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The non-integer local order calculus

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Introduction

Fractional calculus concerns the generalization of differentiation and integration to non-integer (fractional) orders. The subject has a long mathematical history being discussed for the first time already in the correspondence of Leibniz with L'Hopital when this replied "What does $\frac{d^n}{dx^n}f(x)$ mean if n=$\frac{1}{2}$?" in September 30 of 1695. Over the centuries many mathematicians have built up a large body of mathematical knowledge on fractional integrals and derivatives. Although fractional calculus is a natural generalization of calculus, and although its mathematical history is equally long, it has, until recently, played a negligible role in physics. One reason could be that, until recently, the basic facts were not readily accessible even in the mathematical literature (see[1]). The nature of many systems makes that they can be more precisely modeled using fractional differential equations. The differentiation and integration of arbitrary orders have found applications in diverse fields of science and engineering like viscoelasticity, electrochemistry, diffusion processes, control theory, heat conduction, electricity, mechanics, chaos, and fractals (see[1-3]).

The history of differential operators from Newton to Caputo, both local and global, is given in[4] (Chapter 1). Here is the definition of a local derivative with a new parameter, which has a large number of applica- tions. More importantly, section 1.4 concludes: "Therefore, we can conclude that both the Riemann-Liouville operator and the Riemann-Liouville operator Caputo are not derivatives, and therefore they are not fractional derivatives, but fractional operators". We are of in agreement with the result[5] that says "the local fractional operator is not a fractional derivative"(page 24). As mentioned above, these tools are new and have demonstrated their potential and usefulness in solving phenomena and process modeling problems in various fields of science and technology (see[6]). Many different types of fractional operators have been proposed in the literature, here we show that several of these different notions of derivatives can be considered particular cases of our definition and, even more relevant, that it is possible to establish a direct relationship between derivatives global (classical) and local, the latter not very accepted by the mathematical community, under two arguments: its local character and compliance with the Leibniz Rule. In this note we present the recent development of the so-called Non-Integer Order Local Calculus, which is the correct name (sometimes we use the name Generalized Calculus, although it does not illustrate the concept well). To facilitate the understanding of the scope of our objective, we present the best known definitions of differential and integral local operators (for more details you can consult,[7,8] Without much difficulty, we can extend these definitions, for any higher order. We assume that the reader is familiar with the classical Calculus, so we will not present it.

Preliminary results

Local fractional calculus (is also called Fractal calculus) was first introduced by Kolwankar and Gangal, although there were some attempts in the 1960s, this is the first formal definition of a local operator that generalizes the classical derivative. It is explain the behavior of continuous but nowhere differentiable function. They proposed particular notation that they had used in their publication for the local fractional derivative of a function defined on fractal sets.9-11 So we have

Definition 1 *If, for a function $f:[0,1]\to R$, the limit*

$$D^q f(y)=\lim_{x\to y}\frac{d^q\left(f(x)-f(y)\right)}{d(x-y)^q}, \tag{1}$$

exists and is finite, then we say that the local fractional derivative (LFD) of order q, at $x=y$, exists.

To understand the fractal behavior of functions, Parvate and Gangal (see)[12] introduce the fractal derivative as follows:

$$x_0 D_x^\alpha f(y)=\frac{d^\alpha f(x)}{dx^\alpha}(x_0)=F-\lim_{x\to x_0}\frac{f(x)-f(x_0)}{S_F^\alpha(x)-S_F^\alpha(x_0)} \tag{2}$$

where the right hand side is the notion of the limit by the points of the fractal set F.

Definition 2 *Let α be an arbitrary but fixed real number. The integral staircase function $S_F^\alpha(x)$ of order α for a set F is given by:*

$$S_F^\alpha(x)=\begin{cases}\gamma^\alpha[F,a,x] & si \quad x\ge a\\ -\gamma^\alpha[F,a,x] & si \quad x<a\end{cases} \tag{3}$$

and the mass function is defined in this way

Definition 3 *The mass function $\gamma^\alpha[F,a,b]$ can written as (see[13,14]):*

$$\gamma^\alpha[F,a,b]=\lim_{\delta\to 0}\gamma_\delta^\alpha[F,a,b]=\frac{(b-a)^\alpha}{\tilde{A}(1+\alpha)}. \tag{4}$$

Another version can be found at:[15]

$$ {}_{x_0}D_x^{\alpha} f(y) = \frac{d^{\alpha} f(x)}{dx^{\alpha}}(x_0) = \lim_{x \to x_0^{\sigma}} D_{y,-\sigma}^{\alpha}\left[\sigma\left(f(x) - f(x_0)(x)\right]\right. \quad (5) $$

with $\sigma = \pm$ and $D_{y,-\sigma}^{\alpha}$ is the Riemann-Liouville derivative.

In[16] we have the following notion:

$$ {}_{x_0}D_x^{\alpha} f(y) = \frac{d^{\alpha} f(x)}{dx^{\alpha}}(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x^{\alpha} - x_0^{\alpha}}, \quad (6) $$

obtained from (??) under assumption $x^{\alpha} - x_0^{\alpha} = (x - x_0)^{\alpha}$.

He gave a new fractal derivative in theis way:[17]

$$ {}_{x_0}D_x^{\alpha} f(y) = \frac{d^{\alpha} f(x)}{dx^{\alpha}}(x_0) = \lim_{\text{Äx} \to L_0} \frac{f(x) - f(x_0)}{KL_0^{\alpha}}. \quad (7) $$

Taking into account

$$ H^{\alpha}\left(F \cap (x, x_0)\right) = \left(x - x_0^{\alpha}\right) = \frac{K}{\tilde{A}(1+\alpha)} L_0^{\alpha}. $$

Yjis is the unified notation of.[18] In this address we have another definition,[19,20] as follows:

$$ D^{\alpha} f(x) = \frac{d^{\alpha} f(x)}{dx^{\alpha}}(x_0) = \lim_{x \to x_0} \frac{\text{Ä}^{\alpha}\left[f(x) - f(x_0)\right]}{(x - x_0)^{\alpha}}, \quad (8) $$

$(x - x_0)^{\alpha}$ is a measure fractal[20] and $\text{Ä}^{\alpha}\left[f(x) - f(x_0)\right] \cong \tilde{A}(1+\alpha)\text{Ä}\left[f(x) - f(x_0)\right]$. In [68] we have:

$$ D^{\alpha} f(x) = \frac{d^{\alpha} f(x)}{dx^{\alpha}}(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{(x - x_0)^{\alpha}}. \quad (9) $$

All these results, although they do not exactly coincide with the direction of our work, we present them so that readers have a more complete picture and because they have become relevant again in recent years.

Post Kahlil derivative

In[21] a definition of local derivative is presented, which opens a new direction of work, which is what we intend to illustrate here.

So they presented the following definition(see also[22]).

Thus, for a function $f : (0, \infty) \to R$ the conformable derivative of order $0 < \alpha \le 1$ of f at $t > 0$ was defined by

$$ T_{\alpha} f(t) = lim_{\varepsilon \to 0} \frac{f\left(t + \varepsilon t^{1-\alpha}\right) - f(t)}{\varepsilon}, \quad (10) $$

and the fractional derivative at 0 is defined as $T_{\alpha} f(0) = lim_{t \to 0} T_{\alpha} f(t)$.

In a work from the same year (cf.)[58] another conformable derivative is defined in a very similar way. Let f be a function of $(0, \infty) \to \mathbb{R}$, $t > 0$ define the derivative of order α with $0 < \alpha < 1$ as the expression $D^{\alpha} f(t) = lim_{\varepsilon \to 0} \frac{f\left(te^{\varepsilon t^{-\alpha}}\right) - f(t)}{\varepsilon}$, of course, if $D^{\alpha} f(t)$ exists at some $(0, a)$ with $a > 0$ then defines the derivative of order α at 0 as $D^{\alpha} f(0) = lim_{t \to 0} D^{\alpha} f(t)$.

[8]introduces a new twist when it defines a general derivative as follows, $f : \mathbb{R} \to \mathbb{R}$ is a function, α a real number, the derivative of fractional order can be thought of as $f^{\alpha}(t) = lim_{\varepsilon \to 0} \frac{f^{\alpha}(t + \varepsilon) - f(t)}{(t + \varepsilon)^{\alpha} - t^{\alpha}}$.

In 2018 we introduced a new local derivative, with a very distinctive property: when $\alpha \to 1$ we do not get the ordinary derivative. We call this derivative non-conformable, to distinguish it from the previous known ones, since when $\alpha \to 1$ the slope of the tangent line to the curve at the point is not preserved.

Be $\alpha \in (0,1]$ and define a continuous function $f : [t_0, +\infty) \to \mathbb{R}$.

First, let's remember the definition of ${}_1N^{\alpha} f(t)$, a non conformable fractional derivative of a function in a point t defined in[23] and that is the basis of our results, that are close resemblance of those found in classical qualitative theory.

Definition 4 *Given a function* $f : [t_0, +\infty) \to \mathbb{R}$, $t_0 > 0$. *Then the N-derivative of f of order* α *is defined by* ${}_1N^{\alpha} f(t) = lim_{\varepsilon \to 0} \frac{f\left(t + \varepsilon e^{t^{-\alpha}}\right) - f(t)}{\varepsilon}$ *for all* $t > 0$, $\alpha \in (0,1)$. *If f is* α *– differentiable in some* $(0, a)$, *and* $lim_{t \to 0^+} N_1^{(\alpha)} f(t)$ *exists, then define* ${}_1N^{\alpha} f(0) = lim_{t \to 0^+} N_1^{(\alpha)} f(t)$

If the above derivative of the function $x(t)$ of order α exists and is finite in (t_0, ∞) , we will say that $x(t)$ is N_1 -differentiable in $I = (t_0, \infty)$.

Remark 5 *The use in Definition 1 of the limit of a certain incremental quotient, instead of the integral used in the classical definitions of fractional derivatives, allows us to give the following interpretation of the N-derivative. Suppose that the point moves in a straight line in* $\mathbb{R}_+$. *For the moments* $t_1 = t$ *and* $t_2 = t + he^{t^{-\alpha}}$ *where* $h > 0$ *and* $\alpha \in (0,1]$ *and we denote* $S(t_1)$ *and* $S(t_2)$ *the path traveled by point P at time* t_1 *and* t_2 *so we have* $\frac{S(t_2) - S(t_1)}{t_2 - t_1} = \frac{S\left(t + he^{t^{-\alpha}}\right) - S(t)}{he^{t^{-\alpha}}}$ *this is the average* N_1 *-speed of point P over time* $he^{t^{-\alpha}}$. *Let's consider*

$$ \lim_{h \to 0} \frac{S\left(t + he^{t^{-\alpha}}\right) - S(t)}{he^{t^{-\alpha}}}. $$

When $\alpha = 1$, this is the usual instantaneous velocity of a point P at any time $t > 0$. If $\alpha \in (0,1)$ this is the instantaneous q-speed of the point P for any $t > 0$. Therefore, the physical meaning of the N-derivative is the instantaneous q-change rate of the state vector of the considered mechanics or another nature of the system.

Remark 6 *The* N_1 *-derivative solves almost all the insufficiencies that are indicated to the classical fractional derivatives. In particular we have the following result.*

Theorem 7 *(See[24]) Let f and g be N-differentiable at a point* $t > 0$ *and* $\alpha \in (0,1]$. *Then*

1. $N_1^{\alpha}(af + bg)(t) = a_1 N^{\alpha}(f)(t) + b_1 N^{\alpha}(g)(t)$.
2. ${}_1N^{\alpha}\left(t^p\right) = e^{t^{-\alpha}} pt^{p-1}, \; p \in \mathbb{R}$.
3. ${}_1N^{\alpha}(\lambda) = 0, \; \lambda \in \mathbb{R}$.

4. ${}_1N^{\alpha}(fg)(t) = f_1N^{\alpha}(g)(t) + g_1N^{\alpha}(f)(t)$.

5. ${}_1N^{\alpha}\left(\frac{f}{g}\right)(t) = \frac{gN_1^{\alpha}(f)(t) - f_1N^{\alpha}(g)(t)}{g^2(t)}$.

6. If, in addition, f is differentiable then $N_1^{\alpha}(f) = e^{t^{-\alpha}} f'(t)$.

7. Being f differentiable and $\alpha = n$ integer, we have $N_1^{n}(f)(t) = e^{t^{-n}} f'(t)$.

Remark 8 *The relations a), c), d) and (e) are similar to the classical results mathematical analysis, these relationships are not established (or do not occur) for fractional derivatives of global character (see[1,2] and bibliography there). The relation c) is maintained for the fractional derivative of Caputo. Cases c), f) and g) are typical of this non conformable local fractional derivative.*

Now we will present the equivalent result, for ${}_1N^{\alpha}$, of the well-known chain rule of classic calculus and that is basic in the Second Method of Lyapunov, for the study of stability of perturbed motion.

Theorem 9 *(See[24]) Let $\alpha \in (0,1]$, g N-differentiable at $t > 0$ and f differentiable at $g(t)$ then ${}_1N^{\alpha}(f \circ g)(t) = f'(g(t))_1N^{\alpha}g(t)$.*

Definition 10 *The non conformable fractional integral of order α is defined by the expression ${}_1J_{t_0}^{\alpha} f(t) = \int_{t_0}^{t} \frac{f(s)}{e^{s^{-\alpha}}} ds$.*

The following statement is analogous to the one known from the Ordinary Calculus.

Theorem 11 *Let f be N_1-differentiable function in (t_0, ∞) with $\alpha \in (0,1]$. Then for all $t > t_0$ we have*

1. If f is differentiable ${}_1J_{t_0}^{\alpha}\left({}_1N^{\alpha} f(t)\right) = f(t) - f(t_0)$.

2. ${}_1N^{\alpha}\left({}_1J_{t_0}^{\alpha} f(t)\right) = f(t)$.

Proof. See[25]

This derivative, and some variants, proved useful in various application problems (see[26-35]).

The N-derivative.

In[36] a generalized derivative was defined as follows (see also[37,38]).

Definition 12 *Given a function $\psi : [0,+\infty) \to \mathbb{R}$. Then the N-derivative of ψ of order α is defined by*

$$N_F^{\alpha}\psi(\tau) = \lim_{\varepsilon \to 0} \frac{\psi(\tau + \varepsilon F(\tau,\alpha)) - \psi(\tau)}{\varepsilon} \quad (11)$$

for all $\tau > 0$, $\alpha \in (0,1)$ being $F(\tau,\alpha)$ is some function.

If ψ is N-differentiable in some $(0,\alpha)$, and $\lim_{\tau \to 0^+} N_F^{\alpha}\psi(\tau)$ exists, then define $N_F^{\alpha}\psi(0) = \lim_{\tau \to 0^+} N_F^{\alpha}\psi(\tau)$, note that if ψ is differentiable, then $N_F^{\alpha}\psi(\tau) = F(\tau,\alpha)\psi'(\tau)$ where $\psi'(\tau)$ is the ordinary derivative.

Examples. Let's see some particular cases that provide us with new non-conforming derivatives.

1. Mellin-Ross Function. In this case we have

$$E_t(\alpha, a) = t^{\alpha} E_{1,\alpha+1}(at) = t^{\alpha} \sum_{k=0}^{\infty} \frac{(at)^k}{\Gamma(\alpha + k + 1)}$$

with $E_{1,\alpha+1}(.)$ the Mittag-Leffler two-parameter function. So, we obtain $\lim_{\alpha \to 1} N_{E_t(\alpha,a)}^{\alpha} f(t) = f'(t) t E_{1,2}(at)$, i.e.,

$$N_{E_t(1,a)}^{1} f(t) = f'(t) t \sum_{k=0}^{\infty} \frac{(at)^k}{\Gamma(k+2)}$$

2. Robotov's Function. That is to say

$$R_{\alpha}(\beta, t) = t^{\alpha} \sum_{k=0}^{\infty} \frac{\beta^k t^{k(\alpha+1)}}{\Gamma(1+\alpha)(k+1)} = t^{\alpha} E_{\alpha+1,\alpha+1}\left(\beta t^{\alpha+1}\right)$$

like before, $E_{\alpha+1,\alpha+1}(.)$ is the Mittag-Leffler two-parameter function. Now, we obtain $\lim_{\alpha \to 1} N_{R_{\alpha}(\beta,t)}^{\alpha} f(t) = f'(t) t E_{2,2}\left(\beta t^2\right)$ and

$$N_{R_1(\beta,t)}^{1} f(t) = \frac{f'(t)t}{\Gamma(2)} \sum_{k=0}^{\infty} \frac{\beta^k t^{2k}}{(k+1)}$$

3. Let $F(t,\alpha) = E_{1,1}\left(t^{-\alpha}\right)$. In this case we obtain, from Definition 12, the derivative ${}_1N^{\alpha} f(t)$ defined in [18] (and [46]).

4. Be now $F(t,\alpha) = E_{1,1}(t^{-\alpha})_1$, in this case we have $F(t,\alpha) = \frac{1}{t^{\alpha}}$, a new derivative with a remarkable propertie $\lim_{t \to \infty} N_1^{\alpha} f(t) = 0$, i.e., the derived N is annulled to infinity.

5. If we now take the development of function E to order 1, we have $E_{a,b}\left(t^{-\alpha}\right) = 1 + \frac{1}{t^{\alpha}}$. Then $\lim_{t \to \infty} N_F^{\alpha} f(t) = \lim_{t \to \infty} {}_1N^{\alpha} f(t) = f'(t)$, in this case we have the classic derivative at infinit.

Remark 13 *It is easy to check but tedious, following for example, that the general derivative fulfills properties very similar to those known from the classical calculus. As well as its most important consequences, among them the Chain Rule, of vital importance in many applications, among them the Second Method of Lyapunov.*

Remark 14 *The generalized derivative defined above is not fractional (as we noted above), but it does have a very desirable feature in applications, its dual dependency on both α and the kernel expression itself, with $0 < \alpha \leq 1$ in [21] the conformal derivative is defined by putting $F(t,\alpha) = t^{1-\alpha}$, while in[24] the nonconforming derivative is obtained with $F(t,\alpha) = e^{t^{-\alpha}}$ (see also [25]). This generalized derivative, in addition to the aforementioned cases, contains as particular cases practically all known local operators and has proved its utility in various applications, see, for example,[23,30,32-35,39,40-52]*

Remark 15 *One of the characteristics of this generalized derivative is the fact that $N_F^{2\alpha} f(t) \neq N_F^{\alpha}\left(N_F^{\alpha} f(t)\right)$, that is, it is necessary to indicate successive derivatives in the second way. Obviously, if $F \equiv 1$, the ordinary derivative is obtained.*

Remark 16 *From the above definition, it is not difficult to extend the order of the derivative for $0 \leq n-1 < \alpha \leq n$ by putting*

$$N_F^{\alpha} h(\tau) = \lim_{\varepsilon \to 0} \frac{h^{(n-1)}(\tau + \varepsilon F(\tau,\alpha)) - h^{(n-1)}(\tau)}{\varepsilon}. \quad (12)$$

If $h^{(n)}$ exists on some interval $I \subseteq \mathbb{R}$, then we have $N_F^\alpha h(\tau) = F(\tau,\alpha) h^{(n)}(\tau)$, with $0 \le n-1 < \alpha \le n$.

Slightly more recent, in[37] a notion of generalized fractional derivative is defined, which is general from two points of view:

1) Contains as particular cases, both conformable and non-conformable derivatives.

2) It is defined for any order $\alpha > 0$.

Given $s \in \mathbb{R}$, we denote by $[s]$ the *upper integer part* of s, i.e., the smallest integer greater than or equal to s.

Definition 17 *Given an interval* $I \subseteq (0,\infty)$, $f : I \to \mathbb{R}$, $\alpha \in \mathbb{R}^+$ *and a continuous function positive* $T(t,\alpha)$, *the derivative* $G_T^\alpha f$ *of* f *of order* α *at the point* $t \in I$ *is defined by*

$$G_T^\alpha f(t) = \lim_{h\to 0} \frac{1}{h^a} \sum_{k=0}^{a} (-1)^k \binom{a}{k} f(t - khT(t,\alpha)) \quad (13)$$

In 2018, a derivative operator is defined on the real line with a limit process as follows (se[53]). For a given function p of two variables, the symbol $D_p f(t)$ defined by the limit $D_p f(t) = \lim_{\varepsilon\to 0} \frac{f(p(t,\varepsilon)) - f(t)}{\varepsilon}$, as long as the limit exists and is finite, it will be called the derivative p of f at t or the generalized derivative from f to t and, for brevity, we also say that f is p-differentiable in t. In the case that it is a closed interval, we define the p-derivative at the extremes as the respective side derivatives. Starting from this definition, the derivative of order α of a function is constructed as the following limit:

$$D_p^\alpha f(t) = \lim_{\varepsilon\to 0} \frac{f(p(t,\varepsilon,\alpha)) - f(t)}{\varepsilon}, \quad 0 < \alpha < 1 \quad (14)$$

where it is understood that in the case $\alpha = 1$ we have the ordinary derivative. It is clear that if f is differentiable in t, then $D_p^\alpha f(t) = p_h(t,0,\alpha) f'(t), \quad 0 < \alpha < 1$. Note that there are no sign restrictions on the function p nor in its partial derivative $p_h(t,0,\alpha)$.

There is an additional detail that we want to point out, in[36] the following is pointed out.

However, a new local derivative that violates Leibniz's Rule can be constructed, so the violation of this rule cannot be a necessary condition for a given operator to be a fractional derivative, let's go back to (11). It is clear that the violation of this rule does not depend (at least not only) on the incremental quotient, but on a factor that we can add to the increased function, from which the non-symmetry of the product rule would be obtained.

Taking into account[54] we can write from (11) the following derivative $(\alpha + \beta = 1)$:

$$DH_\beta^a f(t) := \lim_{\varepsilon\to 0} \frac{H(\varepsilon,\beta) f(t + \varepsilon F(t,\alpha)) - f(t)}{\varepsilon} \quad (15)$$

with $H(\varepsilon,\beta) \to k$ if $\varepsilon \to 0$. In the case that $k \equiv 1$, we can consider two simple cases:

1. $H(\varepsilon,\beta) = 1 + \varepsilon\beta$ as in[54] and so

$$DL_\beta^\alpha f(t) := \lim_{\varepsilon\to 0} \frac{(1+\varepsilon\beta) f(t + \varepsilon F(t,\alpha)) - f(t)}{\varepsilon}.$$

If $F(t,\alpha) = e^{t^{-\alpha}}$, that is, a generalization of the local fractional derivative presented in example 4 above. In this case we have:

$$NL_2^\alpha f(t) := \lim_{\varepsilon\to 0} \frac{(1+\varepsilon\beta) f\left(t + \varepsilon e^{t^{-\alpha}}\right) - f(t)}{\varepsilon}. \quad (16)$$

2. $H(\varepsilon,\beta) = 1 + \varepsilon\beta^r$, $r > 0$, in this way we obtain

$$DP_\beta^\alpha f(t) := \lim_{\varepsilon\to 0} \frac{(1+\varepsilon\beta^r) f(t + \varepsilon F(t,\alpha)) - f(t)}{\varepsilon}.$$

Refer to our N-derivative of[24] we have:

$$NP_2^\alpha f(t) := \lim_{\varepsilon\to 0} \frac{(1+\varepsilon\beta^r) f\left(t + \varepsilon e^{t^{-\alpha}}\right) - f(t)}{\varepsilon}. \quad (17)$$

If $k \ne 1$, as $e^x = 1 + x + \frac{x^2}{2!} + \ldots$ we can take (as a first possibility):

3. $H(\varepsilon,\beta) = E_{1,1}(\varepsilon\beta)$ and so we have

$$DE_\beta^\alpha f(t) := \lim_{\varepsilon\to 0} \frac{E_{1,1}(\varepsilon\beta) f(t + \varepsilon F(t,\alpha)) - f(t)}{\varepsilon},$$

and regarding our N-derivative of[24] it becomes:

$$NE_\beta^\alpha f(t) := \lim_{\varepsilon\to 0} \frac{E_{1,1}(\varepsilon\beta) f\left(t + \varepsilon e^{t^{-\alpha}}\right) - f(t)}{\varepsilon}. \quad (18)$$

From (15) we can easily obtain the following conclusions:

1. Is a derivative local operator, that is, defined at a point.
2. They are derivative in the strict sense of the word.
3. It does not comply with Leibniz's rule, so for (16) we have (the calculations are similar for (17) and (18)):

$$NL_2^\alpha [f(t) g(t)] = \left(N_2^\alpha f(t)\right) g(t) + f(t) \left(N_F^\alpha g(t)\right).$$

Also for (16) we have (again the calculations for (17) and (18) are very similar):

4. If $\alpha = 0$, $\beta = 1$ then $N_2^\alpha f(t) = N_F^0 f(t) + f(t) = (1+e) f(t)$.
5. If $\alpha = 1$, $\beta = 0$ then

$$N_2^1 f(t) = N^1_{e^{t^{-1}}} f(t) = \lim_{\varepsilon\to 0} \frac{f\left(t + \varepsilon e^{t^{-1}}\right) - f(t)}{\varepsilon} = e^{t^{-1}} \left[\lim_{\varepsilon\to 0} \frac{f\left(t + \varepsilon e^{t^{-1}}\right) - f(t)}{\varepsilon e^{t^{-1}}} \right] = e^{t^{-1}} f'(t)$$

if f is derivable.

6. If the limit exists in (18) then we have

$$NL_\beta^\alpha f(t) = N_F^\alpha f(t) + \beta f'(t). \quad (19)$$

7. Unfortunately, "we lose" the Chain Rule that was valid for our N-derivative (see[24]), so for NL_β^α we obtain:

$$NL_\beta^\alpha [f(g(t))] = N_F^\alpha f(g(t)) + \beta f(g(t)).$$

If $g(t) = t$, the above expression is a generalization of proportional derivative of.[55]

8. From (19) we derive that

$$\lim_{t\to\infty} NL^{\alpha}_{\beta} f(t) = \lim_{t\to\infty} N^{\alpha}_{F} f(t) + \lim_{t\to\infty} \beta f'(t) = f'(t) + \beta f(\infty).$$

Where we can draw the following: if the term $\beta f(\infty)$ exists, then the derivative $N^{\alpha}_{\beta} f(t)$ is only a "translation" of the derivative of the function when $t \to \infty$, so it does not affect the qualitative behavior of the ordinary derivative, this is of vital importance in the study of asymptotics properties of solutions of fractional differential equations with NL^{α}_{β}. Unfortunately, the non-existence of the limit of the function to infinity makes the qualitative study of these fractional differential equations impossible.

9. Let's go back to the equation (15), it is clear that the function $H(\varepsilon, \beta)$ can be generalized although that would complicate the calculations extraordinarily. Of course this does not close the discussion on what terms can be "added" to the increased function that give local fractional derivatives that violate the Leibniz Rule, which would maintain the locality, as a historical inheritance of the derivative, and would default Leibniz's Rule, as a "necessary" condition so that there is a fractional derivative.

Conclusion

In this paper, we have presented a sketch of the latest developments obtained in the Non-Integer Order Calculus. Of course, they are not all, for example in[56] a multi-index derivative is presented that generalizes the previous definitions and includes as a particular case the derivative presented in.[57]

All of the above shows that this topic is a fruitful field and has not finished giving us good results.

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None

Conflicts of Interest

None

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