematics itself, for example Functional Analysis, Complex Analysis, Calculus of Variations, Differential Equations, Discrete Mathematics, Algebraic Geometry, Probability, Code Theory, Graph Theory and Crystallography, but it also finds important applications in other areas such as Medicine, Economics, Physics, Chemistry, Biology, Engineering, Architecture and many more.

Furthermore, in recent years, various extensions and generalizations of the classical concept of convexity, both for sets and functions, have been studied and there is a fairly significant production of works on the subject (interested readers can consult,[1] where a fairly complete panorama of the current development of this concept is presented).

In what follows, I is a real, closed and bounded interval. A function $f : I \subseteq R \to R$ is said to be convex on the Interval I, if the in equality

$$f\left(tx+(1-t)y\right)\leq tf\left(x\right)+(1-t)f\left(y\right) \qquad (1)$$

holds for all x,y∈I and t∈[0,1]. We say that f is concave if -f is convex.

This notion is usually attributed to the Danish mathematician Johan Jensen (1859-1925), who showed that many classical inequalities (Holder's Inequality, Minkowski's Inequality) follow from what is now called Jensen's Inequality. Jensen unified in a functional class, those functions that verify certain properties studied by O. Holder, O. Stolz, J. Hadamard and Ch. Hermite.[2,3]

The classic text by Hardy,[4] Littlewood, and Pólya was influential in increasing research on the study of convex functions, their properties, characterizations, and various inequalities associated with them (inequalities of the Jensen type, of the Hermite-Hadamard type and of Fejer that generalizes this last).

The Hermite-Hadamard Inequality, central to this Review, is presented in this way.

The inequality

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f\left(x\right)dx\leq\frac{f\left(a\right)+f\left(b\right)}{2} \qquad (2)$$

is true for any convex function f in [a,b].

The Hermite-Hadamard Inequality is one of the topics that attracts the most attention in the Mathematical Sciences today (see[5]). This relevance is because this inequality establishes a relationship between the mean of a convex function and its value at the midpoint of an interval. This interest lies for several reasons:

Optimization theory: The Hermite-Hadamard inequality is a fundamental tool in optimization theory, as it provides sufficient and necessary conditions for the convexity of a function. This is crucial in convex optimization, where we seek to minimize or maximize functions subject to constraints.

Functional analysis: In functional analysis, the Hermite-Hadamard inequality is used to study properties of convex functions. This is important in areas such as approximation theory and measure theory.

Economics: In economics, the Hermite-Hadamard inequality is applied in general equilibrium theory and utility theory. It helps to understand and model phenomena related to decision making and the optimal allocation of resources in situations of inequality.

Statistics: In statistics, this inequality is used in estimation theory and statistical inference. It allows establishing limits for the variance of estimators and provides tools for evaluating the quality of the estimators.

In summary, the Hermite-Hadamard inequality is important because it provides a powerful tool for the study of convex functions and their properties, which has applications in a wide range of disciplines, including mathematics, economics, statistics, and optimization.

The left side of the inequality was proven by Jaques Salomon Hadamard in 1893, for the case in which the functions f with increasing derivative on a closed interval of the real line. At that time the notion of convex functions was in the process of construction. Today this inequality is known as Hadamard's inequality. While the right side of the inequality is attributed to Charles Hermite in 1883. Today the inequality is known as the Hermite-Hadamard inequality (more historical details can be consulted at[6]). It gives an estimate of the mean value of a convex function and notes that it also provides an analysis of the inequality of Jensen.

In the last 25 years, we have witnessed a great growth in the number of researchers and their productions, interested in the Hermite-Hadamard Inequality. These productions have focused on the following work directions: 1) Using different notions of convexity (see[7-20]).

2) Refinement of the mesh used (there is a crucial issue in this direction of work, suppose we use instead of a and b, the ends of the interval, the points a, (a+b)/2 and b, then we must ensure that at the midpoint, the integral operator used, does not have a jump, since the result would not be guaranteed in all [a,b]).

3) Improved estimates of the left and right members of (2) (see[21]).

4) Using new generalized and fractional integral operators (see[22-52]).

In the aforementioned works, there are enough references so that the interested reader can form an important database.

In this paper we will present some historical details about the classic Hermite-Hadamard Inequality. Some methodological observations relative to a classic result are added, to illustrate the generalization process that occurs in Mathematics and obtain new versions of this inequality.

History

On 22 November 1881 Ch. Hermite (1822-1901) sent a letter to the journal Mathesis. An extract from that letter was published in Mathesis 3 (1883), p. 82. It reads: "Sur deux limites d'une integrale definie. Soit f(x) une fonction qui varie toujours dans le m me sens de x=a, à x=b. On aura les relations

$$(b-a)f\left(\frac{a+b}{2}\right)<\int_a^b f(x)dx<(b-a)\frac{f(a)+f(b)}{2} \qquad (3)$$

ou bien

$$(b-a)f\left(\frac{a+b}{2}\right)>\int_a^b f(x)dx>(b-a)\frac{f(a)+f(b)}{2} \qquad (4)$$

Suivant que la courbe y=f(x) tourne sa convexité ou sa concavité vers l'axe des abcisses. En faisant dans ces formules f(x)=1/(1+x), a=0, b=x, il vient

$$x-\frac{x}{2+x}<\log(x+1)<x-\frac{x^2}{2+2x} \qquad (5)$$

Hermite's note is not recorded in the referative journal Jahrbuch fiber die Fortschritte der Mathematik, nor in Hermite's collected papers which were published "sous les auspices de l'Académie des sciences de Paris par Émile Picard, membre de l'Institut". E. F. Beckenbach, p. 441, writes that the first inequality in (2) was proved in 1893 by J. Hadamard; see, in particular, pp. 174-176, 186. Beckenbach used great skill in order to recognize that Hadamard obtained the first inequality in (2), although it was explicitly published by Hermite ten years earlier. Beckenbach, though undoubtedly an expert in the history and theory of convex functions, was not aware of Hermite's result.

Methodological remarks

A new way to define an integral operator, and take a first step in generalizing a known result, is to consider a certain weight in the definition of the operator integral, as follows:

Definition 1. (see [3]) Let $\varphi \in L_1[a,b]$ and let w be a continuous and positive function, $w: I \to R$, with first derivative integrables on I°. Then the weighted fractional integrals are defined by (right and left respectively):

$$I^{w}_{a_1+}\phi(t)=\frac{1}{\Gamma(\alpha)}\int_{a_1}^{t} w'\left(\frac{a_2-t}{a_2-a_1}\right)\phi(t)dt, \quad t>a_1 \qquad (6)$$

$$I^{w}_{a_2-}\phi(t)=\frac{1}{\Gamma(\alpha)}\int_{t}^{a_2} w'\left(\frac{t-a_1}{a_2-a_1}\right)\phi(t)dt, \quad t<a_2 \qquad (7)$$

Remark 2. The consideration of the first derivative of the weight function w is given by the nature of the problema to be solved, it can also be considered the second derivative.

Remark 3. To have a clearer idea of the amplitude of the Definition 1, let's consider some particular cases of the weight w':

a) Putting w'≡1, we obtain the classical Riemann integral.

b) If $w'(t)=\frac{t^{(\alpha-1)}}{\Gamma(\alpha)}$, then we obtain the Riemann-Liouville fractional integral.

c) With convenient weight choices w' we can get the k-Riemann-Liouville fractional integral right and left of,[53] the right-sided fractional integrals of a function ψ with respect to another function h on [a,b] (see[54]), the right and left integral operator of,[55] the right and left sided generalized fractional integral operators of[56] and the integral operators of[57] and,[58] can also be obtained from above Definition by imposing similar conditions to w'.

d) Of course there are other known integral operators, fractional or not, that can be obtained as particular cases of the previous one, but we leave it to interested readers (see [41,59]).

In the following result is presented:

Lemma 4. Let f:I°⊆R→R be a differentiable mapping on I°, a,b∈ I°, with a<b. If f'∈L[a,b], then the following equality holds:

$$\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx=\frac{b-a}{2}\int_0^1(1-2t)f'(ta+(1-t)b)dt. \qquad (8)$$

This is probably the main result referred to the Hermite-Hadamard Inequality in the last 30 years, it establishes a working method that has been repeated in the vast majority of known results.

The idea is, from this equality, to be able to estimate the left member of this equality (in the end an estimate of the right member of (2) and the mean value of the function), using known inequalities such as that of Hölder, Young, mean power, etc.

Let's look at a couple of details regarding (8). It is clear that

$$\int_0^1(1-2t)f'(ta+(1-t)b)dt=\int_0^1(1-t)f'(ta+(1-t)b)dt-\int_0^1 tf'(ta+(1-t)b)dt, \qquad (9)$$

making the change of variables u=1-t in the first integral of the right member, we have

$$\int_0^1(1-t)f'(ta+(1-t)b)dt=\int_0^1 uf'((1-u)a+ub)du,$$

so, essentially, (8) could be rewritten as

$$\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx=\frac{b-a}{2}\int_0^1 t\left[f'(ta+(1-t)b)-f'((1-t)a+tb)\right]dt. \qquad (10)$$

Another version in which the equality (8) is presented.

If we take into account the Definition 1, we can provide a

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generalization of (8) in this form:

Lemma 5. *Let* f:I°⊆R→R *be a differentiable mapping on* I°, a,b∈ I°, with a<b. If f'∈L[a,b], *and w has first derivative integrable on* I° *then the following equality holds:*

$$\begin{aligned}&\left(w(1)-w(0)\right)\left(f(b)-f(a)\right)-\frac{1}{b-a}\left[I_{a+}^{w}f(b)+I_{b-}^{w}f(a)\right]\\ =\quad &(b-a)\int_0^1 w(t)\left[f'\left((1-t)a+tb\right)-f'\left(ta+(1-t)b\right)\right]dt.\end{aligned} \tag{11}$$

Proof. It is easy to obtain from the integral of the second member that

$$\begin{aligned}I&=\int_0^1 w(t)\left[f'\left((1-t)a+tb\right)-f'\left(ta+(1-t)b\right)\right]dt\\ =\quad &\int_0^1 w(t)f'\left((1-t)a+tb\right)dt-\int_0^1 w(t)f'\left(ta+(1-t)b\right)dt=I_1-I_2.\end{aligned} \tag{12}$$

For I_2 we have, integrating by parts and changing the variables u=ta+(1-t)b

$$\begin{aligned}I_2&=-\frac{1}{b-a}\left[w(1)f(b)-w(0)f(a)\right]-\left(\frac{1}{b-a}\right)^2\int_a^b w'\left[\frac{u-a}{b-a}\right]f(u)du\\ =\quad &-\frac{1}{b-a}\left[w(1)f(b)-w(0)f(a)\right]-\left(\frac{1}{b-a}\right)^2 J_{a+}^{w}f(b).\end{aligned} \tag{13}$$

Similarly for I_1 we have

$$\begin{aligned}I_1&=-\frac{1}{b-a}\left[w(1)f(a)-w(0)f(b)\right]+\left(\frac{1}{b-a}\right)^2\int_a^b w'\left[\frac{b-u}{b-a}\right]f(u)du\\ =\quad &-\frac{1}{b-a}\left[w(1)f(a)-w(0)f(b)\right]+\left(\frac{1}{b-a}\right)^2 J_{b-}^{w}f(a).\end{aligned} \tag{14}$$

Subtracting I_2 form I_1 we get

$$\begin{aligned}&\frac{1}{b-a}\left(w(1)-w(0)\right)\left(f(a)+f(b)\right)-\left(\frac{1}{b-a}\right)^2\left(J_{b-}^{w}f(a)+J_{a+}^{w}f(b)\right)\\ =\quad &\int_0^1 w(t)f'\left((1-t)a+tb\right)dt-\int_0^1 w(t)f'\left(ta+(1-t)b\right)dt.\end{aligned}$$

After multiplying both members by b-a, we obtain the desired equality.

Remark 6. *The interested reader can verify that putting w(t)=t and taking into account (10), we obtain (8) from the last result.*

Putting $w(t)=\frac{t^{\alpha}}{\Gamma(\alpha)}$ in (11) we obtain the following new result for Riemann-Liouville fractional integrals:

Corollary 7. *Let* f:I⊆R→R *be a differentiable mapping on* I°, a,b∈ I°, with a<b. If f'∈L[a,b], *then the followng equality holds:*

$$\begin{aligned}&\frac{1}{\Gamma(\alpha)}\left(f(b)+f(a)\right)-\frac{\alpha}{(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right]\\ =\quad &\frac{b-a}{\Gamma(\alpha)}\int_0^1 t^{\alpha}\left[f'\left((1-t)a+tb\right)-f'\left(ta+(1-t)b\right)\right]dt.\end{aligned}$$

Proof. It is sufficient to substitute w in (11).

Obviously taking into account the Definition 1 and previous remark, we can provide a new version of Lemma 5 in the following form:

Lemma 8. *Let* f:I°⊆R→R *be a differentiable mapping on* I°, a,b∈ I°, with a<b. If f'∈L[a,b], *and w has first derivative integrable on* $I°$ *then the following equality holds:*

$$\begin{aligned}&\left(w(1)-w(0)\right)\left(f(b)-f(a)\right)+\frac{1}{b-a}\left[I_{a+}^{w}f(b)+I_{b-}^{w}f(a)\right]\\ =\quad &(b-a)\left[\int_0^1 w(1-t)f'\left(ta+(1-t)b\right)-\int_0^1 w(t)f'\left(ta+(1-t)b\right)dt\right].\end{aligned}$$

The second result of[17] and which will serve as a basis for commenting on a second generalization is the following (see Theorem 2.2):

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Theorem 9. *Let* f:I°⊆R→R *be a differentiable mapping on* I°, a,b∈ I°, with a<b. *If* |f'| *is convex on [a,b], then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{(b-a)\left(|f'(a)|+|f'(b)|\right)}{8}. \quad (15)$$

In[3] we presented the following definitions.

Definition 10. *Let h:[0,1]→**R** be a nonnegative function, h≠0 and ψ:I=[0,+∞)⟶ [0,+∞). If inequality*

$$\psi(\tau\xi+m(1-\tau)\varsigma) \le h^s(\tau)\psi(\xi)+m\left(1-h^s(\tau)\right)\psi(\varsigma) \quad (16)$$

is fulfilled for all ξ,ζI and τ∈[0,1], where m∈[0,1], s∈[-1,1]. Then the function is called a (h,m)-convex modified of the first type on I.

Definition 11. *Let h:[0,1]→**R** be a nonnegative function, h≠0 and :I=[0,+∞)⟶ [0,+∞). If inequality*

$$\psi(\tau\xi+m(1-\tau)\varsigma) \le h^s(\tau)\psi(\xi)+m\left(1-h(\tau)\right)^s\psi(\varsigma) \quad (17)$$

is fulfilled for all ξ,ζI and τ∈[0,1], where m∈[0,1], s∈[-1,1]. Then the function ψ is called a (h,m)-convex modified of the second type on I.

Remark 12. *Interested readers can verify that, under different considerations about h, m and s, many notions of convexity known from the literature can be derived from the previous definition.*

Now, taking into account Lemma 5 we can generalize the previous result in this form:

Theorem 13. *Let* f:I°⊆R→R *be a differentiable mapping on* I°, a,b∈ I°, with a<b. *If* |f'| *is (h,m)-convex modified of second type on [a,b], then the following inequality holds:*

$$\begin{aligned}&\left|\left(w(1)-w(0)\right)\left(f(b)-f(a)\right)-\frac{1}{b-a}\left[I^w_{a+}f(b)+I^w_{b-}f(a)\right]\right|\\ \le\;& (b-a)\Big\{\left(|f'(a)|+|f'(b)|\right)\int_0^1 w(t)h^s(t)\,dt \quad (18)\\ +\;& m\left(|f'(a/m)|+|f'(b/m)|\right)\int_0^1 w(t)\left(1-h(t)\right)^s dt\Big\}.\end{aligned}$$

Proof. Using I, I_1 and I_2 as in Lemma 5 we have by properties and using the (h,m)-convexity of |f'|:

$$\begin{aligned}|I_1| &\le \int_0^1 w(t)\left|f'((1-t)a+tb)\right|dt\\ &\le \int_0^1 w(t)\left[m|f'(a/m)|\left(1-h(t)\right)^s+|f'(b)|h^s(t)\right]dt\\ &= m|f'(a/a)|\int_0^1 w(t)\left(1-h(t)\right)^s dt+|f'(b)|\int_0^1 w(t)h^s(t)\,dt.\end{aligned}$$

Analogously for I_2 we have:

$$|I_2| \le |f'(a)|\int_0^1 w(t)h^s(t)\,dt+m|f'(b/m)|\int_0^1 w(t)\left(1-h(t)\right)^s dt.$$

From where we have

$$|I| \le \left(|f'(a)|+|f'(b)|\right)\int_0^1 w(t)h^s(t)\,dt+m\left(|f'(a/m)|+|f'(b/m)|\right)\int_0^1 w(t)\left(1-h(t)\right)^s dt.$$

After multiplying by (b-a) the previous result, the inequality (18) is obtained. This completes the proof.

Remark 14. *Under assumptions w(t)=t, m=s=1 and h(t)=t from Theorem 13 we obtain the Theorem 9.*

A last step in the generalization process is to consider the argument of the function, dependent on a parameter, so instead of working with (1-t)a+tb we would work with $\frac{(1-t)a}{n+1}+\frac{(n+t)b}{n+1}$, for example. So, we have the final result, the proof of which is left to the reader.

Lemma 15. *Let* f:I°⊆R→R *be a differentiable mapping on* I°, a,b∈ I°, with a<b. *If fL[a,b], and w has first derivative integrable on* I° *then the following equality holds:*

$$\begin{aligned}&\left[w(1)\left(f(b)+f(a)\right)-w(0)\left(f\left(\frac{a+nb}{n+1}\right)+f\left(\frac{na+b}{n+1}\right)\right)\right]\\ -\;&\left(\frac{n+1}{b-a}\right)\left\{J^w_{\frac{a+nb}{n+1}+}f(b)+J^w_{\frac{na+b}{n+1}-}f(a)\right\}\\ =\;&\frac{b-a}{n+1}\int_0^1 w(t)\left[f'\left(\frac{(1-t)a}{n+1}+\frac{(n+t)b}{n+1}\right)-f'\left(\frac{(n+t)a}{n+1}+\frac{(1-t)b}{n+1}\right)\right]dt.\end{aligned}$$

Remark 16. *It can be seen that under the considerations* $w(t)=t$ *and* $n=0$, *this Lemma becomes the classic Lemma 5. Obviously the rest of the results of the paper can be generalized following this same idea.*

An example

Following the generalizations of the original result of, we will present an example of the importance of the results obtained.

The following means for positive real numbers m, n, m≠n are known (arithmetic and generalized log-mean, respectively):

$$A(m,n)=\frac{m+n}{2}$$

$$L_p(m,n)=\left[\frac{n^{p+1}-m^{p+1}}{(p+1)(n-m)}\right]^{\frac{1}{p}}.$$

So we can obtain the following result:

Theorem 17. *Let m,n∈R, m<n and p∈N with p>2. Then, the following inequaty holds:*

$$\left|A(m^p,n^p)-L_p^p(m,n)\right| \le \frac{p(n-m)}{4}A\left(|m|^{p-1},|n|^{p-1}\right).$$

Proof. Using Theorem 9 with the convex function f(x)=x^p, x∈*R* we obtain

$$\left|\frac{m^p+n^p}{2}-\frac{1}{n-m}\frac{n^{p+1}-m^{p+1}}{p+1}\right| \le \frac{(n-m)}{8}p\left(|m|^{p-1}+|n|^{p-1}\right).$$

From where the desired inequality is obtained.

Obviously with other generalizations following the idea presented here, new relationships will be obtained, not only between these means.

Conclusion

In this work we have presented some methodological notes, which we have illustrated with a well-known classic result (Lemma 2.1 of the work[60] that has received more than a thousand citations, which clearly speaks of its seminal importance). We have shown the four fundamental steps that allow us to obtain new generalizations of the aforementioned Lemma. First the underlying idea that two functions can be used in said Lemma instead of just one, through a variable change. The introduction of weighted integrals, which allows us to give results for other integral operators, even fractional ones. The third step is the consideration of a new notion of convexity, the modified (h,m)-convex functions of the second type, which encompass many of the known definitions of convexity. And finally, the consideration of a functional argument that depends on such a parameter, instead of giving a new equality, we are giving "families" of equalities, which shows the breadth of the last Lemma presented above.

Obviously this idea does not stop here, it can be used in new directions of work for other integral inequalities. On this last point, we can indicate that this methodology is applicable to the case of Fractional Derivatives, in particular of the Caputo type (see[61]), where instead of the classic derivative, we use

Definition 18. *Let* $\alpha>0$, *and* $\alpha\neq 1,2,3,..$ $n=[\alpha]+1$, $f\in AC^n[a,b]$, *the space of functions that have the* n*-th absolutely continuous derivatives. The weighted Caputo derivatives of the right-hand side and the left-hand side of order* α *are defined as follows:*

$$\left({}_{n}^{C}D_{v_1+}^{w'}f\right)(v_2)=\int_{v_1}^{v_2}w'\left[\frac{\frac{v_2-x}{v_2-v_1}}{r+1}\right]f^{(n)}(x)dx,$$

$$\left({}_{n}^{C}D_{v_2-}^{w'}f\right)(v_1)=\int_{v_1}^{v_2}w'\left[\frac{\frac{x-v_1}{v_2-v_1}}{r+1}\right]f^{(n)}(x)dx.$$

One of the results obtained is the following (see Lemma 9, r is the parameter, because n indicates the order of the derivative considered):

Lemma 19. *Let f be a real function defined on the real Interval [a,b] and differentiable on (a,b). If If f∈L*$_1$*[a,b], and w(t) is a function differentiable on (a,b), then we have the following equality:*

$$\left\{-w(1)\left(f^{(n)}\left(\frac{a+rb}{r+1}\right)+f^{(n)}\left(\frac{ra+b}{r+1}\right)\right)+w(0)\left(f^{(n)}(a)+f^{(n)}(b)\right)\right\}$$

$$\frac{r+1}{b-a}\left[\left({}_{n}^{C}D_{\frac{ra+b}{r+1}-}^{w'}f\right)(a)+\left({}_{n}^{C}D_{\frac{a+rb}{r+1}+}^{w'}f\right)(b)\right]$$

$$=\frac{b-a}{r+1}\int_0^1 w(t)\left[f^{(n+1)}\left(\frac{t}{r+1}a+\frac{r+1-t}{r+1}b\right)-f^{(n+1)}\left(\frac{t}{r+1}b+\frac{r+1-t}{r+1}a\right)\right]dt.$$

Another of the inequalities where this methodology can be applied is the case of the Milne Inequality, an integral inequality involves integrals of functions and provides bounds or inequalities for these integrals based on conditions or assumptions on the certain integrands and the integration domain. Thus, in[62] we obtain new versions of this inequality on fractal sets, using the weighted fractal derivative:

Definition 20. *Let*φ *be a local fractional continuous on [a,b] and let* $w(x)\in I_x^{\delta}[a,b]$. *The right and left local fractional weighted integral of* φ *of order* δ *are given by*

$${}^{w}J_{a+}^{\delta}\phi(b)=\frac{1}{\Gamma(\delta+1)}\int_a^b w^{(\delta)}\left(\frac{t-a}{b-a}\right)\phi(t)dt^{\delta}$$

and

$${}^{w}J_{b-}^{\delta}\phi(a)=\frac{1}{\Gamma(\delta+1)}\int_a^b w^{(\delta)}\left(\frac{b-t}{b-a}\right)\phi(t)dt^{\delta}$$

and the following result is obtained (see Lemma 2):

Lemma 21. *Let* $\phi:[0,\infty)\to\mathbb{R}$ *be a local fractional differentiable function, with* $\phi^{(\delta)}\in L_1[a,b]$, $0\le a<b$ *and let* $w(x)\in {}_aJ_x^{\delta}[a,b]$. *If* $f\left(\frac{a}{m}\right)\in[a,b]$, *then we will have*

$$\left(\frac{n+2}{b-a}\right)^{\delta}\left\{w(1)\left[\varnothing\left(\frac{na+2b}{n+2}\right)-\varnothing\left(\frac{2a+nb}{n+2}\right)\right]\right.$$

$$\left.-w(0)\left[\varnothing\left(\frac{(n+1)a+b}{n+2}\right)-\varnothing\left(\frac{a+(n+1)b}{n+2}\right)\right]\right\}$$

$$-\left(\frac{n+2}{b-a}\right)^{2\delta}\left[{}^{w}J_{\frac{(n+1)a+b}{n+2}+}^{\delta}\varnothing\left(\frac{na+2b}{n+2}\right)+{}^{w}J_{\frac{a+(n+1)b}{n+2}-}^{\delta}\varnothing\left(\frac{2a+nb}{n+2}\right)\right]$$

$$=\int_0^1 w(t)\left[\varnothing^{\delta}\left(\frac{n+1-t}{n+2}a+\frac{1+t}{n+2}b\right)-\varnothing^{\delta}\left(\frac{1+t}{n+2}a+\frac{n+1-t}{n+2}b\right)\right]dt^{\delta}$$

From the point of view of concrete applications, in,[34] using this methodology, we obtain two Propositions that present estimates between the arithmetic mean and the logarithmic mean (see Propositions 3.1 and 3.2).

To conclude, we want to point out that these generalizations can be applied in new directions of work, linked to new notions of convexity. For example, in[63] exponentially convex functions are defined and new versions of the Hermite-Hadamard Inequality are obtained in this framework. Obviously, we can state and prove, using weighted integrals, new results for this new functional class.

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References

1. Nápoles Valdes JE, Rabossi F, Samaniego AD. Convex functions: Ariadne's thread or charlotte's spiderweb?. *Advanced Mathematical Models & Applications.* 2020;5(2):176–191.
2. Hadamard J. Étude sur les propriétés des fonctions entières et en particulier d'une fonction considerée par Riemann. *J Math Pures Appl.* 1893;58:171–215.
3. Hermite C. Sur deux limites d'une intégrale définie. *Mathesis.* 1883;3:82.
4. Hardy GH, Littlewood JE. George Pólya, Inequalities, Cambridge University Press, 1952.
5. Nápoles Valdés JE. A Review of Hermite–Hadamard Inequality. *Partners Universal International Research Journal (PUIRJ).* 2022;1(4):98–101.
6. Mitrinovic DS, Lackovic IB. Hermite and convexity. *Aequationes Mathematicae.* 1985;28:229–232.
7. Ali MA, Nápoles JE V, Kashuri A, et al. Fractional non conformable Hermite–Hadamard inequalities for generalized –convex functions. *Fasciculi Mathematici.* 2020;64:5–16.
8. B Bayraktar, JE Nápoles V. A note on Hermite–Hadamard integral inequality for (h, m)–convex modified functions in a generalized framework. Archives des sciences / éditées par la Société de physique et d'histoire naturelle de Genève. 2024;74(3):118–122.
9. B Bayraktar, JE Nápoles Valdés. New generalized integral inequalities via (h,m)–convex modified functions. *Izvestiya Instituta Matematiki i Informatiki Udmurtskogo Gosudarstvennogo Universiteta*. 2022;60:3–15.

10. B Bayraktar, JE Nápoles Valdés. Hermite–Hadamard weighted integral inequalities for (h,m)–convex modified functions. *Fractional Differential Calculus.* 2022;12(2):235–248.
11. B Bayraktar, Juan E Nápoles Valdés. Integral inequalities for mappings whose derivatives are (h,m,s)–convex modified of second type via Katugampola integrals. *Annals of the University of Craiova, Mathematics and Computer Science Series.* 2022;49(2):2022.
12. B Bayraktar, JE Nápoles Valdés. Florencia Rabossi, et al. Samaniego, Some extensions of the Hermite–Hadamard inequalities for quasi–convex functions via weighted integral. *Proyecciones Journal of Mathematics.* 2023;42(5):1221–1239.
13. Galeano Delgado J, Lloreda J, Nápoles JEV, et al. Certain integral inequalities of hermite–hadamard type for h–convex functions. *Journal of Mathematical Control Science and Applications.* 2021;7(2):129–140.
14. Juan Gabriel Galeano Delgado, Juan E Nápoles Valdés, Edgardo Pérez Reyes. Some inequalities of the Hermite–Hadamard type for two kinds of convex functions. *Revista Colombiana de Matemáticas.* 2023;57:43–55.
15. Paulo M Guzmán, Juan E Nápoles Valdes, Vuk Stojiljkovic. New extensions of the Hermite–Hadamard inequality. *Contrib Math.* 2023;7:60–66.
16. Artion Kashuri, Juan E Nápoles Valdés, Muhammad Aamir Ali, et al. New integral inequalities using quasi–convex functions via generalized integral operators and their applications. *Applied Mathematics E–Notes.* 2022;22: 221–231.
17. Kórus P, Nápoles Valdés JE. On some integral inequalities for (h, m)–convex functions in a generalized framework. Carpathian Journal of Mathematics. *Carpathian Math Publ.* 2023;15(1):137–149.
18. Lugo LM, Nápoles Valdés JE. Hermite–Hadamard Type Inequalities with Generalized Integrals for various Kinds of Convexity. *Contrib Math.* 2022;5:45–51.
19. Nápoles Valdes JE, Florencia Rabossi, Hijaz Ahmad. Inequalities of the hermite–hadamard type, for functions (h,m)convex modified of the second type. *Commun Combin Cryptogr & Computer Sci.* 2021;1:33–43.
20. Miguel Vivas–Cortez, S Kermausuor, JE Nápoles Valdés. Hermite–Hadamard Type Inequalities for Coordinated Quasi–Convex Functions via Generalized Fractional Integrals, in P. Debnath et al. (eds.), Fixed Point Theory and Fractional Calculus: Recent Advances and Applications, Forum for Interdisciplinary Mathematics, Springer Nature Singapore Pte Ltd. 2022.
21. B Bayraktar, SI Butt, JE Nápoles. Some new estimates of integral inequalities and their applications. 2024;76(2):159–178.
22. Bahtiyar B, Nápoles JE V. New integral inequalities of Hermite–Hadamard type in a generalized context. *Punjab University journal of mathematics.* 2021.
23. B Bayraktar, JE Nápoles V, F Rabossi. On generalizations of integral inequalities. *Probl Anal Issues Anal.* 2022;11(29)3–23.
24. B Bayraktar, JE Nápoles, F Rabossi. Some refinements of the hermite–hadamard inequality with the help of weighted integrals. *Ukrains'kyi Matematychnyi Zhurnal.* 2023;75(6):2023.
25. S Bermudo, P Kórus, JE Nápoles V. On q–Hermite–Hadamard inequalities for general convex functions. *Acta Math Hungar.* 2020;162:364–374.
26. M Bohner, A Kashuri, PO Mohammed, et al. Hermite–Hadamard–type inequalities for conformable integrals. *Hacet J Math Stat.* 2022;51(3):775–786.
27. Saad Ihsan Butt, Bahtiyar Bayraktar, Juan E Nápoles Valdes. Some new inequalities of Hermite–Hadamard type via Katugampola fractional integral. *Punjab University Journal of Mathematics.* 2023;55(7–8):269–289.
28. Galeano Delgado J, Nápoles Valdés JE, Edgardo Pérez Reyes. A note on some integral inequalities in a generalized framework. *Int J Appl Math Stat.* 2021;60(1):45–52.
29. Galeano Delgado J, Nápoles Valdés JE, Edgardo Pérez Reyes. Several Integral Inequalities For Generalized Riemann–Liouville Fractional Operators. *Commun Fac Sci Univ Ank Ser A1 Math Stat.* 2021;70(1):269–278.
30. Galeano Delgado J, Nápoles Valdés JE, Pérez Reyes E. New Hermite–Hadamard inequalities in the framework of generalized fractional integrals, Annals of the University of Craiova. *Mathematics and Computer Science Series.* 2021;48(2):319–327.
31. Galeano Delgado J, Nápoles Valdés JE, Pérez Reyes E. Concerning the generalized Hermite–Hadamard integral inequality. *Sigma J Eng Nat Sci.* 2023;41(2):226–231.
32. Juan Gabriel Galeano Delgado, Juan E Nápoles Valdés, Edgardo Pérez Reyes. New integral inequalities involving generalized Riemann–Liouville fractional operators. *Stud Univ Babe¸s–Bolyai Math.* 2023;68(3):481–487.
33. Yusif S Gasimov, Juan Eduardo Nápoles Valdés. Some refinements of hermite–hadamard inequality using –fractional caputo derivatives. *Fractional Differential Calculus.* 2022;12(2):209–221.
34. Guzmán PM, Lugo LM, Nápoles JE, et al. On a new generalized integral operator and certain operating properties. *Axioms.* 2020;9:69.
35. Guzmán PM, Nápoles JE, Gasimov Y. Integral inequalities within the framework of generalized fractional integrals. *Fractional Differential Calculus.* 2021;11(1):69–84.
36. Kórus P, Lugo LM, Nápoles Valdés JE. Integral inequalities in a generalized context. *Studia Scientiarum Mathematicarum Hungarica.* 2020;57(3):312–320.
37. Kórus P, Nápoles Valdés J E. Some hermite–hadamard inequalities involving weighted integral operators via–convex functions. *TWMS J App And Eng Math.* 2023;13(4):1461–1471.
38. Peter Kórus, Juan Eduardo Nápoles Valdés, Bahtiyar Bayraktar. Weighted Hermite–Hadamard integral inequalities for general convex functions. *Math Biosci Eng.* 2023;20(11):19929–19940.
39. Péter Kórus, Juan E Nápoles Valdés, María N. Hermite–Hadamard type inequalities via weighted integral operators. *Proyecciones Journal of Mathematics.* 2023;42(6):1499–1519.
40. Mehmood S, Nápoles Valdés JE, Fatima N, et al. Some integral inequalities via fractional derivatives. *Adv Studies: Euro–Tbilisi Math J.* 2022;15(3):31–44.
41. Mehmood S, Nápoles Valdés JE, Fatima N, et al. Some New Inequalities Using Conformable Fractional Integral of Order. *Journal of Mathematical Extension.* 2021;15:1–22.
42. Nápoles Valdés JE. A Generalized–Proportional Fractional Integral Operators with General Kernel, submited.
43. Nápoles Valdés JE. Some integral inequalities in the framework of Generalized–Proportional Fractional Integral Operators with General Kernel. *Honam Mathematical J.* 2021;4:587–596.
44. Nápoles Valdés JE. On the Hermite–Hadamard type inequalities involving generalized integrals. *Contrib Math.* 2022;5:45–51.
45. Nápoles Valdés JE, Bahtiyar Bayraktar. On the generalized inequalities of the hermite–hadamard type. *Filomat.* 2021;35(14):4917–4924.
46. Nápoles JE, Bahtiyar Bayraktar. New extensions of Hermite–Hadamard inequality using–fractional Caputo derivatives. *Advanced Studies: Euro–Tbilisi Mathematical Journal.* 2023;16(2):11–27.
47. Nápoles Valdés JE, Bahtiyar Bayraktar, Saad Ihsan Butt. New integral inequalities of Hermite–Hadamard type in a generalized context. *Punjab University Journal of Mathematics.* 2021;53(11):765–777.
48. Juan E Nápoles Valdes, Florencia Rabossi. Generalized fractional operators and inequalities integrals, in Bipan Hazarika, Santanu Acharjee, H. M. Srivastava (eds.), Advances in Mathematical Analysis and its Applications. Chapman and Hall/CRC, New York, 2022.

49. Nápoles Valdés JE, Rodríguez JM, Sigarreta JM. On Hermite–Hadamard type inequalities for non–conformable integral operators. *Symmetry.* 2019;11:1108.

50. Edgardo Pérez Reyes, Juan E Nápoles Valdés, Bahtiyar Bayraktar. On the hermite–hadamard inequality via generalized integrals. *Turkish J Ineq.* 2023;7(1):1–11.

51. Miguel Vivas–Cortez, Juan EJ E Nápoles Valdés, JA Guerrero. Some Hermite–Hadamard Weighted Integral Inequalities for (h,m)–Convex Modified Functions. *Appl Math Inf Sci.* 2022;16(1):25–33.

52. Miguel Vivas–Cortez, Juan E J E Nápoles Valdés, Praveen Agarwal, et al. Concerning the Inequality of Hermite–Hadamard Generalized. *Appl Math Inf Sci.* 2023;17(4):649–657.

53. Mubeen S, Habibullah GM. Fractional integrals and applications. *Int J Contemp Math Sci.* 2021;7:89–94.

54. Akkurt A, Yildirim ME, Yildirim H. On some integral inequalities for (k,h)–RiemannLiouville fractional integral. *NTMSCI.* 2016;4(1):138–146.

55. Jarad T, Abdeljawad T. On the weighted fractional operators of a function with respect to another function. *Fractals.* 2020;28(8):2040011.

56. Sarikaya MZ, Ertugral F. On the generalized Hermite–Hadamard inequalities. *Annals of the University of Craiova. Mathematics and Computer Science Series.* 2020;47(1):193–213.

57. Ugurlu E, Abdeljawad T, Baleanu D. On a new class of fractional operators. *Adv Differ Equ.* 2017;2017: 247.

58. Khan TU, Khan MA. Generalized conformable fractional integral operators. *J Comput Appl Math.* 2019;346: 378–389.

59. Tomar M, Set E, Sarikaya MZ. Hermite–Hadamard type Riemann–Liouville fractional integral inequalities for convex functions. *AIP Conf Proc.* 2016;1726:020035.

60. Dragomir SS, Agarwal RP. Two Inequalities for Differentiable Mappings and Applications to Special Means of Real Numbers and to Trapezoidal Formula. *Appl Math Lett.* 1998;11(5):91–95.

61. Guzmán PM, Nápoles JE, Murat Cancan, et al. Some Integral Inequalities for Differentiable S–Convex and (H, M)–Convex Functions Through Generalized Caputo–Type Derivatives. *Power system technology.* 2024;48(1):1188–1201.

62. Nápoles JE, Guzmán PM, Bayraktar B. Milne–type integral inequalities for modified–convex functions on fractal sets. *Probl Anal Issues Anal.* 2024;13(31).

63. Kadakal M, Iscan I. Exponential type convexity and some related inequalities. *J Inequal Appl.* 2020;82.

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