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Fixed point theorems for relation-theoretic *F*-interpolative in branciari distance with an application

Abstract

This paper proves fixed point theorems for relation-theoretic F -interpolative mapping endowed with binary relation in Branciari Distance. Henceforth, the results obtained will be verified with the help of illustrative examples. Also, we demonstrate the results with an application in matrix equations.

Mathematics Subject Classification 2010: 47H10, 54H25.

Keywords: Fixed point, interpolative mapping, F -contraction, metric space, binary relation, matrix equation

Volume 7 Issue 2 - 2023

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Received: April 12, 2023 | **Published:** May 22, 2023

Introduction

Fixed point theory is a fascinating area of research for the researchers studying non-linear phenomena. It has many applications for non-linear functional analysis, Approximation theory, Optimization Theory (Saddle function), Variation inequalities, Game theory (Nash equilibrium) and Economics (Black Scholes theorem). Fixed point theory is quite and sequel to the existing theory of Differential, Integral, Partial, Fractional differential, functional equations and matrix equations. Fixed point theory as well as Banach contraction principle have been studied and generalized in different spaces and various fixed point theorems are developed. In 1968 Kannan[1] introduced a discontinuity of contraction mappings that can possess a fixed point on a complete metric space by filling the gap created by Banach for more than thirty years. Reich proved the fixed point theorem using three metric points by combining the concept of Banach and Kanann on complete metric space. Dass-Gupta proved the results of the fixed point theorem of the rational type operator by using contraction mapping in metric space.

In 2000, Branciari[2] introduced a class of generalized metric spaces by replacing triangular inequality with similar ones which involve four or more points instead of three and improved Banach contraction mapping principle. In 2008, Azam and Arshad[3] using the concept of Branciari investigated the mappings given by Kannan by applying the rectangular properties in a generalized metric space. In 2011, Moradi and Alimohammadi[4] generalized Kannan's results, by using the sequentially convergent mappings and rectangular properties in metric space. Furthermore, Morandi and Alimohammadi investigated and extended Kannan's mapping by using the ideal due to Branciari. Since then, several authors involved in investigations of Banach's contraction mappings using rectangular properties in different spaces.

In 2004, Ran and Reurings[5] proved an order-theoretic analogue of Banach contraction principle which marks the beginning of a vigorous research activity. This result was discovered while investigating the solutions to some special matrix equations.. In continuation of Ran and Reurings, Nieto and Rodríguez-López[6] who proved two very useful results and used them to solve some differential equations.

In 2012, Wardowski[7] initiated the study of fixed points of a new type of contractive mappings in complete metric spaces. In 2014, Wardowski and Dung,[8] proved fixed points of F -weak contractions on complete metric spaces. Acar *et al.*[9] and Altun *et al.*[10] gave Generalized multivalued F -contractions on complete metric spaces. In 2014, Minak *et al.*,[11] proved Ćirić type generalized F -contractions on complete metric spaces and fixed point results. Paesano and Vetro[12] gave the proof on Multi-valued $F-$contractions in 0-complete partial metric spaces with application to Volterra type integral equation. Piri and Kumam[13] proved some fixed point theorems concerning $F-$contraction in complete metric spaces. Sawangsup and Sintunavarat[14] proved the fixed point theorems for $F_{\mathfrak{R}}$ contractions with applications to the solution of non-linear matrix equations, Tomar and Sharma[15] proved some coincidence and common fixed point theorems concerning $F-$contraction and applications and Bashir[16] proved the fixed point results of a generalized reversed $F-$contraction mapping and its application.

On the other hand, Alam and Imdad[17] gave a generalizatiFor more res on of the Banach contraction principle in a complete metric space equipped with binary relation. Their results show that the contraction condition holds only for those elements linked with the binary relation, not for every pair of elements. Recently, Kannan's and Reich's[18] fixed point theorems have been studied and extended in several directions, Karapinar[19] modified the classical Kannan contraction phenomena to an interpolative Kannan contraction one to maximize the rate of convergence of an operator to a unique fixed point. However, by giving a counter-example, Karapinar and Agarwal[20] pointed out a gap in the paper about the assumption of the fixed point being unique and came up with a corrected version. They provided a counter-example to verify that the fixed point need not be unique and invalidate the assumption of a unique fixed point. Since then, several results for variants of interpolative mapping proved for single and multivalued in various abstract spaces.

Further, Karapinar and Agarwal[21] proved interpolative Rus-Reich-Ćirić type contractions via simulation functions. Errai *et al.*[22] gave some new results of interpolative Hardy-Rogers and Reich-Rus-Ćirić type contraction in-metric spaces to prove the existence of the coincidence point. Mishra *et al.*[23] proved the common fixed point theorems for interpolative Hardy-Rogers and Reich-Rus-Ćirić type contraction on quasi partial b-metric space. Aydi *et al.*[24] proved ω -interpolative Reich-Rus-Ćirić type contractions on metric spaces.

Aydi *et al.*[25] proved an interpolative Ćirić-Reich-Rus type contractions via the Branciari distance. Gautam *et al.*[26] proved the fixed point of interpolative Rus-Reich-Ćirić contraction mapping on rectangular quasi-partial b-metric space.

This manuscripts prove a fixed points theorem for relation-theoretic F-contraction mappings via an arbitrary binary relation concept in Branciari distance metric space. In particular, we improve and extend the works due to Alam and Imdad,[27] Ahmadullah *et al.*,[28] Ahmadullah *et al.*,[29] Eke *et al.*,[30] Sawangsup and Sintunavarat,[14] Aydi *et al.*[31] and Karapinar *et al.*[20] In doing so, we will generalize several other works in the literature having the same setting.

Material and methods

This section introduces some definitions, theorems and preliminary results, which will help develop the main result.

The concept of a Branciari distance space has been introduced by Brianciari where the triangular inequality is replaced by a quadrilateral one, which states as follows:

Definition 1[2]. *Let $\mathcal{M}$ be a non-empty set. Suppose that the mapping $d:\mathcal{M}\times\mathcal{M}\to[0,\infty)$ be a function for all $\mu,\nu\in\mathcal{M}$ and all distinct points $z,w\in\mathcal{M}$, each distinct from μ and ν.*

i. $d(\mu,\nu)\geq 0$ *and* $d(\mu,\nu)=0$ *if and only if* $\mu=\nu$;

ii. $d(\mu,\nu)=d(\mu,\nu)$;

iii. $d(\mu,\nu)\leq d(\mu,w)+d(w,z)+d(z,\nu)$.

Then d is called a Branciari distance and the pair $(\mathcal{M},d)$ is called a Branciari distance space.

Definition 2[2]. *Let $(\mathcal{M},d)$ be a metric space. A mapping $\Gamma:\mathcal{M}\to\mathcal{M}$ is said to be sequentially convergent if we have, for every sequence $\{\nu_n\}$, if $\{\Gamma\nu_n\}$ is convergence then $\{\nu_n\}$ also is convergence. $\mathcal{M}$ is said to be subsequentially convergent if we have, for every sequence $\{\nu_n\}$, if $\{\Gamma\nu_n\}$ is convergence then $\{\nu_n\}$ has a convergent subsequence.*

Definition 3[36]. *Let $(\mathcal{M},d)$ be a Branciari distance space and $\{\mu_n\}$ be a sequence in $\mathcal{M}$.*

i. A sequence $\{\mu_n\}$ *is converges to point $\mu^\star\in\mathcal{M}$ if* $\lim_{n\to\infty}d\left(\mu_n,\mu^\star\right)=0$.

ii. *A sequence $\{\mu_n\}$ is said to be Cauchy if for every $\epsilon>0$, there exists a positive integer $\mathbb{N}=\mathbb{N}(\epsilon)$ such that $d(\mu_n,\mu_m)<\epsilon$, for all $\boldsymbol{n},\boldsymbol{m}>\mathbb{N}$.*

iii. *We say that $(\mathcal{M},d)$ is complete if each Cauchy sequence in $\mathcal{M}$ is convergent.*

Lemma 1[36]. *Let $(\mathcal{M},d)$ be a Branciari distance space. A mapping $\Gamma:\mathcal{M}\to\mathcal{M}$ is continuous at $\mu^\star\in\mathcal{M}$, if we have $\Gamma\mu_n\to\Gamma\mu^\star$ or $\lim_{n\to\infty}d\left(\Gamma\mu_n,\Gamma\mu^\star\right)=0$, for any sequence $\{\mu_n\}$ in $\mathcal{M}$ converges to $\mu^\star\in\mathcal{M}$, that is $\mu_n\to\mu^\star$.*

Proposition 1[37]. *Suppose μ_n is a Cauchy sequence in a Branciari distance space such that $\lim\limits_{n\to\infty} d\left(\mu_n,\mu^\star\right)=d\left(\mu_n,w^\star\right)=0$, where $\mu^\star,w^\star\in\mathcal{M}$. Then $\mu^\star=w^\star$.*

Another noted attempt to extend the Banach contraction principle is essentially due to Wardowski.

The following explanations for developing the F-contraction definition was obtained from Wardowski,[7] Wardowski and Van Dung,[8] and Cosentino *et al.*[32]

Let $F:\mathbb{R}^+\to\mathbb{R}$ be a mapping satisfying:

(F1) $\boldsymbol{F}$ is strictly increasing, i.e. for all $\mathcal{J},\mathcal{K}\in\mathbb{R}^+,\mathcal{J}<\mathcal{K}$ implies $F(\mathcal{J})<F(\mathcal{K})$;

(F2) For each sequence $\{\mathcal{J}_n\}_{n\in\mathbb{N}}$ of positive numbers, $\lim_{n\to\infty}\mathcal{J}_n=0$ if and only if $\lim_{n\to\infty}F(\mathcal{J}_n)=-\infty$;

(F3) There exists $z\in(0,1)$ satisfying $\lim_{\mathcal{J}_n\to 0^+}\mathcal{J}_n^zF(\mathcal{J}_n)=0$.

We denote the family of all functions F satisfying conditions $(F1-F3)$ by $\mathcal{F}$. Some examples of functions $F\in\mathcal{F}$ are:

$$F_1(c)=\ln c\Rightarrow\frac{d(\Gamma\mu,\Gamma\nu)}{d(\mu,\nu)}\leq e^{-\eta};$$

$$F_2(c)=c+\ln c\Rightarrow\frac{d(\Gamma\mu,\Gamma\nu)}{d(\mu,\nu)}\leq e^{-\eta+d(\mu,\nu)-d(\Gamma\mu,\Gamma\nu)};$$

$$F_3(c)=-\frac{1}{\sqrt{c}}\Rightarrow\frac{d(\Gamma\mu,\Gamma\nu)}{d(\mu,\nu)}\leq\frac{1}{\left(1+\eta\sqrt{d(\mu,\nu)}\right)^2};$$

$$F_4(c)=\ln\left(c^2+c\right)\Rightarrow\frac{d(\Gamma\mu,\Gamma\nu)(1+d(\Gamma\mu,\Gamma\nu))}{d(\mu,\nu)(1+d(\mu,\nu))}\leq e^{-\eta}.$$

Wardowski introduced a generalization of the Banach contraction principle in metric spaces as follows:

Definition 4[7]. *Let $(\mathcal{M},d)$ be a metric space. A self-mapping Γ on $\mathcal{M}$ is called an F-contraction mapping if there exists $F\in\mathcal{F}$ and $\eta\in\mathbb{R}^+$ such that for all $\mu,\nu\in\mathcal{M}$, $d(\Gamma\mu,\Gamma\nu)>0\Rightarrow\eta+F\left(d(\Gamma\mu,\Gamma\nu)\right)\leq F\left(d(\mu,\nu)\right)$.*

Wardowski[7] proved the following fixed point theorem:

Theorem 1[7]. *Let $(\mathcal{M},d)$ be a complete metric space and $\Gamma:\mathcal{M}\to\mathcal{M}$ be a F-contraction mapping. If there exist $\eta>0$ such that for all $\mu,\nu\in\mathcal{M},d(\Gamma\mu,\Gamma\nu)>0$, implies $\eta+F\left(d(\Gamma\mu,\Gamma\nu)\right)\leq F\left(d(\mu,\nu)\right)$, then Γ has a unique fixed point.*

Kannan[1] proved the following theorem:

Theorem 2[1]. *Let $(\mathcal{M},d)$ be a complete metric space and a self-mapping $\Gamma:\mathcal{M}\to\mathcal{M}$ be a mapping such that $d(\Gamma\mu,\Gamma\nu)\leq\eta\{d(\mu,\Gamma\mu)+d(\nu,\Gamma\nu)\}$, for all $\mu,\nu\in\mathcal{M}$ and $0\leq\eta\leq\frac{1}{2}$. The Γ has a unique fixed point $\delta\in\mathcal{M}$ and for any $\mu\in\mathcal{M}$ the sequence of iterate $\left\{\Gamma^n\mu\right\}$ converges to δ.*

The following results for interpolative Kannan contraction have been proved in as follows:

Definition 5[19]. *Let $(\mathcal{M},d)$ be a metric space, the mapping $\Gamma:\mathcal{M}\to\mathcal{M}$ is said to be interpolative Kannan contraction mappings if $d(\Gamma\mu,\Gamma\nu)\leq\eta\left[d(\mu,\Gamma\mu)\right]^\delta.\left[d(\nu,\Gamma\nu)\right]^{1-\delta}$, for all $\mu,\nu\in\mathcal{M}$ with $\mu\neq\Gamma\mu$, where $\eta\in[0,1)$ and $\delta\in(0,1)$.*

Theorem 3[19]. *Let $(\mathcal{M},d)$ be a complete metric space and Γ be an interpolative Kannan type contraction. Then Γ has a unique fixed point in $\mathcal{M}$.*

In 2018, Karapinar *et al.*[21] proved an interpolative Reich-Rus-Ćirić type contractions fixed point result on partial metric space as follows:

DOI: 10.15406/paij.2023.07.00296

Theorem 4[21]. *Let $(\mathcal{M},d)$ be a complete metric space. $\Gamma:\mathcal{M}\to\mathcal{M}$ be a mapping such that $p(\Gamma\mu,\Gamma\nu) \leq \eta[p(\mu,\nu)]^{\delta}.[p(\mu,\Gamma\mu)]^{\alpha}.[p(\nu,\Gamma\nu)]^{1-\alpha-\delta}$, for all $\mu,\nu\in\mathcal{M}\setminus Fix(\Gamma)$ where $Fix(\Gamma)=\{\mu\in\mathcal{M},\Gamma\mu=\mu\}$. Then Γ has a fixed point in $\mathcal{M}$.*

Binary relation-theoretic in metric spaces

In this part, we will recall some definitions of relation theoretic notion related to binary relation with relevant relation-theoretical variants of some metrical concepts such as completeness and continuity which will be useful in developing our main results.

In the following discussion $\mathcal{R}$ stands for a nonempty binary relation while $\mathbb{N}_0$ denotes the set of whole numbers, i.e., $\mathbb{N}\cup\{0\}$.

Definition 6[38]. *A binary relation on a non-empty set $\mathcal{M}$ is defined as a subset of $\mathcal{M}\times\mathcal{M}$, which will be denoted by $\mathcal{R}$. We say that μ relates to ν under $\mathcal{R}$ iff $(\mu,\nu)\in\mathcal{R}$.*

Definition 7[17]. *Let $\mathcal{R}$ be a binary relation defined on a non-empty set $\mathcal{M}$ and $\mu,\nu\in\mathcal{M}$. We say that μ and ν are $\mathcal{R}$-comparative if either $(\mu,\nu)\in\mathcal{R}$ or $(\nu,\mu)\in\mathcal{R}$. We denote it by $[\mu,\nu]\in\mathcal{R}$.*

Definition 8[38]. *Let $\mathcal{R}$ be a binary relation defined on a non-empty set $\mathcal{M}$. Then the symmetric closure of $\mathcal{R}$ is defined as the smallest symmetric relation containing $\mathcal{R}\left(i.e.\mathcal{R}^{s}:=\mathcal{R}\cup\mathcal{R}^{-1}\right)$, where $\mathcal{R}^{-1}=\left\{(\mu,\nu)\in\mathcal{M}^2:(\nu,\mu)\in\mathcal{R}\right\}$.*

Proposition 2[17]. *If $\mathcal{R}$ is a binary relation defined on a non-empty set $\mathcal{M}$, then $(\mu,\nu)\in\mathcal{R}^{s} \Leftrightarrow [\mu,\nu]\in\mathcal{R}$.*

Definition 9[17]. *Let $\mathcal{R}$ be a binary relation defined on a non-empty set $\mathcal{M}$. Then a sequence $\{\mu_n\}\subset\mathcal{M}$ is called $\mathcal{R}$-preserving if $(\mu_n,\mu_{n+1})\in\mathcal{R},\ \forall\ n\in\mathbb{N}_0$.*

Definition 10[17]. *Let $\mathcal{M}$ be a non-empty set and Γ a self-mapping on $\mathcal{M}$. A binary relation $\mathcal{R}$ on X is called Γ-closed if for any $\mu,\nu\in\mathcal{M}$, $(\mu,\nu)\in\mathcal{R} \Rightarrow (\Gamma\mu,\Gamma\nu)\in\mathcal{R}$.*

Definition 11[27]. *Let $(\mathcal{M},d)$ be a metric space and $\mathcal{R}$ a binary relation on $\mathcal{M}$. We say that $(\mathcal{M},d)$ is $\mathcal{R}$-complete if every $\mathcal{R}$-preserving Cauchy sequence in $\mathcal{M}$ converges.*

Definition 12[17]. *Let $(\mathcal{M},d)$ be a metric space. A binary relation $\mathcal{R}$ defined on $\mathcal{M}$ is called d-self closed if whenever $\{\mu_n\}$ is an $\mathcal{R}$-preserving sequence and $\mu_n\xrightarrow{d}\mu$, then there is a sub sequence $\left\{\mu_{n_k}\right\}$ of $\{\mu_n\}$ with $\left[\mu_{n_k},\mu\right]\in\mathcal{R}$ for all $k\in\mathbb{N}_0$.*

Definition 13[17]. *Let $\mathcal{M}$ be a non-empty set and $\mathcal{R}$ a binary relation on $\mathcal{M}$. A subset D of $\mathcal{M}$ is called $\mathcal{R}$-directed if for each $\mu,\nu\in D$, there exists z in $\mathcal{M}$ such that $(\mu,z)\in\mathcal{R}$ and $(\nu,z)\in\mathcal{R}$.*

Definition 14[39]. *Let $\mathcal{M}$ be a non-empty set and $\mathcal{R}$ be a binary relation defined on a non-empty set $\mathcal{M}$. Let k be a natural number, a path $\mathcal{R}$ from μ to ν is a finite sequence $\{z_0,z_1,z_2,...,z_k\}\in\mathcal{M}$ which satisfies the following conditions:*

i. $z_0=\mu$ *and* $z_1=\mu$;

ii. $[z_i,z_{i+1}]\in\mathcal{R}$ *for each* $i\in\{0,1,2,3,...k-1\}$ *for all* $\mu,\nu\in\mathcal{M})$;

iii. $\mathcal{M}(\tilde{A},\mathcal{R})$: *the collection of all points* $\imath\in\mathcal{M}$ *such that* $(\imath,\tilde{A}\imath)\in\mathcal{R}$;

Let us denote $\gamma(\mu,\nu,\mathcal{R})$: the collection of all paths $\{z_0,z_1,z_2,...,z_k\}$ joining μ to ν in $\mathcal{R}$ such that $[z_i,\Gamma z_i]\in\mathcal{R}$ for each $i\in\{1,2,3,...k-1\}$.

Further, we state some preliminary results which will be helpful to develop our main results.

Ahmadullah *et al.*[33] proved the results in metric-like spaces as well as partial metric spaces equipped with a binary relation. Sawangsup and Sintunavarat[14] by combining the concepts of Wardowski and proved the fixed point theorems for $F_{\mathfrak{R}}$-contractions in metric space with applications to the solution of non-linear matrix equations with binary relation as follows:

Theorem 5[34]. *Let $(\mathcal{M},d)$ be a complete metric space, $\mathcal{R}$ a binary relation on $\mathcal{M}$ and let Γ be a self-mapping on $\mathcal{M}$. Suppose that the following conditions hold:*

i. $\mathcal{M}(\Gamma,\mathcal{R})$ *is non-empty,*

ii. $\mathcal{R}$ *is* Γ*-closed,*

iii. either Γ *is continuous or* $\mathcal{R}$ *is* $\boldsymbol{G}$*-self-closed,*

iv. *there exists* $\boldsymbol{F}\in\boldsymbol{\mathcal{F}}$ *and* $\boldsymbol{\varsigma}\in\mathbb{R}^{+}$ *such that for all* $\mu,\nu\in\boldsymbol{\mathcal{M}}$ *with* $(\mu,\nu\in\boldsymbol{\mathcal{R}})$, $d(\Gamma\mu,\Gamma\nu)>\Rightarrow\eta+F(d(\Gamma\mu,\Gamma\nu)) \leq F(d(\mu,\nu))$.

Then Γ has a fixed point. Moreover, for each $x_0\in\mathcal{M}(\Gamma,\mathcal{R})$ the Picard sequence $\{\Gamma^n x_0\}$ is convergent to the fixed point Γ.

Results and discussion

Now, we prove the main results using interpolative Reich-Rus-Ćirić- F-contraction mapping concepts via binary relation in generalized metric spaces.

Theorem 6. *Let $(\mathcal{M},d)$ be a complete metric space, $\mathcal{R}$ a binary relation on $\mathcal{M}$ and let Γ be an interpolative Reich-Rus-Ćirić type contractions mapping on $\mathcal{M}$. Suppose that the following conditions hold:*

i. $(\mathcal{M},d)$ *is* Γ *complete,*

ii. $\mathcal{M}(\Gamma,\mathcal{R})$ *is non-empty,*

iii. $\mathcal{R}$ *is* Γ*-closed,*

iv. *the sequence* $\{\mu_n\}$ *is* $\mathcal{R}$*-preserving,*

v. either Γ *is continuous or* $\mathcal{R}$ *is* $\boldsymbol{d}$*-self closed,*

vi. *there exists a constant* $\eta>0$ *such that* $\forall\mu,\nu\in\mathcal{M}$ *with* $(\mu,\nu\in\mathcal{R})$ $\eta+F(d(\Gamma\mu,\Gamma\nu)) \leq F(\mathcal{M}_{\mathcal{R}}(\mu,\nu))$, $\eta+F(d(\Gamma\mu,\Gamma\nu)) \leq F(\mathcal{M}_{\mathcal{R}}(\mu,\nu))$, *where* $\mathcal{M}_{\mathcal{R}}(\mu,\nu) = [d(\mu,\nu)]^{\delta}.[d(\mu,\Gamma\mu)]^{\alpha}.[d(\nu,\Gamma\nu)]^{1-\alpha-\delta}$, *for all* $\mu,\nu\in\mathcal{M}\setminus Fix(\Gamma)$ *where* $Fix(\Gamma)=\{\mu\in\mathcal{M},\Gamma\mu=\mu\}$. *Then* Γ *has a fixed point. Also, if* Γ *is subsequentially convergent then for every* $\mu_{n-1}\in\mathcal{M}$ *the sequence of iterate* $\{\Gamma^n\mu_{n-1}\}$ *converges to this fixed point. Moreover, if*

vii. $\gamma(\mu,\nu,\mathcal{R}^{s})$ *is non-empty, for each* $\mu,\nu\in\mathcal{M}$. *Then* Γ *has a unique fixed point.*

Proof. Assume x_0 be an arbitrary point in $\mathcal{M}(\Gamma,\mathcal{R})$. We construct a sequence $\{\mu_n\}$ of Picard iterates such that $\mu_n=\Gamma^n\mu_0=\Gamma\mu_{n-1}$ for all $n\in\mathbb{N}$. By condition (iii) of Theorem 6, we have $(\mu_0,\Gamma\mu_0)\in\mathcal{R}$ and $\mathcal{R}$ is Γ-closed, therefore

$$\left(\Gamma\mu_{n-1},\Gamma^{n+1}\mu_{n-1}\right),\left(\Gamma^{n+1}\mu_{n-1},\Gamma^{n+2}\mu_{n-1}\right),\ldots,\left(\Gamma^{n}\mu_{n-1},\Gamma^{n+2}\mu_{n-1}\right).$$

Using $(\ref{Eqt 3.3})$, we note that

$$\left(\Gamma^{n}\mu_{n-1},\Gamma^{n+1}\mu_{n-1}\right)\in\mathcal{R},$$

$\forall n\in\mathbb{N}_0$. Therefore the sequence $\{\mu_n\}$ is $\mathcal{R}$-preserving.

If there exists n such that $\mu_n = \mu_{n+1}$, then μ_n is a fixed pint of Γ. The proof is completed. For that case, we assume that $\mu_n \neq \mu_{n+1}$ for each $n \geq 0$. Therefore

$$\lim_{n\to\infty} d(\mu_n, \mu_{n+1}) = 0.$$

To show this, let $\mu = \mu_{n-1}$ and $\nu = \mu_n$, using $(\ref{eqt 3.1})$ for all $n \in \mathbb{N}_0$, we deduce that

$$\eta + F\left(d\left(\Gamma\mu_{n-1}, \Gamma\mu_n\right)\right) \leq F\left(\mathcal{M}_{\mathcal{R}}\left(\mu_{n-1}, \mu_n\right)\right),$$

where

$$\begin{aligned}\mathcal{M}_{\mathcal{R}}(\mu_{n-1},\mu_n) &= \left[d(\mu_{n-1},\mu_n)\right]^{\delta}.\left[d(\mu_{n-1},\Gamma\mu_{n-1})\right]^{\alpha}.\left[d(\mu_n,\Gamma\mu_n)\right]^{1-\alpha-\delta},\\ &\leq \left[d(\mu_{n-1},\mu_n)\right]^{\delta}.\left[d(\mu_{n-1},\mu_n)\right]^{\alpha}.\left[d(\mu_n,\mu_{n+1})\right]^{1-\alpha-\delta},\\ &= \left[d(\mu_{n-1},\mu_n)\right]^{\alpha+\delta}.\left[d(\mu_n,\mu_{n+1})\right]^{1-\alpha-\delta}.\end{aligned}$$

Taking $(\ref{Eqt 3.7})$ into $(\ref{Eqt 3.6})$, we obtain

$$\eta + F\left(d\left(\Gamma\mu_{n-1}, \Gamma\mu_n\right)\right) \leq F\left(\left[d(\mu_{n-1},\mu_n)\right]^{\alpha+\delta}.\left[d(\mu_n,\mu_{n+1})\right]^{1-\alpha-\delta}\right).$$

By the continuity property of F, $F1$ and $(\ref{Eqt 3.8})$, we get

$$\begin{aligned}d(\Gamma\mu_{n-1},\Gamma\mu_n) &\leq \left[d(\mu_{n-1},\mu_n)\right]^{\alpha+\delta}.\left[d(\mu_n,\mu_{n+1})\right]^{1-\alpha-\delta},\\ \left(d(\mu_n,\mu_{n+1})\right)^{1-(1-\alpha-\delta)} &\leq \left[d(\mu_{n-1},\mu_n)\right]^{\alpha+\delta},\\ \left(d(\mu_n,\mu_{n+1})\right)^{\alpha+\delta} &\leq \left[d(\mu_{n-1},\mu_n)\right]^{\alpha+\delta}.\end{aligned}$$

So, we conclude that

$$d(\mu_n,\mu_{n+1}) \leq d(\mu_{n-1},\mu_n),$$

for all $n \geq 1$.

Consequently, we have

$$\eta + F\left(d(\mu_n,\mu_{n+1})\right) \leq F\left(d(\mu_{n-1},\mu_n)\right).$$

Equivalent to

$$F\left(d(\mu_n,\mu_{n+1})\right) \leq F\left(d(\mu_{n-1},\mu_n)\right) - \eta.$$

Similar, let $\mu = \mu_n$, $\nu = \mu_{n+1}$, using $(\ref{eqt 3.1})$ and $(\ref{Eqt 3.9})$ for all $n \in \mathbb{N}_0$, we get

$$F\left(d(\mu_{n+1},\mu_{n+2})\right) \leq F\left(d(\mu_{n-1},\mu_n)\right) - 2\eta.$$

Proceeding this way, by induction we deduce

$$F\left(d(\mu_n,\mu_{n+1})\right) \leq F\left(d(\mu_{n-1},\mu_n)\right) - n\eta, \forall n \geq 1.$$

That is $d(\mu_{n-1},\mu_n)$ is non-increasing sequence with non-negative terms. We denote $\mathcal{J}_n = d(\mu_n,\mu_{n+1})$, for all $n \in \mathbb{N}_0$. Since Γ is an F-$\mathcal{R}$-interpolative contraction mapping.

From $(\ref{Eqt 3.11})$, we obtain

$$F(\mathcal{J}_n) \leq F(\mathcal{J}_{n-1}) - \eta \leq F(\mathcal{J}_{n-2}) - 2\eta \leq \ldots \leq F(\mathcal{J}_0) - n\eta,$$

for all $n \in \mathbb{N}_0$.

By $(F2)$, we have

$$\lim_{n\to\infty} \mathcal{J}_n = 0.$$

If and only if

$$\lim_{n\to\infty} F(\mathcal{J}_n) = -\infty.$$

From $(F3)$ and $(\ref{Eqt 3.12})$, there exists $z \in (0,1)$ such that

$$\begin{aligned}\mathcal{J}_n^z F(\mathcal{J}_n) &\leq \mathcal{J}_n^z\left(F(\mathcal{J}_{n-1}) - \eta\right) \leq \ldots \leq \mathcal{J}_n^z\left(F(\mathcal{J}_0) - n\eta\right),\\ \mathcal{J}_n^z F(\mathcal{J}_n) &\leq \mathcal{J}_n^z\left(F(\mathcal{J}_0) - n\eta\right) \leq 0,\\ \mathcal{J}_n^z F(\mathcal{J}_n) &\leq \mathcal{J}_n^z F(\mathcal{J}_0) - \mathcal{J}_n^z n\eta \leq 0,\\ \mathcal{J}_n^z F(\mathcal{J}_n) - \mathcal{J}_n^z F(\mathcal{J}_0) &\leq -\mathcal{J}_n^z n\eta \leq 0,\\ \mathcal{J}_n^z\left(F(\mathcal{J}_n) - F(\mathcal{J}_0)\right) &\leq -\mathcal{J}_n^z n\eta \leq 0.\end{aligned}$$

Letting $n \to \infty$ in $(\ref{Eqt 3.15})$, we obtain that

$$\lim_{n\to\infty} \mathcal{J}_n^z n = 0.$$

Now, from $(\ref{Eqt 3.16})$ there exist $n_1 \in \mathbb{N}_0$ such that $\mathcal{J}_n^z n \leq 1$ for all $n \geq n_1$.

Consequently, we have that

$$\begin{aligned}\mathcal{J}_n^z n &\leq 1,\\ \mathcal{J}_n^z &\leq \frac{1}{n},\\ \mathcal{J}_n &\leq \frac{1}{n^{\frac{1}{z}}},\\ \mathcal{J}_n &\leq n^{-\frac{1}{z}}.\end{aligned}$$

Therefore, $\sum_{n=0}^{\infty} d(\mu_n,\mu_{n+1}) = 0$ converges.

Next, we claim that $\{\mu_n\}$ is Cauchy sequence, that is, $\lim_{n\to\infty} d(\mu_n,\mu_m) = 0$ $\forall n,m \in \mathbb{N}_0$ such that $m \geq n$, by using the rectangular property we have

$$\begin{aligned}d(\mu_n,\mu_m) &\leq d(\mu_n,\mu_{n+1}) + d(\mu_{n+1},\mu_{n+2}) + \ldots + d(\mu_{m-1},\mu_m),\\ &\leq \mathcal{J}_n + \mathcal{J}_{n+1} + \mathcal{J}_{n+2} + \ldots + \mathcal{J}_{m-1},\\ &= \sum_{i=n}^{m-1} \mathcal{J}_i,\\ &\leq \sum_{i=n}^{m-1} n^{-\frac{1}{z}}.\end{aligned}$$

Since $\sum_{i=n}^{m-1} n^{-\frac{1}{z}} < \infty$, we get that $\{\mu_n\}$ is a Cauchy sequence in $\mathcal{M}$. Since $(\mathcal{M}, d)$ is complete, there exists $\mu^\star \in \mathcal{M}$ such that

$$d(\mu_n,\mu^\star) = \lim_{n\to\infty} d(\mu_n,\mu^\star) = 0.$$

Now, by the continuity of Γ, we get $\Gamma\mu^\star = \mu^\star$. We show that $\mu^\star$ is a fixed point of Γ. Assume that $\Gamma\mu^\star \neq \mu^\star$ such that $\Gamma\mu_n \neq \mu_n$ $\forall n \geq \mathbb{N}_0$. By letting $\mu = \mu_n$ and $\nu = \mu^\star$ in $(\ref{eqt 3.1})$, we obtain

$$\eta + F\left(d\left(\Gamma\mu_n, \Gamma\mu^\star\right)\right) \leq F\left(\mathcal{M}_{\mathcal{R}}\left(\mu_n, \mu^\star\right)\right),$$

where

$$\begin{aligned}\mathcal{M}_{\mathcal{R}}(\mu_n,\mu^\star) &= \left[d(\mu_n,\mu^\star)\right]^{\delta}.\left[d(\mu_n,\Gamma\mu_n)\right]^{\alpha}.\left[d(\mu^\star,\Gamma\mu^\star)\right]^{1-\alpha-\delta},\\ &\leq \left[d(\mu^\star,\mu^\star)\right]^{\delta}.\left[d(\mu^\star,\mu^\star)\right]^{\alpha}.\left[d(\mu^\star,\mu^\star)\right]^{1-\alpha-\delta},\\ &= \left[d(\mu^\star,\mu^\star)\right]^{(\alpha+\delta)+(1-\alpha-\delta)},\\ &= d(\mu^\star,\mu^\star).\end{aligned}$$

Taking $(\ref{Eqt 3.20})$ into $(\ref{Eqt 3.19})$, we get

$$\begin{aligned}\eta + F\left(d(\Gamma\mu^\star,\Gamma\mu^\star)\right) &\leq F\left(d(\mu^\star,\mu^\star)\right),\\ \eta + F(0) &\leq F(0),\\ \eta &\leq 0,\end{aligned}$$

which is a contradiction. Hence, $d(\mu^\star,\Gamma\mu^\star) = 0$ therefore $\mu^\star = \Gamma\mu^\star$, which shows that $\mu^\star$ is a fixed point of Γ. Also Γ is subsequentially convergent on $\mathcal{M}$. To observe this, let $\mu = \mu_{n_{k-1}}$ and $\nu = \mu_{n_k}$, using $(\ref{eqt 3.1})$ we obtain

$$\eta + F\left(d\left(\Gamma\mu_{n_{k-1}}, \Gamma\mu_{n_k}\right)\right) \leq F\left(\mathcal{M}_{\mathcal{R}}\left(\mu_{n_{k-1}}, \mu_{n_k}\right)\right),$$

where

$$\mathcal{M}_{\mathcal{R}}(\mu_{n_{k-1}},\mu_{n_k}) = \left[d(\mu_{n_{k-1}},\mu_{n_k})\right]^{\delta}.\left[d(\mu_{n_{k-1}},\Gamma\mu_{n_{k-1}})\right]^{\alpha}.\left[d(\mu_{n_k},\Gamma\mu_{n_k})\right]^{1-\alpha-\delta},$$

Citation: Wangwe L. Fixed point theorems for relation-theoretic F-interpolative in branciari distance with an application. *Phys Astron Int J*. 2023;7(2):122–129. DOI: 10.15406/paij.2023.07.00296

$$\leq \left[d\left(\mu_{n_{k-1}},\mu_{n_k}\right)\right]^{\delta}.\left[d\left(\mu_{n_{k-1}},\mu_{n_k}\right)\right]^{\alpha}.\left[d\left(\mu_{n_k},\mu_{n_{k+1}}\right)\right]^{1-\alpha-\delta},$$
$$= \left[d\left(\mu_{n_{k-1}},\mu_{n_k}\right)\right]^{\alpha+\delta}.\left[d\left(\mu_{n_k},\mu_{n_{k+1}}\right)\right]^{1-\alpha-\delta}.$$

Using $(\ref{Eqt 3.22})$ in $(\ref{Eqt 3.21})$, we get

$$\eta+F\left(d\left(\Gamma\mu_{n_{k-1}},\Gamma\mu_{n_k}\right)\right) \leq F\left(\left[d\left(\mu_{n_{k-1}},\mu_{n_k}\right)\right]^{\alpha+\delta}.\left[d\left(\mu_{n_k},\mu_{n_{k+1}}\right)\right.\right)$$

By the property of F and $F1$ with $(\ref{Eqt 3.23})$, we get

$$d\left(\Gamma\mu_{n_{k-1}},\Gamma\mu_{n_k}\right) \leq \left[d\left(\mu_{n_{k-1}},\mu_{n_k}\right)\right]^{\alpha+\delta}.\left[d\left(\mu_{n_k},\mu_{n_{k+1}}\right)\right]^{1-\alpha-\delta},$$
$$\left(d\left(\mu_{n_k},\mu_{n_{k+1}}\right)\right)^{1-(1-\alpha-\delta)} \leq \left[d\left(\mu_{n_{k-1}},\mu_{n_k}\right)\right]^{\alpha+\delta},$$
$$\left(d\left(\mu_{n_k},\mu_{n_{k+1}}\right)\right)^{\alpha+\delta} \leq \left[d\left(\mu_{n_{k-1}},\mu_{n_k}\right)\right]^{\alpha+\delta},$$

which is equivalent to

$$F\left(d\left(\mu_{n_k},\mu_{n_{k+1}}\right)\right) \leq F\left(d\left(\mu_{n_{k-1}},\mu_{n_k}\right)\right)-\eta.$$

Due to continuity of Γ, it implies that

$$\lim_{n\to\infty}\Gamma\mu_{n_k} = \Gamma\mu^\star=\mu^\star.$$

This shows that Γ is subsequentially convergent.

Consider the hypothesis in Theorem 6, we prove assertion (vii) as follows: we observe that $\mathcal{M}(\Gamma,\mathcal{R})$ is non-empty, so let us take a pair of elements say $(\mu^\star,w^\star)$ in $\mathcal{M}(\Gamma,\mathcal{R})$ such that

$$\Gamma\mu=\mu^\star,$$
$$\Gamma\nu=w^\star.$$

Next, we claim that $\mu^\star\neq w^\star$. By the above equalities, there exists a S-path (say, $z_0,z_1,z_2,...,z_l$) of length l in $\mathcal{R}^s$ from $\Gamma\mu$ to $\Gamma\nu$, with

$$\Gamma z_0=\Gamma\mu,$$
$$\Gamma z_l=\Gamma\nu,$$

such that

$$\left[\Gamma z_i,\Gamma z_{i+1}\right]\in\mathcal{R}^s\subseteq\mathcal{R},$$

for all $i\in 0,1,2,3,...l-1$.

Define two constant sequences such that

$$z_n^0=\mu \ and \ z_n^l=\nu$$

By using ([equation 4.20]), for all $n\in\mathbb{N}$, we have

$$\Gamma z_n^0 = \Gamma\mu=\mu^\star,$$
$$\Gamma z_n^l = \Gamma\nu=w^\star.$$

By usual substitution for $z_0^i=z_i$ for each $i\in 0,1,2,...l$, that is

$$z_0^1 = z_1,$$
$$z_0^2 = z_2,$$
$$z_0^3 = z_3,$$
$$z_0^4 = z_4,$$
$$z_0^{l-1} = z_{l-1}.$$

Thus we construct a sequence

$$\left\{z_n^1\right\},\left\{z_n^2\right\},\left\{z_n^3\right\},\dots,\left\{z_n^i\right\}\in\mathcal{M}.$$

Corresponding to each z_i, we have $\left[\Gamma z_0^i,\Gamma z_1^i\right]\in\mathcal{R}$ from ([equation 4.20]), ([equation 4.21]) and $\mathcal{R}$ is Γ-closed, we get

$$\lim_{n\to\infty}d\left(\Gamma z_n^i,\Gamma z_n^{i+1}\right) = 0,$$

for each $i\in 1,2,3,...l-1$ ar d for all $n\in\mathbb{N}$.

Define $d_n^i=d\left(\Gamma z_n^i,\Gamma z_n^{i+1}\right)$, for each $i\in 0,1,2,3,...l-1$ and for all $n\in\mathbb{N}$. We assert that, $\lim_{n\to\infty}d_n^i>0$.

Since $\left[\Gamma z_n^i,\Gamma z_n^{i+1}\right]\in\mathcal{R}$, either $\left[\Gamma z_n^i,\Gamma z_n^{i+1}\right]\in\mathcal{R}$ or $\left[\Gamma z_n^{i+1},\Gamma z_n^i\right]\in\mathcal{R}$.

If $\left[\Gamma z_n^i,\Gamma z_n^{i+1}\right]\in\mathcal{R}$, for $\mu=z_n^i$ and $\nu=z_n^{i+1}$. Then applying the condition $(\ref{eqt 3.1})$, we have

$$\eta+F\left(d\left(\Gamma z_n^i,\Gamma z_n^{i+1}\right)\right) \leq F\left(\mathcal{M}_{\mathcal{R}}\left(z_n^i,z_n^{i+1}\right)\right),$$

where

$$\mathcal{M}_{\mathcal{R}}\left(z_n^i,z_n^{i+1}\right) = \left[d\left(z_n^i,z_n^{i+1}\right)\right]^{\delta}.\left[d\left(z_n^i,\Gamma z_n^i\right)\right]^{\alpha}.\left[d\left(z_n^{i+1},\Gamma z_n^{i+1}\right)\right]^{1-\alpha-\delta},$$
$$\leq \left[d\left(z_n^i,z_n^{i+1}\right)\right]^{\delta}.\left[d\left(z_n^i,z_n^{i+1}\right)\right]^{\alpha}.\left[d\left(z_n^{i+1},z_n^{i+2}\right)\right]^{1-\alpha-\delta},$$
$$= \left[d\left(z_n^i,z_n^{i+1}\right)\right]^{\alpha+\delta}.\left[d\left(z_n^{i+1},z_n^{i+2}\right)\right]^{1-\alpha-\delta}.$$

Substituting $(\ref{Eqt 3.30})$ in $(\ref{Eqt 3.29})$, we get

$$\eta+F\left(d\left(z_n^{i+1},z_n^{i+2}\right)\right) \leq F\left(\left[d\left(z_n^i,z_n^{i+1}\right)\right]^{\alpha+\delta}.\left[d\left(z_n^{i+1},z_n^{i+2}\right)\right]^{1-\alpha-\delta}\right).$$

By the property of F, we have

$$d\left(z_n^{i+1},z_n^{i+2}\right) \leq \left[d\left(z_n^i,z_n^{i+1}\right)\right]^{\alpha+\delta}.\left[d\left(z_n^{i+1},z_n^{i+2}\right)\right]^{1-\alpha-\delta},$$
$$d\left(z_n^{i+1},z_n^{i+2}\right)^{1-(1-\alpha-\delta)} \leq \left[d\left(z_n^i,z_n^{i+1}\right)\right]^{\alpha+\delta},$$
$$d\left(z_n^{i+1},z_n^{i+2}\right)^{\alpha+\delta} \leq \left[d\left(z_n^i,z_n^{i+1}\right)\right]^{\alpha+\delta},$$
$$d\left(z_n^{i+1},z_n^{i+2}\right) \leq d\left(z_n^i,z_n^{i+1}\right).$$

Which is equivalent to

$$\eta+F\left(d\left(z_n^{i+1},z_n^{i+2}\right)\right) \leq F\left(d\left(z_n^i,z_n^{i+1}\right)\right),$$
$$F\left(d\left(z_n^{i+1},z_n^{i+2}\right)\right) \leq F\left(d\left(z_n^i,z_n^{i+1}\right)\right)-\eta.$$

Taking lim as $i\to\infty$ and using $\lim_{i\to\infty}d_n^i=d$, we get

$$d\left(\Gamma z_n^i,\Gamma z_n^{i+1}\right) \leq 0.$$

Implies that

$$\eta \leq 0,$$

which is a contradiction and hence

$$\lim_{i\to\infty}d_n^i=d=0$$

The same for rectangular property (iii), if $\left(\Gamma z_n^i,\Gamma z_n^{i+1}\right)\in\mathcal{R}$, we have

$$\lim_{i\to\infty}d_n^i=\lim_{i\to\infty}d\left(\Gamma z_n^i,\Gamma z_n^{i+1}\right)=0,$$

for $i\in 0,1,2,...l-1$.

Using ([equation 4.21]), $\lim_{i\to\infty}d_n^i=0$ and ((iii)), we have

$$d\left(\mu^\star,w^\star\right)=d\left(z_n^i,z_n^j\right) \leq \sum_{i=0}^{l-1}d\left(z_n^i,z_n^{i+1}\right),$$
$$\leq \sum_{i=0}^{l-1}d_n^i,$$
$$\to 0 \, as \ n\to\infty.$$

So that

$$d\left(\mu^\star,w^\star\right) = 0\Rightarrow$$
$$\mu^\star = w^\star.$$

Therefore

$$\Gamma\mu^\star = \Gamma w^\star,$$

which is a contradiction. Thus $\mu^{\star}$ is a unique fixed point of Γ. Thus the proof is completed.

Due to the generalization of Theorem 6, we can deduce the corollary as follows:

Corollary 1. *Let $(\mathcal{M},d)$ be a complete metric space and let $\Gamma:\mathcal{M}\to\mathcal{M}$ be F -interpolative type mapping such that the following hypothesis hold:*

(i) $\mathcal{M}$ is Γ is closed in $(\mathcal{M},d)$,

(ii) there exists a constant $\eta\in[0,1)$ such that

$$\eta+F\left(d(\Gamma\mu,\Gamma\nu)\right) \quad \leq \quad F\left(\left[d(\mu,\Gamma\mu)\right]^{\delta}.\left[d(\nu,\Gamma\nu)\right]^{1-\delta}\right),$$

for all $\mu,\nu\in\mathcal{M}$ with $\mu\neq\Gamma\mu$, where $\eta\in[0,1)$ and $\delta\in(0,1)$.

Proof. The proof of the above corollary follows similar steps of Theorem 6. Therefore, the proof is completed.

Next, we give the following similar example from Moradi and Alimohammadi[4] for illustration of the hypothesis of Theorem 6.

Example 1. *Consider $\mathcal{M}=\{0\}\cup\left\{0,1,\frac{1}{2},\frac{1}{3}\right\}$ and d be a Euclidean metric on $\mathcal{M}$. Then $(\mathcal{M},d)$ is a complete metric space. The mapping $\Gamma:\mathcal{M}\to\mathcal{M}$ be determined as*

$$\begin{aligned}\Gamma(0) &= 0 \quad \forall n=0,\\ \Gamma(\mu) &= \frac{1}{\mu^{n}+1}, \forall n\geq 1.\end{aligned}$$

Define a binary relation $\mathcal{R}=\{(\mu,\nu)\in\mathbb{R}^2\},\mathcal{R}\in\mathbb{R}^2$ and $\mathcal{R}=\left\{(0,1),\left(0,\frac{1}{2}\right),\left(0,\frac{1}{3}\right),\left(1,\frac{1}{3}\right),\left(1,\frac{1}{2}\right),\left(\frac{1}{2},\frac{1}{3}\right)\right\}$ on $\mathcal{M}$. Then $\mathcal{M}$ is $\mathcal{R}$ -complete.

Weclaimthat $\mathcal{M}$ isnoteither $\mathcal{R}$ -completeor Γ -closed.Toverifythis, we show that

$$F_3(c) \quad = \quad -\frac{1}{\sqrt{c}}\Rightarrow\frac{d(\Gamma\mu,\Gamma\nu)}{\mathcal{M}_{\mathcal{R}}(\mu,\nu)}\leq\frac{1}{\left(1+\eta\sqrt{\mathcal{M}_{\mathcal{R}}(\mu,\nu)}\right)^2}$$

satisfy all the hypothesis of Theorem 6.

We complete the following metrics Using all of the above equalities, we obtain

$$\mathcal{M}_{\mathcal{R}}(\mu,\nu) \quad = \quad [|\mu-\nu|]^{\delta}.\left[\left|\frac{\mu^{n+1}+\mu-1}{\mu^{n}+1}\right|\right]^{\alpha}.\left[\left|\frac{\nu^{n+1}+\nu-1}{\nu^{n}+1}\right|\right]^{1-\alpha-\delta},$$

By substituting $(\ref{Eqt 3.41})$ in $(\ref{Eqt 3.36})$, we obtain

$$\frac{\left|\frac{\mu^{n}-\nu^{n}}{(\mu^{n}+1)(\nu^{n}+1)}\right|}{[|\mu-\nu|]^{\delta}.\left[\left|\frac{\mu^{n+1}+\mu-1}{\mu^{n}+1}\right|\right]^{\alpha}.\left[\left|\frac{\nu^{n+1}+\nu-1}{\nu^{n}+1}\right|\right]^{1-\alpha-\delta}} \quad \leq \quad \frac{1}{(1+\eta\sqrt{\mathcal{M}_{\mathcal{R}}(\mu,\nu)})^2}.$$

If we take $\delta=0.2,\alpha=0.5,\eta=\frac{1}{3}$ and $n=1$ in the above inequality, for all $(\mu,\nu)\in\mathcal{R}$, such that $\delta+\alpha\leq 1$. We conclude that $\mathcal{M}$ is either $\mathcal{R}$ -complete or Γ -closed. Which is a contradiction to our claim. Hence, all the hypotheses of Theorem 6 are satisfied.

An application to non linear matrix equations

In this section, we prove the existence of the solution for the nonlinear matrix equation. We use one application to utilize the results obtained in Theorem 6, where a fixed point solution is applied to complete the Branciari distance. We refer to the study of the nonlinear matrix equation from Ran and Reurings[5] who proved a fixed point theorem in partially ordered sets and some applications to matrix equations. The Hermitian solution of the equation $X=Q+\mathcal{N}X^{-1}\mathcal{N}^{*}$ is the matrix equation arising from the Gaussian process. The equation admits both definite positive solution and definite negative solution if and only if $\mathcal{N}$ is non-singular. If $\mathcal{N}$ is singular, no definite negative solution exists. Nonlinear matrix equations play an important role in several problems that arise in the analysis of control theory and system theory. The main concern of this section is to apply Theorem 6 to study the following nonlinear matrix equations, which are motivated by Jain *et al.*,[34] Lim *et al.*,[35] Sawangsup and Sintunavara,[14] Ran and Reurings[5] and several others.

$$\begin{aligned}\mu &= Q+\sum_{i=1}^{n}\mathcal{N}_i^{*}\Gamma(\mu)\mathcal{N}_i,\\ Q &= \mu-\mathcal{N}_1^{*}\Gamma(\mu)\mathcal{N}_1-\ldots-\mathcal{N}_n^{*}\Gamma(\mu)\mathcal{N}_n,\end{aligned}$$

where $\mathcal{H}(n)$ is a set of $n\times n$ Hermitian matrices, $\mathfrak{p}(n)$ is a set of $n\times n$ positive definite matrices and $\mathfrak{p}(n)\subseteq\mathcal{H}(n)$, $Q\in\mathfrak{p}(n)$ is a Hermitian positive definite matrix, $\mathcal{N}_i$ is $n\times n$ matrices and $\Gamma;\mathfrak{p}(n)\to\mathfrak{p}(n)$ is a continuous order-preserving map such that $\Gamma(0)=0$.

The set $\mathcal{H}(n)$ equipped with the trace norm $\|\cdot\|_{tr}$ is a complete metric space and partially ordered with partial ordering $\preceq$, where $\mu\preceq\nu$ equivalently $\nu\succeq\mu$.

We use the following lemmas from Ran and Reurings[5] that will be useful for developing our results.

Lemma 2[5]. *If $\mu,\nu\succeq 0$ are $n\times n$ matrices, then* $0\leq tr(\mu,\nu)\leq\|\nu\|\,|tr(\mu)|$.

Lemma 3[5]. *If $\mu,\nu\preceq I_n$, then* $\|\mu\|\leq 1$.

Now, we prove a fixed point for self-mappings for the following nonlinear matrix equation in Branciari distance.

$$\mu \quad = \quad Q+\sum_{i=1}^{n}\mathcal{N}_i^{*}\Gamma(\mu)\mathcal{N}_i,$$

where $Q\in\mathfrak{p}(n)$, $\mathcal{N}_i$ is $n\times n$ matrices, $\mathcal{N}_i^{*}$ stands for conjugate transpose of $\mathcal{N}_i\in\mathcal{H}(n)$ and $\Gamma;\mathfrak{p}(n)\to\mathfrak{p}(n)$ is a continuous order-preserving map such that $\Gamma(0)=0$.

Theorem 7. *Consider the class of nonlinear matrix Equation* [equation 3.5] *and suppose the following condition holds.*

(i) *there exists $Q\in\mathfrak{p}(n)$ with* $Q \quad = \quad Q+\sum_{i=1}^{n}\mathcal{N}_i^{*}\Gamma(Q)\mathcal{N}_i,$

(ii) *for all* $\mu,\nu\in\mathfrak{p}(n)$, $\mu\leq\nu \quad\Rightarrow\quad \sum_{i=1}^{n}\mathcal{N}_i^{*}\Gamma(\mu)\mathcal{N}_i\leq\sum_{i=1}^{n}\mathcal{N}_i^{*}\Gamma(\nu)\mathcal{N}_i,$

(iii)*Thereexist $\delta,\alpha\in(0,1)$ forwhich* $\sum_{i=1}^{n}\mathcal{N}_i^{*}\mathcal{N}_i<\delta\mathcal{I}_n$ *and* $\sum_{i=1}^{n}\mathcal{N}_i^{*}\Gamma(Q)\mathcal{N}_i>0$

such that for all $\mu\leq\nu$ we have $\mu\leq\nu \quad\Rightarrow\quad \sum_{i=1}^{n}\mathcal{N}_i^{*}\Gamma(\mu)\mathcal{N}_i\leq\sum_{i=1}^{n}\mathcal{N}_i^{*}\Gamma(\nu)\mathcal{N}_i,$

and $\sum_{i=1}^{n}\mathcal{N}_i^{*}\Gamma(\mu)\mathcal{N}_i \quad\neq\quad \sum_{i=1}^{n}\mathcal{N}_i^{*}\Gamma(\nu)\mathcal{N}_i,$

(iv)*thereexist $\mu,\nu\in\mathcal{P}(n)$ and $\vartheta\leq 1$ such that* $\|\Gamma\mu-\Gamma\nu\|_{tr} \;\leq\; \vartheta\frac{\|\mu-\nu\|_{tr}}{\left(1+\eta\sqrt{\|\mu-\nu\|_{tr}}\right)^2},$

where

$$\|\mu-\nu\|_{tr}=d(\mu,\nu) \quad = \quad \mathcal{M}_{\mathcal{R}}(\mu,\nu)=\left[d(\mu,\nu)\right]^{\delta}.\left[d(\mu,\Gamma\mu)\right]^{\alpha}.\left[d(\nu,\Gamma\nu)\right]^{1-\alpha-\delta},$$

and $\vartheta=\sum_{i=1}^{n}\mathcal{N}_i^{*}\mathcal{N}_i.$

Then, the non linear matrix equation $(\ref{equation 3.5})$ has a solution in $\mathfrak{p}(n)\subseteq\mathcal{H}(n)$.

Proof. Define $\Gamma:\mathfrak{p}(n)\to\mathfrak{p}(n)$ by

$$\Gamma(x) \quad = \quad Q+\sum_{i=1}^{n}\mathcal{N}_i^{*}\Gamma(x)\mathcal{N}_i,$$

Citation: Wangwe L. Fixed point theorems for relation-theoretic *F*-interpolative in branciari distance with an application. *Phys Astron Int J.* 2023;7(2):122‒129. DOI: 10.15406/paij.2023.07.00296

for all $x \in \mathfrak{p}(n)$. Then the fixed point of the mapping Γ is a solution of the matrix equation $(\ref{equation 3.5})$.

The Branciari metric $d : \mathfrak{p}(n) \times \mathfrak{p}(n) \to \mathbb{R}_+$ is defined by

$$d(\mu,\nu) \quad = \quad \|\mu-\nu\|.$$

Let Γ be well defined on $\mathfrak{p}(n)$ and Γ-closed. For $\mu,\nu \in \mathfrak{p}(n)$ with $\mu \preceq \nu$, then $\Gamma(\mu) \preceq \Gamma(\nu)$. We claim that Γ is not an F-$\mathcal{R}$-contraction mapping with respect to $\eta > 0$ and $d(\mu,\nu) > 0$ and by using $(i)-(iv)$ we get

$$\begin{aligned}
d(\Gamma\mu,\Gamma\nu) &= \|\Gamma\mu,\Gamma\nu\|_{tr} \Rightarrow \|\Gamma x,\Gamma y\|, \\
&= \left\|\sum_{i=1}^{n}\mathcal{N}_i^*\Gamma(\mu)\mathcal{N}_i - \sum_{i=1}^{n}\mathcal{N}_i^*\Gamma(\nu)\mathcal{N}_i\right\| \\
&= \left\|\sum_{i=1}^{n}\mathcal{N}_i^*\mathcal{N}_i\left[\Gamma(\mu)-\Gamma(\mu)\right]\right\| \\
&= \left\|\sum_{i=1}^{n}\mathcal{N}_i^*\mathcal{N}_i\right\| \|\Gamma\mu-\Gamma\nu\| \\
&= \left\|\sum_{i=1}^{n}\mathcal{N}_i^*\mathcal{N}_i\right\| \frac{\|\mu-\nu\|_{tr}}{\left(1+\eta\sqrt{\|\mu-\nu\|_{tr}}\right)^2}, \\
&= \vartheta\frac{\|\mu-\nu\|_{tr}}{\left(1+\eta\sqrt{\|\mu-\nu\|_{tr}}\right)^2}, \\
&= \vartheta\frac{\mathcal{M}_{\mathcal{R}}(\mu,\nu)}{\left(1+\eta\sqrt{\mathcal{M}_{\mathcal{R}}(\mu,\nu)}\right)^2}, \\
\frac{d(\Gamma\mu,\Gamma\nu)}{\mathcal{M}_{\mathcal{R}}(\mu,\nu)} &\le \frac{1}{\left(1+\eta\sqrt{\mathcal{M}_{\mathcal{R}}(\mu,\nu)}\right)^2},
\end{aligned}$$

which is a contradiction. Hence Γ is a contraction. Therefore, from $\sum_{i=1}^{n}\mathcal{N}_i^*\Gamma(Q)\mathcal{N}_i > 0$, we have $Q \le \Gamma(Q)$. Thus, by using Theorem 6 we conclude that Γ has a unique fixed point in $\mathfrak{p}(n)$ and $\mathfrak{p}(n) \in \mathcal{M}$.

Conclusion

The new concept of relation-theoretic F-interpolative mapping endowed with binary relation in Branciari Distance in metric spaces has been introduced. In particular, we improved and extended the works due to Alam and Imdad[17], Ahmadullah *et al.*[28], Ahmadullah *et al*[29]. , Eke *et al.*[30], Sawangsup and Sintunavarat[14] ,Aydi *et al.*[24] and Karapinar *et al*[20]. . In doing so, we generalized several other works in the literature having the same setting. Henceforth, the results obtained will be verified with the help of illustrative examples. Also, we demonstrate the results with an application in matrix equations.

Compliance with ethical standards

Conflict of interest: Authors declare that they have no conflicts of interest.

Research involving human participants and/or animals: The author declares that there are no human participants and/or animals involved in this research.

Funding

Authors declare that there is no funding available for this research.

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Citation: Wangwe L. Fixed point theorems for relation-theoretic *F*-interpolative in branciari distance with an application. *Phys Astron Int J.* 2023;7(2):122–129. DOI: 10.15406/paij.2023.07.00296