Review Article

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Combination of He method and Kharrat-Toma transform for solving boundary and initial value problems

Abstract

This paper discusses and implements a newly developed technique using the He method with Kharrat-Toma Transform. The main aim is to solve some initial and boundary problems. This combination exhibits an accurate strategy to obtain a precise solution for linear and nonlinear problems. To validate the proposed Hybrid method, a 4- examples are discussed, these including: Burger's equation a differential equation used in fluid mechanics to represented the flow of a fluid, telegraph equation that used in electric waves, Kelin-Gordan equation used in quantum mechanics, modified Boussinesq equation. The obtained results improve the exactness and the accuracy of the proposed combinations, and the proposed method is capable to solve a large number of linear and nonlinear initial and boundary value problems.

Keywords: Kharrat-Toma Transform, Burger's equation, telegraph equation, Kelin-Gordan equation, modified Boussinesq equation, He's Polynomial

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Introduction

Burger's equation was presented for the time by Bateman in1915.[1] It followed by Hradyesh kumar Mishra and Atulya K. Nagar and it is solved using He-laplace method in 2012,[2] then it followed by Mahgoub, MAM and Al Shikhit's solved using Mahgoub transform in 2017,[3] Mohand, Mohamed Zebir solved it via Mohand transform in 2021,[4] then it followed by Sarah Rabie, Bachir Nour Kharrat, Ghada Joujeh, Abd Alulkader Joukhadar, solved using He-Mohand method in 2023,[5] then it followed by Sarah Rabie, Bachir Nour Kharrat, Ghada Joujeh, Abd Alulkader Joukhadar, solved using He-Sawi method in 2023.[6]

In work[7] Muhammad Nadeem and fengquanlil using He-laplace method to solve telegraph equation in2019, then it followed by Sarah Rabie, Bachir Nour Kharrat, Ghada Joujeh, Abd Alulkader Joukhadar, solved using He-Mohand method in 2023,[5] then it followed by Sarah Rabie, Bachir Nour Kharrat, Ghada Joujeh, Abd Alulkader Joukhadar, solved using He-Sawi method in 2023.[6]

In 2010[8] MAJafari and Aminataei followed Homotopy Perturbation method (HPM) to solve Kelin-Gorden equation, then in 2012[2] Hradyesh kumar Mishra and Atulya K. Nagar and it is solved using He-laplace method, then it solved by Sarah Rabie, Bachir Nour Kharrat, Ghada Joujeh, Abd Alulkader Joukhadar, solved using He-Sawi method in 2023.[6] Modified Boussinesq equation solved by Muhammad Nadeem and fengquanlil using He-laplace method in 2019.[7]

Basic concepts

This section provides review some of the basic concepts, which needed for this paper:

3.1.Definition of Kharrat Toma transform:

Kharrat-Toma Transform of the function G(s); s>0 was proposed by the Syrian resrarchers Kharrat and Toma,[9] in 2020 is given as:

$$B\left[f(x)\right]=G(s)=s^3\int_0^{\infty} f(x)e^{-\frac{x}{s^2}}dx \quad ,x\geq 0 \qquad (1)$$

$$f(x)=B^{-1}\left[G(s)\right]=B^{-1}\left[s^3\int_0^{\infty} f(x)e^{-\frac{x}{s^2}}dx\right]$$

Where: ($\mathcal{B}$) is Kharrat-Toma Transform operator.

3.2.Some properties of Kharrat-Toma transform:[9]

If $B\{f(x)\}=G(s)$

$$B\left\{f^{(n)}(x)\right\}=\frac{1}{s^{2n}}G(s)-\sum_{k=0}^{n-1}s^{-2n+2k+5}f^{(k)}(0)\ ;n\geq 1$$

Table 1 Shows the Kharrat-Toma of some elementary functions:

G(s)	$B\{f(x)\}=G(s)$
1	s^5
x^n	$s^{2n+5}n!$
e^{at}	$\frac{1}{v(1-av)}$
$\sin(kx)$	$\frac{k\,s^7}{1+k^2s^4}$
$\cos(kt)$	$\frac{s^5}{1+k^2s^4}$
$sh(kx)$	$\frac{k\,s^7}{1-k^2s^4}$
$\cosh(kx)$	$\frac{s^5}{1-k^2s^4}$

Analysis of the proposed combined method

In order to explain the proposed method let's consider the following nonlinear functional equation:

$$L(u(x))+N(u(x))=g(x) \qquad (2)$$

Where: L and N are linear and nonlinear operator respectively.

g(x): is analytical function.

Taking the Kharrat-Toma Transform of equation (2) and obtain:

$$B\{L(u(x))+N(u(x))-g(x)\}=0 \quad (3)$$

Then multiplying the (3) equation with Lagrange multiplier, say $\lambda(s)$, we get:

$$\lambda(s)B\{L(u(x))+N(u(x))-g(x)\}=0 \quad (4)$$

Therefore, the recurrence relation becomes:

$$u_{n+1}(x,s)=u_n(x,s)+\lambda(s)\{B\{L(u_n(x))\}+B\{N(\tilde{u}_n(x))-g(x)\}\} \quad (5)$$

Taking the variation of equation (5) results in:

$$\delta u_{n+1}(x,s)=\delta u_n(x,s)+\lambda(s)\delta\{s\{L(u_n(x))\}+s\{N(\tilde{u}_n(x))-g(x)\}\} \quad (6)$$

To identify the value of Lagrange multiplier $\lambda(s)$ with the help of Kharrat-Toma. Transform, it is revealed that $\tilde{u}_n$ is a restricted variable, i,e, $\delta\tilde{u}_n=0$ taking the inverse of Kharrat-Toma Transform of equation (5) this results in:

$$u_{n+1}(x,s)=u_n(x,s)+B^{-1}\{\lambda(s)\{B\{L(u_n(x))\}+B\{N(\tilde{u}_n(x))-g(x)\}\}\} \quad (7)$$

Test examples

The following section presents a descriptive examples of the proposed method.

Conider Burger's equation:

$$u_t=u_{xx}-uu_x \quad (8)$$

With initial condition of:

$$u(x,0)=1-\frac{2}{x} \quad (9)$$

taking the Kharrat-Toma transform of equation (8):

$$B\{u_t-u_{xx}+uu_x\}=0 \quad (10)$$

Multiplying the equation (10) with $\lambda(s)$ results in:

$$\lambda(s)B\{u_t-u_{xx}+uu_x\}=0$$

The recurrence relation takes the form:

$$u_{n+1}(x,s)=u_n(x,s)+\lambda(s)B\left\{\frac{\partial u_n}{\partial t}-\frac{\partial^2 u_n}{\partial x^2}+u_n\frac{\partial u_n}{\partial x}\right\} \quad (11)$$

taking the variation of equation (11):

$$\delta u_{n+1}(x,s)=\delta u_n(x,s)+\lambda(s)\delta\left\{\frac{1}{s^2}u_n(x,s)-s^3\tilde{u}_n(x,0)\right\}$$

$$+\lambda\delta B\left\{-\frac{\partial^2\tilde{u}_n}{\partial x^2}+u_n\frac{\partial\tilde{u}_n}{\partial x}\right\}$$

$$\delta u_{n+1}(x,s)=\delta u_n(x,s)+\lambda\frac{1}{s^2}\delta u_n$$

In turn gives the value of λ becomes as follows:

$$0=1+\frac{1}{s^2}\lambda$$

$$\lambda=-s^2$$

Which: $\tilde{u}_n$ is a restricted variable $\delta\tilde{u}_n=0$ and $\frac{\delta u_{n+1}}{\delta u_n}=0$ using the value of $\lambda=-s^2$, will result in:

$$u_{n+1}(x,s)=u_n(x,s)-s^2B\left\{\frac{\partial u_n}{\partial t}-\frac{\partial^2 u_n}{\partial x^2}+u_n\frac{\partial u_n}{\partial x}\right\} \quad (12)$$

Taking the inverse Kharrat-Toma Transform of equation (12):

$$u_{n+1}(x,t)=u_n(x,t)-B^{-1}\left\{s^2\,B\left\{-\frac{\partial^2 u_n}{\partial x^2}+u_n\frac{\partial u_n}{\partial x}\right\}\right\}$$

Applying He's polynomial formula, yields:

$$u_0+pu_1+..=u_n$$

$$-pB^{-1}\left\{-s^2B\left\{\left(-\frac{\partial^2 u_0}{\partial x^2}+u_0\frac{\partial u_0}{\partial x}\right)+p\left(-\frac{\partial^2 u_1}{\partial x^2}+u_1\frac{\partial u_0}{\partial x}+u_0\frac{\partial u_1}{\partial x}\right)+\ldots\right\}\right\}$$

Equating highest power of p will result in:

$$u_0=1-\frac{2}{x}$$

$$u_1=-B^{-1}\left\{-s^2B\left\{\left(-\frac{\partial^2 u_0}{\partial x^2}+u_0\frac{\partial u_0}{\partial x}\right)\right\}\right\}=-\frac{2}{x^2}t$$

$$u_2=-B^{-1}\left\{-s^2B\left\{\left(-\frac{\partial^2 u_1}{\partial x^2}+u_1\frac{\partial u_0}{\partial x}+u_0\frac{\partial u_1}{\partial x}\right)\right\}\right\}=-\frac{2}{x^3}t^2$$

Hence the series solution can expressed as:

$$u(x,t)=u_0+u_1+\ldots=1-\frac{2}{x}-\frac{2}{x^3}t^2-\ldots=1-\frac{2}{x-t}$$

4-2: Consider the following Telegraph's equation:

$$u_{xx}=\frac{1}{3}u_{tt}+\frac{4}{3}u_t+u \quad (13)$$

With initial conditions:

$$u(x,0)=e^x+1 \qquad u_t(x,0)=-3 \quad (14)$$

and boundary conditions:

$$u(0,t)=e^{-3t}+1 \qquad u_x(0,t)=1 \quad (15)$$

Taking the Kharrat-Toma Transform of equation (13):

$$B\left\{-u_{xx}+\frac{1}{3}u_{tt}+\frac{4}{3}u_t+u\right\}=0 \quad (16)$$

Multiplying the equation (16) with $\lambda(s)$:

$$\lambda(s)B\left\{-u_{xx}+\frac{1}{3}u_{tt}+\frac{4}{3}u_t+u\right\}=0 \quad (17)$$

The recurrence relation takes the form:

$$u_{n+1}(x,s)=u_n(x,s)+\lambda(s)B\left[\frac{1}{3}\frac{\partial^2 u_n}{\partial t^2}-\frac{\partial^2 u_n}{\partial x^2}+u_n+\frac{4}{3}\frac{\partial u_n}{\partial t}\right] \quad (18)$$

Taking the variation of equation (18):

$$\delta u_{n+1}=\delta u_n+\lambda(s)\delta B\left[\frac{1}{3}\frac{\partial^2 u_n}{\partial t^2}-\frac{\partial^2 u_n}{\partial x^2}+u_n+\frac{4}{3}\frac{\partial u_n}{\partial t}\right]$$

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$$\delta u_{n+1}=\delta u_n+\frac{\lambda\delta}{3}\left\{\left(\frac{1}{s^4}u_n(x,s)-s^3\tilde{u}'_n(x,0)^{`}-s\tilde{u}_n(x,0)\right)\right\}+\lambda\delta B\left\{-\frac{\partial^2\tilde{u}_n}{\partial x^2}+\tilde{u}_n+\frac{4}{3}\frac{\partial\tilde{u}_n}{\partial t}\right\}$$

$$\delta u_{n+1}=\delta u_n+\lambda\frac{1}{3s^4}\delta u_n$$

In turn gives the value of λ becomes as follows:

$$0=1+\lambda\frac{1}{3s^4}$$

$$\lambda=-3s^4$$

Which: $\tilde{u}_n$ is a restricted variable $\delta\tilde{u}_n=0$ and $\frac{\delta u_{n+1}}{\delta u_n}=0$ using the value of $\lambda(s)=-3s^4$ in equation (18), will result in:

$$u_{n+1}(x,s)=u_n(x,s)-3s^4B\left[\frac{1}{3}\frac{\partial^2u_n}{\partial t^2}-\frac{\partial^2u_n}{\partial x^2}+u_n+\frac{4}{3}\frac{\partial u_n}{\partial t}\right] \quad (19)$$

Taking the inverse Kharrat-Toma Transform of equation (19):

$$u_{n+1}(x,t)=u_n(x,t)-B^{-1}[3S^4B\left[-\frac{\partial^2u_n}{\partial x^2}+u_n+\frac{4}{3}\frac{\partial u_n}{\partial t}\right] \quad (20)$$

Applying He's polynomial formula, yields:

$$u_0+pu_1+p^2u_2+\ldots=$$

$$u_n-pB^{-1}\{3s^4B\left\{\left(-\frac{\partial^2u_0}{\partial x^2}+u_0+\frac{4}{3}\frac{\partial u_0}{\partial t}\right)+p\left(-\frac{\partial^2u_1}{\partial x^2}+u_1+\frac{4}{3}\frac{\partial u_1}{\partial t}\right)+p^2\left(-\frac{\partial^2u_2}{\partial x^2}+u_2+\frac{4}{3}\frac{\partial u_2}{\partial t}\right)+\ldots\right\}$$

Equating highest power of p will result in:

$$p^0: u_0=e^x+1-3t$$

$$p^1: u_1=-B^{-1}\left\{3s^4B\left\{\left(-\frac{\partial^2u_0}{\partial x^2}+u_0+\frac{4}{3}\frac{\partial u_0}{\partial t}\right)\right\}\right\}=\frac{9t^2}{2}+\frac{3t^3}{2}$$

$$p^2: u_2=-B^{-1}\left\{3s^4B\left\{\left(-\frac{\partial^2u_1}{\partial x^2}+u_1+\frac{4}{3}\frac{\partial u_1}{\partial t}\right)\right\}\right\}=-6t^3-\frac{21}{8}t^4-\frac{9}{40}t^5$$

Hence the series solution can expressed as:

$$u(x,t)=u_0+u_1+u_2+\ldots=e^x+1-3t+\frac{9t^2}{2}-\frac{9t^3}{2}+\frac{27}{8}t^4+\ldots=e^x+e^{-3t}$$

Consider the following Kelin-Gorden equation:

$$\frac{\partial^2u}{\partial t^2}+u+\frac{\partial^2u}{\partial x^2}=0 \quad (21)$$

With initial conditions:

$$u(x,0)=e^{-x}+x \qquad \frac{\partial u}{\partial t}(x,0)=0 \quad (22)$$

Taking the Kharrat-Toma Transform of equation (21):

$$B\left\{\frac{\partial^2u}{\partial t^2}+u+\frac{\partial^2u}{\partial x^2}\right\}=0 \quad (23)$$

Multiplying the equation (23) with $\lambda(s)$:

$$\lambda(s)\ B\left\{\frac{\partial^2 u}{\partial t^2}+u+\frac{\partial^2 u}{\partial x^2}\right\}=0 \qquad (24)$$

The recurrence relation takes the form:

$$u_{n+1}=u_n+\lambda\ B\left\{\frac{\partial^2 u_n}{\partial t^2}+u_n+\frac{\partial^2 u_n}{\partial x^2}\right\} \qquad (25)$$

Taking the variation of equation (25):

$$\delta u_{n+1}=\delta u_n+\lambda\delta\left\{\left(\frac{1}{s^4}u_n(x,s)-s^3\,\tilde{u}'_n(x,0)^{`}-s\,\tilde{u}_n(x,0)\right)\right\}+\lambda\,\delta B\left\{\tilde{u}_n+\frac{\partial^2 \tilde{u}_n}{\partial x^2}\right\}$$

$$\delta u_{n+1}=\delta u_n+\lambda\frac{1}{s^4}\delta u_n$$

in turn gives the value of λ becomes as follows:

$$0=1+\lambda\frac{1}{s^4}$$

$$\lambda=-s^4$$

Which: $\tilde{u}_n$ is a restricted variable $\delta\tilde{u}_n=0$ and $\frac{\delta u_{n+1}}{\delta u_n}=0$ using the value of $\lambda(v)=-s^4$, will result in:

$$u_{n+1}=u_n-s^4\,B\left\{\frac{\partial^2 u_n}{\partial t^2}+u_n+\frac{\partial^2 u_n}{\partial x^2}\right\} \qquad (26)$$

Taking the inverse of Kharrat-Toma Transform of equation (26):

$$u_{n+1}=u_n-B^{-1}\left\{s^4\,B\left\{\frac{\partial^2 u_n}{\partial t^2}+u_n+\frac{\partial^2 u_n}{\partial x^2}\right\}\right\}$$

Applying He's polynomial formula, yields:

$$u_0+pu_1+p^2u_2+\ldots=u_n-pB^{-1}\{s^4B\left\{\left(u_0+\frac{\partial^2 u_0}{\partial x^2}\right)+p\left(u_1+\frac{\partial^2 u_1}{\partial x^2}\right)+\ldots\right\}$$

Equating highest power of p will result in:

$$u_0=e^{-x}+x$$

$$u_1=-x\frac{t^2}{2}$$

$$u_2=x\frac{1}{4!}t^4$$

Hence the series solution can expressed as:

$$u(t)=u_0+u_1+u_2+u_3+\ldots=e^{-x}+x-x\frac{t^2}{2}+x\frac{1}{4!}t^4=e^{-x}+x\cos(t)$$

4-4: Consider modified Boussinesq equation:

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$$u_{tt} - u_{xxxx} - \left(u^3\right)_{xx} = 0 \qquad (27)$$

With initial condition:

$$u(x,0) = \sqrt{2}\operatorname{sech}(x) \qquad u_t(x,0) = \sqrt{2}\operatorname{sech}(x)\tanh(x)$$

Taking the Kharrat-Toma transform of equation (27):

$$B\left\{\frac{\partial^2 u}{\partial t^2} - \frac{\partial^4 u}{\partial x^4} - \frac{\partial^2 u^3}{\partial x^2}\right\} = 0 \qquad (28)$$

Multiplying the equation (28) with $\lambda(s)$ results in:

$$ëB\left\{\frac{\partial^2 u}{\partial t^2} - \frac{\partial^4 u}{\partial x^4} - \frac{\partial^2 u^3}{\partial x^2}\right\} = 0$$

The recurrence relation takes the form:

$$u_{n+1}(xës) = u_n(x,s) + \quad B\left\{\frac{\partial^2 u_n}{\partial t^2} - \frac{\partial^4 u_n}{\partial x^4} - \frac{\partial^2 u_n{}^3}{\partial x^2}\right\} \qquad (29)$$

taking the variation of equation (29):

$$\delta u_{n+1}(xëä) = \delta u_n(x,s) + \quad \left\{\frac{1}{s^4} u_n 0(x\ s) - s^3 \tilde{u}_n 0(x\quad) - s\,\tilde{u}_n(x\quad)\right\}$$

$$-ëäB\left\{\frac{\partial^4 \tilde{u}_n}{\partial x^4} + \frac{\partial^2 \tilde{u}_n{}^3}{\partial x^2}\right\}$$

$$\delta u_{n+1}(x,s) = \delta u_n(x,s) + \frac{1}{s^4}\lambda \delta u_n(x,s)\ ;\delta u_{n+1}(s) = 0$$

In turn gives the value of λ becomes as follows:

$$\lambda = -\frac{1}{s^4}$$

Which: $\tilde{u}_n$ is a restricted variable $\delta \tilde{u}_n = 0$ and $\frac{\delta u_{n+1}}{\delta u_n} = 0$ using the value of $\lambda = -\frac{1}{s^4}$ will result in:

$$u_{n+1}(x,s) = u_n(x,s) - \frac{1}{s^4} B\left\{\frac{\partial^2 u_n}{\partial t^2} - \frac{\partial^4 u_n}{\partial x^4} - \frac{\partial^2 u_n{}^3}{\partial x^2}\right\}$$

Taking the inverse Kharrat-Toma Transform of equation (12):

$$u_{n+1}(x,t) = u_n(x,t) - B^{-1}\left\{\frac{1}{s^4}\mathcal{L}\left\{\frac{\partial^2 u_n}{\partial t^2} - \frac{\partial^4 u_n}{\partial x^4} - \frac{\partial^2 u_n{}^3}{\partial x^2}\right\}\right\}$$

Applying He’s polynomial formula, yields:

$$u_0 + pu_1 + p^2 u_2 + \ldots$$

$$= u_n(x,t) + pB^{-1}\left\{\frac{1}{s^4}\left\{\left(\frac{\partial^4 u_0}{\partial x^4} + \frac{\partial^2 u_0{}^3}{\partial x^2}\right) + p\left(\frac{\partial^4 u_1}{\partial x^4} + 3\frac{\partial^2\left(u_0^2 u_1\right)}{\partial x^2}\right) + \ldots\right\}\right\}$$

Equating highest power of p will result in:

$$p^1 : u_1 = \frac{1}{2}t^2(-\sqrt{2}\mathrm{sech}^5(x)$$
$$+\sqrt{2}\mathrm{sech}(x)\tanh^4(x)) + \frac{1}{6}t^3\left(-5\sqrt{2}\mathrm{sech}^5(x)\tanh(x) - 4\sqrt{2}\mathrm{sech}^3(x)\tanh^3(x) + \sqrt{2}\mathrm{sech}(x)\tanh^5(x)\right)$$
$$+\frac{1}{2}t^4\left(2\sqrt{2}\mathrm{sech}^7(x) - 19\sqrt{2}\mathrm{sech}^5(x)\tanh^2(x) + 9\sqrt{2}\mathrm{sech}^3(x)\tanh^4(x)\right)$$
$$+\frac{1}{10}t^5\left(2\sqrt{2}\mathrm{sech}^7(x)\tanh(x) - 9\sqrt{2}\mathrm{sech}^5(x)\tanh^3(x) + 3\sqrt{2}\mathrm{sech}^3(x)\tanh^5(x)\right)$$

Hence the series solution can expressed as:

$$u(x,t) = u_0 + u_1 + \ldots = \sqrt{2}\mathrm{sech}(x-t)$$

Conclusion

For most of the applications which have been studied in literature, the present study has provided more precise solutions with fewer iteration, compared to other methods. For future research work, it is recommended to combines He-Kharrat-Toma method with other integral transform such as: Foks, Abood, sumdu and Elzaki.

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