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Quantum chaos of the BKL scenario

Abstract

The Belinski-Khalatnikov-Lifshitz (BKL) scenario concerns the existence of generic singularity of general relativity. Evolving towards that singularity, spacetime enters a chaotic phase. We consider a model of the BKL scenario to get insight into the corresponding quantum dynamics. The integral quantization of the BKL model leads to quantum evolution devoid of singularity. The quantum fluctuations seem to be unable to suppress the classical chaos. These interesting features of quantum dynamics result from the never vanishing variance of considered quantum dynamics. We suggest that these results generalize to a quantum model (to be constructed) of the original BKL scenario.

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Introduction

Based on the assumption that the universe is spatially isotropic and homogeneous, Alexander Friedmann in 1922 derived simple dynamics from Einstein's field equations. The solution to this dynamics includes gravitational singularity. However, in 1946, Evgeny Lifshitz found that the isotropy in Friedmann's universe is unstable in the evolution towards the singularity.[1,2] This discovery initiated an extensive examination of the dynamics of anisotropic but homogeneous models, in particular the Bianchi IX (BIX), which is the most sophisticated in the class of the Bianchi-type models.[3] The result of these investigations carried out by Belinski, Khalatnikov and Lifshitz (BKL), led to the conclusion that general relativity includes the generic solution with the singularity.[4,5] That analytical result was supported, to some extent, by numerical simulations of the approach to the singularity in vacuum spacetimes with no symmetries.[6] The BKL scenario was identified in string theory in the low energy limit of bosonic sectors of superstring models.[7] Roughly speaking, by generic solution we mean that it corresponds to a non-zero measure subset of all initial data, is stable against perturbation of the initial data, and depends on arbitrary functions of space.

Quite independently, Roger Penrose proved that under some conditions spacetime may include incomplete geodesics.[8] They are called singular despite they do not imply that the invariants diverge. This theorem states little about the dynamics of the gravitational field near the end points of such pathological geodesics. On the contrary, the BKL scenario describes the evolution towards the gravitational singularity characterized by both incomplete geodesics and diverging curvature invariants. In that dynamics, the terms with temporal derivatives dominate over the terms with spatial derivatives when approaching the singularity. Consequently, the points in space decouple and the dynamics become, to some extent, similar to the evolution of general Bianchi IX model.

The presence of the generic singularity in solutions to Einstein's equations signals the existence of the limit of validity of general relativity and means that this classical theory is incomplete. It is expected that the imposition of quantum rules onto general relativity may lead to quantum theory devoid of singularities.

The BKL scenario presents highly complicated dynamics so to deal with it further we use models. There exist two satisfactory models of the BKL scenario. The vacuum BIX called the mixmaster universe,[9,10] and the massive model[11,12] derived from a general BIX. The former is an exact model, but its dynamics are the same far away and close to the singularity, and it is non-integrable.[13] The latter presents asymptotic dynamics near the singularity, includes effectively some contribution from the matter field and has an analytical special solution.[14] The massive model of the BKL scenario has support from the numerical simulations of the general BIX dynamics near the singularity,[15,16] and analytical studies.[17] The dynamics of both models were compared within the dynamical systems method, and it was found that the topologies of the corresponding spaces of critical points are quite different.[18]

In what follows we focus our attention on the dynamics of the massive model. It is defined by the following system of ordinary differential equations:[11]

$$\frac{d^2\ln a}{d\tau^2}=\frac{b}{a}-a^2,\quad \frac{d^2\ln b}{d\tau^2}=a^2-\frac{b}{a}+\frac{c}{b},\quad \frac{d^2\ln c}{d\tau^2}=a^2-\frac{c}{b} \qquad (1)$$

$$\frac{d\ln a}{d\tau}\frac{d\ln b}{d\tau}+\frac{d\ln a}{d\tau}\frac{d\ln c}{d\tau}+\frac{d\ln b}{d\tau}\frac{d\ln c}{d\tau}=a^2+\frac{b}{a}+\frac{c}{b} \qquad (2)$$

where $a=a(\tau)>0$, $b=b(\tau)>0$ and $c=c(\tau)>0$ are the so-called directional scale factors, and τ is a monotonic function of proper time. The scale factors depend implicitly on the matter field.[11] Equations (1) and (2) define a highly nonlinear coupled system of equations. There exists an exact solution to this dynamics:[14]

$$\tilde{a}(t)=\frac{3}{t-t_0},\quad \tilde{b}(t)=\frac{30}{(t-t_0)^3},\quad \tilde{c}(t)=\frac{120}{(t-t_0)^5} \qquad (3)$$

where $t>t_0$, and where $t_0<0$ is an arbitrary real number.

However, the special solution (3) is unstable against small perturbations:

$$a\ t=\tilde{a}\ t+\epsilon\alpha\ t,\quad b\ t=\tilde{b}\ t+\epsilon\beta\ t,\quad c\ t=\tilde{c}\ t+\epsilon\gamma\ t\ . \qquad (4)$$

We have found an explicit form of α,β, and γ.[14]

Intriguingly, the relative perturbations $\alpha/\tilde{a},\beta/\tilde{b}$ and $\gamma/\tilde{c}$ all grow as $\exp\left(\frac{1}{2}\theta\right)$, where $\theta=\ln(t-t_0)$. The multiplier $1/2$ plays the role of the Lyapunov exponent, describing the rate of divergences. Since it is positive, the evolution of the system towards the gravitational singularity $(\theta\rightarrow+\infty)$ is likely to be chaotic. In general, the positivity of the Lyapunov exponent supports the chaoticity of the dynamics, but does not guarantee its occurrence.[19] Further examination is needed (see, e.g., [20-22]).

The space $\mathcal{M}$ of the constants parameterizing the perturbations α, β and γ is a submanifold of $\mathbb{R}^5$, which is the space of all the initial data for the dynamics (1)–(2). Thus, the perturbations are general as the measure of $\mathcal{M}$ is nonzero.[14] The instability results from strong nonlinearity of the dynamics and growing curvature of spacetime (increasing effectively the nonlinearity) in the evolution towards the singularity. This result is consistent with the original BKL scenario.[4,5]

In what follows, we quantize the massive model of the BKL scenario by making use of the so-called integral quantization method (IQM). Roughly speaking, the IQM method consists in ascribing to a phase space of the considered system the affine group $\mathrm{Aff}(\mathbb{R})$ (or Cartesian product of such groups). It is essential that this group has an irreducible unitary representation in the Hilbert space $\mathcal{H} = L^2(\mathbb{R}_+, d\nu(x))$, where $d\nu(x) = dx/x$, and where $\mathbb{R}_+ = \{x \in \mathbb{R} | x \rangle 0\}$. That representation enables us to define the family of coherent states in $\mathcal{H}$. The irreducibility of that representation leads to the resolution of the unity operator in $\mathcal{H}$, which can be used for ascribing a Hermitian operator to almost any classical observable.[23] The IQM applies both to cosmology[24-26] and astrophysics.[27,28]

We already quantized Hamilton's dynamics[29] of that model ignoring its chaotic phase.[24,25] Our results strongly suggest that the classical singularity turns into the quantum bounce and quantum evolution is unitary across quantum bounce. Here, we are mainly concerned with the issue of the imposition of quantum rules onto the presumably classical chaotic dynamics of that model. There are two novelties in our approach: (i) we do not quantize Hamilton's dynamics, but the explicit solution to that dynamics both unperturbed (3) and perturbed (4), and (ii) we quantize temporal and spatial variables to support the general covariance of general relativity. We already applied successfully that approach to the quantization of the Schwarzschild spacetime,[27] showing that the quantum operator corresponding to the scalar curvature called the Kretschmann invariant, does not diverge at the quantum level. We have also quantized, using the IQM method, a thin matter shell in a vacuum, obtaining the result that the quantum shell bounces above the horizon.[28]

There are two basic characteristics of a quantum observable: (i) expectation value - which corresponds to classical values of measured observable, and (ii) variance - which describes quantum smearing of observable. The first feature leads directly to the conditions for a family of quantum states $|\Psi_\eta\rangle \in \mathcal{H}$ parameterized by a set $\eta = (\eta_1, \eta_2, \ldots)$. We require the states $|\Psi_\eta\rangle$ to satisfy:[26]

$$\langle \Psi_\eta | \hat{t} | \Psi_\eta \rangle = t, \; \langle \Psi_\eta | \hat{a} | \Psi_\eta \rangle = a(t), \; \langle \Psi_\eta | \hat{b} | \Psi_\eta \rangle = b(t), \; \langle \Psi_\eta | \hat{c} | \Psi_\eta \rangle = c(t) \qquad (5)$$

where the mark "hat" over an observable denotes the corresponding quantum operator. The above equations represent the constraints to be satisfied. The parameter η should be a function of time t as the right-hand sides of equations (5) depend on time. The solution to (5) allows constructing the vector state dependent on classical time, $|\Psi_{\eta(t)}\rangle \in \mathcal{H}$. Therefore, Eqs. (5) define effectively the "quantum equations of motion", i.e., the quantum dynamics of our system. Choosing in Eq. (5) unperturbed (3) and perturbed (4) solutions enables finding the corresponding vector states. For the purpose of characteristics of the considered quantum system, we calculate the variances of the quantum observables. The variance is a stochastic deviation from the expectation value of quantum observable. It determines the value of smearing of quantum observable and can be used to define quantum fluctuations. In the quantum state $|\psi\rangle \in \mathcal{H}$, the variance of an operator $\hat{B}$ is defined as follows:

$$var(\hat{B};\psi) = \langle \hat{B}^2;\psi \rangle - \langle \hat{B};\psi \rangle^2 \qquad (6)$$

where $\langle \hat{B};\psi \rangle = \langle \psi | \hat{B} | \psi \rangle$.

To be more specific, we consider a model of vector states to be the Gaussian wave packets:

$$\Psi_n(x;\tau,\gamma_k) = N_k x^n \exp\left[i\tau x - \frac{\gamma_k^2 x^2}{2}\right], \quad N_k^2 = \frac{2\gamma_k^n}{(n-1)!}, \quad n = 1,2,\ldots \qquad (7)$$

which are dense in considered Hilbert space $\mathcal{H}$. The subscript $k = 1,2,3$ correspond to the three scale factors of the dynamics (1)–(2). In this case $\eta = (\tau, \gamma_1, \gamma_2, \gamma_3)$. One can verify that the constraints (5) are satisfied if $\tau = t$ and γ_k are proportional to the unperturbed (3) or perturbed (4) solutions.[26] The corresponding variances of the scale factors in the states (7) are proportional to the squares of the solutions (3) and (4). Having calculated the variances of quantum observables corresponding to perturbed $\{a,b,c\}$ and unperturbed $\{\tilde{a},\tilde{b},\tilde{c}\}$ solutions, we can define the quantum fluctuations as follows:[26]

$$\kappa_k = \frac{var(\hat{\xi}_k;\Psi_p) - var(\hat{\xi}_k;\Psi_{unp})}{var(\hat{\xi}_k;\Psi_{unp})}, \quad k = 1,2,3 \qquad (8)$$

where $\hat{\xi}_1 = \hat{a},\ \hat{\xi}_2 = \hat{b},\ \hat{\xi}_3 = \hat{c}$, and where Ψ_p and Ψ_{unp} denote perturbed and unperturbed wave packets, respectively.

It turns out that in the linear approximation in ϵ, these quantum instabilities read:[26]

$$\kappa_1 = 2\epsilon\,\alpha(t)/\tilde{a}(t), \quad \kappa_2 = 2\epsilon\,\beta(t)/\tilde{b}(t), \quad \kappa_3 = 2\epsilon\,\gamma(t)/\tilde{c}(t). \qquad (9)$$

Therefore, the instabilities (9) are proportional to the relative classical perturbations describing classical chaos.

One can show that the relative perturbations (8) for considered Gaussian wave packets and the vector states defined in terms of coherent states packets are the same.[26]

The conclusions are the following:

a. The quantum instability reproduces classical instability in the lowest order of perturbation.

b. The structure of classical dynamics is likely to create deterministic chaos. Never vanishing variances of observables of the corresponding quantum dynamics enhance that classical chaos.

c. The relative quantum and classical perturbations have similar time evolutions. Thus, it is likely that quantization will not suppress the chaos implied by the original BKL scenario.[4,5]

d. As calculated variances are always non-zero, the probability of obtaining divergencies of quantum observables corresponding to classical gravitational singularity equals zero.[26] This confirms the result obtained within quantized Hamilton's dynamics of the massive model of the BKL scenario.[24,25]

e. Our integral quantization method seems to be powerful enough to suppress gravitational singularity, but preserve the chaotic instability which occurs at the classical level.

We suggest that our quantum description of the massive model of the BKL scenario may be generalized to the quantum model of the original BKL scenario.

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Conflicts of interest

None.

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