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Lie algebra classification, conservation laws and invariant solutions for modification of the generalization of the Emden–Fowler equation

Abstract

We obtain the optimal system's generating operators associated to a modification of the generalization of the Emden–Fowler Equation. Using those operators we characterize all invariant solutions associated to a generalized. Moreover, we present the variational symmetries and the corresponding conservation laws, using Noether's theorem and Ibragimov's method. Finally, we classify the Lie algebra associated to the given equation.

Keywords: Invariant solutions, Lie symmetry group, Optimal system, Lie algebra classification, Variational symmetries, Conservation laws, Ibragimov's method, Noether's theorem.

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Introduction

In,[1] Ibragimov presents the following equation

$$y_x x = -y^{(-1)}y_x^2 - 3x^{(-1)}y_x, \tag{1}$$

with its respective solution

$$y(x) = \pm\sqrt{C_1 - \frac{C_2}{x^2}}, where C_1, C_2 are constants. \tag{2}$$

This solution is obtained using the integrating factor method. In,[2] Muriel and Romero, calculate the λ −Symmetries associated to integrating factors of (1). In,[3] Polyanin and Zaitsev present a solution of (1) of the form

$$y(x) = C_2 exp(C_1 |x|^4), where C_1, C_2 are constants. \tag{3}$$

The purpose of this work is: $i)$ to calculate the Lie symmetry group, $ii)$ to present the optimal algebra (optimal system) for (1), $iii)$ making use of all elements of the optimal algebra, to propose invariant solutions for (1), then $iv)$ to construct the Lagrangian with which we could determine the variational symmetries using Noether's theorem, and thus to present conservation laws associated, and $iv)$ also using Ibragimov's method build some non-trivial conservation laws, and finally $v)$ to classify the Lie algebra associated to (1), corresponding to the symmetry group. we note that equation (1) can be considered as a modification of the generalization of the Emden–Fowler Equation.

Continuous group of Lie symmetries

In this section we study the Lie symmetry group for (1). The main result of this section can be presented as follows:

Proposition 1 *The Lie symmetry group for the equation (1) is generated by the following vector fields:*

$$\Pi_1 = x\frac{\partial}{\partial x}, \Pi_2 = x^3\frac{\partial}{\partial x}, \Pi_3 = xy^2\frac{\partial}{\partial x} + (-y^3)\frac{\partial}{\partial y}, \Pi_4 = x^3y^2\frac{\partial}{\partial x} \tag{4}$$

$$\Pi_5 = y\frac{\partial}{\partial y}$$

Proof. A general form of the one-parameter Lie group admitted by (1) is given by

$$x \to x + \in \xi(x,y) + O(\in^2) \quad and \quad y \to y + \in \eta(x,y) + O(\in^2)$$

where ϵ is the group parameter. The vector field associated with the group of transformations shown above can be written as $\Gamma = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y'}$, where ξ, η are differentiable functions in $\mathbb{R}^2$. Applying its second prolongation

$$\Gamma^{(2)} = \Gamma + \eta_{[x]}\frac{\partial}{\partial y_x} + \eta_{[xx]}\frac{\partial}{\partial y_{xx}}, \tag{5}$$

to eq.(1), we must find the infinitesimals ξ, η satisfying the symmetry condition

$$\xi(-3x^{-2}y_x) + \eta(-y_x^2 y^{-2}) + \eta_{[x]}(2y^{-1}y_x + 3x^{-1}) + \eta_{[xx]} = 0, \tag{6}$$

associated with (1). Here $\eta_{[x]}, \eta_{[xx]}$ are the coefficients in $\Gamma^{(2)}$ given by:

$$\eta_{[x]} = D_x[\eta] - (D_x[\xi])y_x = \eta_x + (\eta_y - \xi_x)y_x - \xi_y y_x^2.$$

$$\eta_{[xx]} = D_x[\eta_{[x]}] - (D_x[\xi])y_{xx},$$

$$= \eta_{xx} + (2\eta_{xy} - \xi_{xx})y_x + (\eta_{yy} - 2\xi_{xy})y_x^2 - \xi_y y y_x^3$$

$$+(\eta_y - 2\xi_x)y_{xx} - 3\xi_y y_x y_{xx}. \tag{7}$$

Being D_x is the total derivative operator: $D_x = \partial_x + y_x\partial_y + y_{xx}\partial_{yx} + \ldots$. Replacing (7) into (6) and using (1) we obtain:

$$(5y^{(-1)}\xi_y - \xi_{yy})y_x^3 + (y^{(-1)}\eta_y - \eta y^{(-2)} - 2\xi_{xy} + 6x^{-1}\xi_y + \eta_y y)y_x^2$$

$$+(-3x^{-2}\xi + 2y^{-1}\eta_x + 3x^{-1}\xi_x + 2\eta_{xy} - \xi_{xxx})y_x + (\eta_{x\,x} + 3x^{-1}\eta_x) = 0.$$

From (8), canceling the coefficients of the monomials variables in derivatives $1, y_x^3, y_x^2$ and y_x we obtain the determining equations for the symmetry group of (1), with $x, y \neq 0$. That is:

$$5\xi_y - y\xi_{yy} = 0 \tag{8a}$$

$$xy\eta_y - x\eta - 2xy^2\xi_{xy} + 6y^2\xi_y + xy^2\eta_{yy} = 0, \tag{8b}$$

$$-3y\xi + 2x^2\eta_x + 3xy\xi_x + 2x^2 y\eta_{xy} - x^2 y\xi_{xxx} = 0. \tag{8c}$$

$$x\eta_{xx} + 3\eta_x = 0. \tag{8d}$$

Solving the system of equations (8a)-(8d) for ξ and η we get

$$\xi = c_1 x + c_2 x^3 + c_3 xy^2 + c_4 x^3 y^2,$$

$$\eta = -c_3 y^3 + c_5 y.$$

Thus, the infinitesimal generators of the group of symmetries of (1) are the operators $\Pi_1 - \Pi_5$ described in the statement of the Proposition 1; thus having the proposed result.

Optimal algebra

Taking into account,[1,4-6] we present in this section the optimal algebra associated to the symmetry group of (1), that shows a systematic way to classify the invariant solutions. To obtain the optimal algebra, we should first calculate the corresponding commutator table, which can be obtained from the operator

$$[\Pi_\alpha, \Pi_\beta] = \Pi_\alpha\Pi_\beta - \Pi_\beta\Pi_\alpha = \Sigma_{i=1}^{n}(\Pi_\alpha(\xi_\beta^i) - \Pi_\beta(\xi_\alpha^i))\frac{\partial}{\partial x^i}, \tag{9}$$

where $i = 1, 2$, with $\alpha, \beta = 1, \ldots 5$ and $\xi_\alpha^i, \xi_\beta^i$ are the corresponding coefficients of the infinitesimal operators Π_α, Π_β. After applying the operator (12) to the symmetry group of (1), we obtain the operators that are shown in the following table !

Table 1 Commutators table associated to the symmetry group of (1)

	Π_1	Π_2	Π_3	Π_4	Π_5
Π_1	0	$2\Pi_2$	0	$2\Pi_4$	0
Π_2	$-2\Pi_2$	0	$-2\Pi_4$	0	0
Π_3	0	$2\Pi_4$	0	0	$-2\Pi_3$
Π_4	$-2\Pi_4$	0	0	0	$-2\Pi_4$
Π_5	0	0	$2\Pi_3$	$2\Pi_4$	0

Now, the next thing is to calculate the adjoint action representation of the symmetries of (1) and to do that, we use Table 1 and the operator.

$Ad(exp(\lambda\Pi))H = \Sigma_{n=0}^{\infty}\frac{\lambda^n}{n!}(ad(\Pi))^n G$ for the symmetries Π and G

Making use of this operator, we can construct the Table 2, which shows the adjoint representation for each Π_i. !

Table 2 Adjoint representation of the symmetry group of (1)

adj[,]	Π_1	Π_2	Π_3	Π_4	Π_5
Π_1	Π_1	$e^{-2\lambda}\Pi_2$	Π_3	$e^{-2\lambda}\Pi_4$	Π_5
Π_2	$\Pi_1 + 2\lambda\Pi_2$	Π_2	$\Pi_3 + 2\lambda\Pi_4$	Π_4	Π_5
Π_3	Π_1	$\Pi_2 - 2\lambda\Pi_4$	Π_3	Π_4	$\Pi_5 + 2\lambda\Pi_3$
Π_4	$\Pi_1 + 2\lambda\Pi_4$	Π_2	Π_3	Π_4	$\Pi_5 + 2\lambda\Pi_4$
Π_5	Π_1	Π_2	$e^{-2\lambda}\Pi_3$	$e^{-2\lambda}\Pi_4$	Π_5

Proposition 2 *The optimal algebra associated to the equation (1) is given by the vector fields*

$$\Pi_4, a_2\Pi_2, a_3\Pi_3, a_1\Pi_1 + a_3\Pi_3, a_2\Pi_2 + \Pi_3\Pi_3 + b_5\Pi_4, a_1\Pi_1 + b_6\Pi_4, \Pi_2 + b_7\Pi_4,$$

$$b_3\Pi_3 + \Pi_5, a_2\Pi_2 + \Pi_5, a_1\Pi_1 + \Pi_5, -2\Pi_1 + b_1\Pi_4 + \Pi_5, a_2\Pi_2 + a_3\Pi_3 + b_4\Pi_4,$$

$$\Pi_1 + b_8\Pi_2 + b_9\Pi_4, a_2\Pi_2 + \frac{a_4}{a_2}\Pi_3 + b_2\Pi_4 + \Pi_5.$$

Proof. To calculate the optimal algebra system, we start with the generators of symmetries (4) and a generic nonzero vector. Let

$$G = a_1\Pi_1 + a_2\Pi_2 + a_3\Pi_3 + a_4\Pi_4 + a_5\Pi_5. \tag{13}$$

The objective is to simplify as many coefficients a_i as possible, through maps adjoint to G, using Table (2).

1. Assuming $a_5 = 1$ in (10) we have that $G = a_1\Pi_1 + a_2\Pi_2 + a_3\Pi_3 + a_4\Pi_4 + \Pi_5$. Applying the adjoint operator to Π_1, G and (Π_5, G) we don't have any reducción, on the other hand applying the adjoint operator to (Π_2, G) we get

$$G_1 = Ad(\exp(\lambda_1\Pi_2))G = a_1\Pi_1 + (a_2 + 2a_1\lambda_1)\Pi_2 + a_3\Pi_3 + (a_4 + 2a_3\lambda_1)\Pi_4 + \Pi_5 \tag{11}$$

1. 1) **Case** $a_1 \neq 0$ **.** Using $\lambda_1 = \frac{-a_2}{2a_1}$, with $a_1 \neq 0$, in (11), Π_2 is eliminated, therefore $G_1 = a_1\Pi_1 + a_3\Pi_3 + b_1\Pi_4 + \Pi_5$, where $b_1 = a_4 + \frac{a_3}{a_1}$. Now, applying the adjoint operator to (Π_3, G_1), we get $G_2 = Ad(\exp(\lambda_2\Pi_3))G_1 = a_1\Pi_1 + (a_3 + 2\lambda_2)\Pi_3 + b_1\Pi_4 + \Pi_5$.

Using $\lambda_2 = \frac{-a_3}{2}$, is eliminated Π_3, then $G_2 = a_1\Pi_1 + b_1\Pi_4 + \Pi_5$. Applying the adjoint operator to (Π_4, G_2), we get

$$G_3 = Ad(\exp\lambda_3\Pi_4))G_2 = a_1\Pi_1 + (b_1 + 2\lambda_3(a_1 + 2))\Pi_4 + \Pi_5. \tag{12}$$

1. 1. A) **Case** $a_1 + 2 \neq 0$ **.** Using $\lambda_3 = \frac{1}{2(a_1 + 2)}$, with $a_1 + 2 \neq 0$, in (12), Π_4 is eliminated, therefore $G_3 = a_1\Pi_1 + \Pi_5$. Then, we have the first element of the optimal system.

$$G_3 = a_1\Pi_1 + \Pi_5, \text{with } a_1 \neq 0 \text{ and } a_1 + 2 \neq 0. \tag{13}$$

This is how the first reduction of the generic element (10) ends.

1. 1. B) **Case** $a_1 + 2 \neq 0$ **.** We get $G_3 = -2\Pi_1 + b_1\Pi_4 + \Pi_5$. Then, we have other element of the optimal system.

$$G_3 = -2\Pi_1 + b_1\Pi_4 + \Pi_5. \tag{14}$$

This is how other reduction of the generic element (13) ends.

1. 2) **Case** $a_1 = 0$ **.** We get $G_1 = a_2\Pi_2 + a_3\Pi_3 + (a_4 + 2a_3\lambda_1) + \Pi_5$.

1. 2. A) **Case** $a_3 \neq 0$ **.** Using $\lambda_1 = \frac{-a_4}{2a_3}$, with $a_3 \neq 0$, is eliminated Π_4, then $G_1 = a_2\Pi_2 + a_3\Pi_3 + \Pi_5$. Applying the adjoint operator to (Π_3, G_1), we get

$$G_4 = Ad(\exp\lambda_4\Pi_3))G_1 = a_2\Pi_2 + (a_3 + 2\lambda)\Pi_3 - 2\lambda_4\Pi_4 + \Pi_5. \tag{15}$$

Using $\lambda_4 = \frac{-a_3}{2}$, is eliminated Π_3, then $G_4 = a_2\Pi_2 + a_3\Pi_4 + \Pi_5$. Now applying the adjoint operator to (Π_4, G_4), we have

$$G_5 = Ad(\exp\lambda_5\Pi_4))G_4 = a_2\Pi_2 + (a_3 + 2\lambda_5)\Pi_4 + \Pi_5. \tag{16}$$

Using $\lambda_5 = \frac{-a_3}{2}$, is eliminated Π_4, then we have other element of the optimal system.

$$G_5 = a_2\Pi_2 + \Pi_5. \qquad (17)$$

This is how other reduction of the generic element (10) ends.

1.2.B) **Case** $a_3 = 0$. We get $G_1 = a_2\Pi_2 + a_4\Pi_4 + \Pi_5$. Now applying the adjoint operator to (Π_3, G_1), we have

$$G_6 = Ad(\exp\lambda_6\Pi_3))G_1 = a_2\Pi_2 + 2\lambda_6\Pi_3 + (a_4 - 2a_2\lambda_6)\Pi_4 + \Pi_5. \quad (18)$$

1.2.B.1) **Case** $a_2 \neq 0$. Using $\lambda_6 = \frac{a_4}{2a_2}$, with $a_2 \neq 0$, is eliminated Π_4, then $G_6 = a_2\Pi_2 + \frac{a_4}{a_2}\Pi_3 + \Pi_5$. Now applying the adjoint operator to (Π_4, G_4), we get

$$G_7 = Ad(\exp(\lambda_7\Pi_4))G_6 = a_2\Pi_2 + \frac{a_4}{a_2}\Pi_3 + 2\lambda_7\Pi_4 + \Pi_5. \qquad (19)$$

It's clear that we don't have any reduction, then using $\lambda_7 = \frac{b_2}{2}$, then we have other element of the optimal system.

$$G_7 = a_2\Pi_2 + \frac{a_4}{a_2}\Pi_3 + b_2\Pi_4 + \Pi_5. \qquad (20)$$

This is how other reduction of the generic element (10) ends.

1.2.B.2) **Case** $a_2 = 0$. We get $G_6 = 2\lambda_6\Pi_3 + a_4\Pi_4 + \Pi_5$. It is clear that we don't have any reduction, then using $\lambda_6 = \frac{b_3}{2}$, we have $G_6 = b_3\Pi_3 + a_4\Pi_4 + \Pi_5$. Now applying the adjoint operator to (Π_4, G_6), we have

$$G_8 = Ad(\exp(\lambda_8\Pi_4))G_6 = b_3\Pi_3 + (a_4 + 2\lambda_8)\Pi_4 + \Pi_5. \qquad (21)$$

Using $\lambda_8 = \frac{-a_4}{2}$, is eliminated Π_4, then we have other element of the optimal system.

$$G_8 = b_3\Pi_3 + \Pi_5. \qquad (22)$$

This is how other reduction of the generic element (10) ends.

2. Assuming $a_5 = 0$ and $a_4 = 1$ in (10), we have that $G = a_1\Pi_1 + a_2\Pi_2 + a_3\Pi_3 + \Pi_4$. Applying the adjoint operator to (Π_1, G) and (Π_5, G) we don't have any reduction, on the other hand applying the adjoint operator to (Π_2, G) we get

$$G_9 = Ad(\exp(\lambda_9\Pi_2))G = a_1\Pi_1 + (a_2 + 2a_1\lambda_9)\Pi_2 + a_3\Pi_3 + (1 + 2a_3\lambda_9)\Pi_4. \quad (23)$$

2.1) **Case** $a_1 \neq 0$. Using $\lambda_9 = \frac{-a_2}{2a_1}$, with $a_1 \neq 0$, in (26), Π_2 is eliminated, therefore $G_9 = a_1\Pi_1 + a_3\Pi_3 + b_4\Pi_4$, where $b_3 = 1 - \frac{a_3a_2}{a_1}$. Now, applying the adjoint operator to (Π_3, G_9), we don't have any reduction, after applying the adjoint operator to (Π_4, G_9), we get $G_{10} = Ad(\exp(\lambda_{10}\Pi_4))G_9 = a_1\Pi_1 + a_3\Pi_3 + (b_3 + 2a_1\lambda_{10})\Pi_4$. How $a_1 \neq 0$, we can use $\lambda_{10} = \frac{-b_3}{2a_1}$, is eliminated Π_4, thus we have other element of the optimal system.

$$G_{10} = a_1\Pi_1 + a_3\Pi_3. \qquad (24)$$

This is how other reduction of the generic element (10) ends.

2.2) **Case** $a_1 = 0$. We get $G_9 = a_2\Pi_2 + a_3\Pi_3 + (1 + 2a_3\lambda_9)\Pi_4$.

2.2.A) **Case** $a_3 \neq 0$. Using $\lambda_9 = \frac{-1}{2a_3}$, with $a_3 \neq 0$, Π_4 is eliminated, therefore $G_9 = a_2\Pi_2 + a_3\Pi_3$. Now, applying the adjoint operator to (Π_3, G_9), we get $G_{11} = Ad(\exp(\lambda_{11}\Pi_3))G_9 = a_2\Pi_2 + a_3\Pi_3 - 2a_1\lambda_{11}\Pi_4$.

2.2.A.1) **Case** $a_2 \neq 0$. It's clear that we don't have any reduction, using $\lambda_{11} = \frac{-b_4}{2a_2}$, with $a_2 \neq 0$, we get $G_{11} = a_2\Pi_2 + a_3\Pi_3 + b_4\Pi_4$. Now, applying the adjoint operator to (Π_4, G_{11}), we don't have any reduction, thus we have other element of the optimal system.

$$G_{11} = a_2\Pi_2 + a_3\Pi_3 + b_4\Pi_4. \qquad (25)$$

This is how other reduction of the generic element (10) ends.

2.2.A.2) **Case** $a_2 = 0$. We get $G_{11} = a_3\Pi_3$. Now, applying the adjoint operator to (Π_4, G_{11}), we don't have any reduction, thus we have other element of the optimal system.

$$G_{11} = a_3\Pi_3. \qquad (26)$$

This is how other reduction of the generic element (10) ends.

2.2.B) **Case** $a_3 = 0$ We get $G_9 = a_2\Pi_2 + \Pi_4$. Now, applying the adjoint operator to (Π_3, G_9), we have $G_{12} = Ad(\exp(\lambda_{12}\Pi_3))G_9 = a_2\Pi_2 + (1 - 2a_2\lambda_{12})\Pi_4$.

2.2.B.1) **Case** $a_2 \neq 0$. Using $\lambda_{12} = \frac{1}{2a_2}$, with $a_2 \neq 0$, is eliminated Π_4, then $G_{12} = a_2\Pi_2$. Now, applying the adjoint operator to (Π_4, G_{12}), we don't have any reduction, thus we have other element of the optimal system.

$$G_{12} = a_2\Pi_2. \qquad (27)$$

This is how other reduction of the generic element (10) ends.

2.2.B.2) **Case** $a_2 = 0$. We get $G_{12} = \Pi_4$. Now, applying the adjoint operator to (Π_4, G_{12}), we don't have any reduction, thus we have other element of the optimal system.

$$G_{12} = \Pi_4. \qquad (28)$$

This is how other reduction of the generic element (10) ends.

3. Assuming $a_5 = a_4 = 0$ and $a_3 = 1$ in (10), we have that $G = a_1\Pi_1 + a_2\Pi_2 + \Pi_3$.. Applying the adjoint operator to (Π_1, G) and (Π_5, G) we don't have any reduction, on the other hand applying the adjoint operator to (Π_2, G) we get

$$G_{13} = Ad(\exp(\lambda_{13}\Pi_2))G = a_1\Pi_1 + (a_2 + 2a_1\lambda_{13})\Pi_2 + \Pi_3 + 2\lambda_{13}\Pi_4. \quad (29)$$

3.1) **Case** $a_1 \neq 0$. Using $\lambda_{13} = \frac{-a_2}{2a_1}$, with $a_1 \neq 0$, in (29), Π_2 is eliminated, therefore $G_{13} = a_1\Pi_1 + \Pi_3 + b_3\Pi_4$, where $b_3 = \frac{a_2}{a_1}$. Now, applying the adjoint operator to (Π_3, G_{13}), we don't have any reduction, after applying the adjoint operator to (Π_4, G_9), we get $G_{14} = Ad(\exp(\lambda_{14}\Pi_4))G_{13} = a_1\Pi_1 + \Pi_3 + (b_3 + 2a_1\lambda_{13})\Pi_4$. A s $a_1 \neq 0$, we can use $\lambda_{13} = \frac{-b_3}{2a_1}$, is eliminated Π_4, then we have other element of the optimal system.

$$G_{14} = a_1\Pi_1 + \Pi_3. \quad (30)$$

This is how other reduction of the generic element (10) ends.

3. 2) **Case** $a_1 = 0$ **.** We get $G_{13} = a_2\Pi_2 + \Pi_3 + 2\lambda_{13}\Pi_4$, using $\lambda_{13} = \frac{b_5}{2}$, then $G_{13} = a_2\Pi_2 + \Pi_3 + b_5\Pi_4$. Now, applying the adjoint operator to (Π_3, G_{13}), we get $G_{14} = Ad(\exp\lambda_{14}\Pi_3))G_{13} = a_2\Pi_2 + \Pi_3 + (b_5 - 2a_2\lambda_{14})\Pi_4$.

3. 2. A) **Case** $a_2 \neq 0$ **.** Using $\lambda_{14} = \frac{b_5}{2a_2}$, with $a_2 \neq 0$, is eliminated Π_4, then $G_{14} = a_2\Pi_2 + \Pi_3$. Now applying the adjoint operator to (Π_4, G_{14}) we don't have any reduction, then we have other element of the optimal system.

$$G_{14} = a_2\Pi_2 + \Pi_3. \quad (31)$$

This is how other reduction of the generic element (13) ends.

3. 2. B) **Case** $a_2 = 0$ **.** We get $G_{14} = \Pi_3 + b_5\Pi_4$. Now applying the adjoint operator to (Π_4, G_{14}) we don't have any reduction, then we have other element of the optimal system.

$$G_{14} = \Pi_3 + b_5\Pi_4. \quad (32)$$

This is how other reduction of the generic element (10) ends.

4. Assuming $a_3 = a_4 = a_3 = 0$ and $a_2 = 1$ in (10), we have that $G = a_1\Pi_1 + \Pi_2$. Applying the adjoint operator to (Π_1, G) and (Π_5, G) we don't have any reduction, on the other hand applying the adjoint operator towe get

$$G_{15} = Ad(\exp(\lambda_{15}\Pi_2))G = a_1\Pi_1 + (1 + 2a_1\lambda_{15})\Pi_2. \quad (33)$$

4. 1) **Case** $a_1 \neq 0$ **.** Using $\lambda_{15} = \frac{-1}{2a_1}$, with $a_1 \neq 0$, is eliminated Π_2, then $G_{15} = a_1\Pi_1$. . Now applying the adjoint operator to (Π_3, G_{15}) we don't have any reduction, on the other hand applying the adjoint operator to (Π_4, G_{15}) we get $G_{16} = Ad(\exp(\lambda_{16}\Pi_4))G_{15} = a_1\Pi_1 + 2a_1\lambda_{16}\Pi_4$. It is clear that we don't have any reduction, then using $\lambda_{16} = \frac{b_6}{2a_1}$, with $a_1 \neq 0$, we have other element of the optimal system.

$$G_{16} = a_1\Pi_1 + b_6\Pi_4. \quad (34)$$

This is how other reduction of the generic element (10) ends.

4. 2) **Case** $a_1 = 0$ **.** We get $G_{15} = \Pi_2$. Now applying the adjoint operator to (Π_3, G_{15}) we get $G_{17} = Ad(\exp(\lambda_{17}\Pi_3))G_{15} = \Pi_2 - 2\lambda_{17}\Pi_4$. It is clear that we don't have any reduction, then using $\lambda_{17} = \frac{-b_7}{2}$, then $G_{17} = \Pi_2 + b_7\Pi_4$. Now applying the adjoint operator to (Π_4, G_{17}), we don't have any reduction, after we have other element of the optimal system.

$$G_{17} = \Pi_2 + b_7\Pi_4. \quad (35)$$

This is how other reduction of the generic element (10) ends.

5. Assuming $a_5 = a_4 = a_3 = a_2 = 0$ and $a_1 = 1$ in (10), we have that $G = \Pi_1$. Applying the adjoint operator to (Π_1, G), (Π_3, G) and (Π_5, G) we don't have any reduction, on the other hand applying the adjoint operator to (Π_2, G) we get

$$G_{18} = Ad(\exp(\lambda_{18}\Pi_2))G = \Pi_1 - 2\lambda_{18}\Pi_2. \quad (36)$$

It's clear that we don't have any reduction, then using $\lambda_{18} = \frac{b_8}{2}$, we get $G_{18} = \Pi_1 + b_8\Pi_2$. Now applying the adjoint operator to (Π_4, G_{18}), we have

$$G_{19} = \Pi_1 + b_8\Pi_2 + 2\lambda_{19}\Pi_2. \quad (37)$$

It's clear that we don't have any reduction, then using $\lambda_{19} = \frac{b_9}{2}$, we have other element of the optimal system.

$$G_{19} = \Pi_1 + b_8\Pi_2 + b_8\Pi_4. \quad (38)$$

This is how other reduction of the generic element (10) ends.

4 Invariant solutions by the generators of the optimal algebra

In this section, we characterize the invariant solutions taking into account all operators that generate the optimal algebra presented in Proposition 2. For this purpose, we use the method of invariant curve condition[5] (presented in section 4.3), which is given by the following equation

$$Q(x, y, y_x) = \eta = y_x\,\xi = 0. \quad (39)$$

Using the element Π_4 from Proposition 2, under the condition (42), we obtain that $Q = \eta_4 - y_x\,\xi_4 = 0$ which implies $(0) - y_x(x^3y^2) = 0$. After, we get $y(x) = c$, where c is a constant, which is an invariant solution for (1), using an analogous procedure with all of the elements of the optimal algebra (Proposition 2), we obtain both implicit and explicit invariant solutions that are shown in the Table 3, with c being a constant.

Variational symmetries and conserved quantities

In this section, we present the variational symmetries of (1) and we are going to use them to define conservation laws via Noether's theorem.[7] First of all, we are going to determine the Lagrangian using the Jacobi Last Multiplier method, presented by Nucci in,[8] and for this reason, we are urged to calculate the inverse of the determinant Δ,

$$\Delta = \begin{vmatrix} x & y_x & y_{xx} \\ \Pi_{1,x} & \Pi_{1,y} & \Pi_1^{(1)} \\ \Pi_{2,x} & \Pi_{2,y} & \Pi_2^{(1)} \end{vmatrix} = \begin{vmatrix} x & y_x & y_{xx} \\ x & 0 & -y_x \\ x^3 & 0 & -3x^2y_x \end{vmatrix},$$

where $\Pi_{1,x}, \Pi_{1,y}, \Pi_{2,x}$, and $\Pi_{2,y}$ are the components of the symmetries Π_1, Π_2 shown in the Proposition 4 and $\Pi_1^{(1)}, \Pi_2^{(1)}$ as its first prolongations. Then we get $\Delta = 2x^3y_x$ which implies that $M = \frac{1}{\Delta} = \frac{x^{-3}}{2y_x}$. Now, from,[8] we know that M can also be written as

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$M = L_{yxyx}$ which means that $L_{yxyx} = \frac{x^{-3}}{2y_x}$, then integrating twice with respect to y_x we obtain the Lagrangian

$$L(x,y,y_x) = \frac{x^{-3}}{2} y_x In(y_x) - \frac{x^{-3}}{2} y_x + y_x f_1(x,y) + f_2(x,y), \tag{40}$$

where f_1, f_2 are arbitrary functions. From the preceding expression we can consider $f_1 = f_2 = 0$. It is possible to find more Lagrangians for (1) by considering other vector fields given in the Proposition 4. We then calculate

$$\xi(x,y)L_x + \xi_x(x,y)L + \eta(x,y)L_y + \eta_{[x]}(x,y)L_{yx} = D_x\left[f(x,y)\right],$$

using (40) and (7). Thus we get

$$\xi\left(\frac{-3x^{-4}}{2} y_x In(yx) + \frac{3x^{-4}}{2} y_x\right) + \xi_x\left(\frac{x^{-3}}{2} y_x In(y_x) - \frac{x^{-3}}{2} y_x\right)$$

$$+(\eta_x + (\eta_y - \xi_x)y_x - \xi_y y_x^2))\left(\frac{x^{-3}}{2} In(y_x)\right) - f_x - y_x f_y = 0.$$

From the preceding expression, rearranging and associating terms with respect to $1, y_x, y_x In(y_x), y_x^2 In(y_x)$ and $In(y_x)$, we obtain the following determinant equations

$$\xi_y = \eta_x = f_x = 0, \tag{41a}$$

$$-3\xi + x\eta_y = 0, \tag{41b}$$

$$3\xi - x\xi_x - 2x^4 f_y = 0. \tag{41c}$$

Solving the preceding system for ξ, η and f we obtain the infinitesimal generators of Noether's symmetries

$$\eta = a_2, \ \xi = 0, \text{ and } f(y) = a_4. \tag{42}$$

with a_2 and a_4 arbitrary constants. Then, the Noether symmetry group or variational symmetries is

$$V_1 = \frac{\partial}{\partial y}, \tag{43}$$

According to,[9] in order to obtain the conserved quantities or conservation laws, we should solve

$$I = (X y_x - Y)L_{yx} - XL + f,$$

so, using (43), (47) and (48). Therefore, the conserved quantities are given by

$$I_1 = -\frac{x^{-3} In(y_x)}{2} + a_4, \tag{44}$$

Nonlinear self-adjointness

In this section we present the main definitions in the N. Ibragimov's approach to nonlinear self-adjointness of differential equations adopted to our specific case. For further details the interested reader is directed to.[6,10,11]

Consider second order differential equation

$$\mathfrak{F}(x,y,y_{(1)},y_2...,y_{(s)}) = 0, \tag{45}$$

With independent variables x and a dependent variable y, where $y_{(1)}, y_{(2)}, ... y_{(s)}$ denote the collection of $1,2,...,s-th$ order derivatives of y.

Definition 1 *Let $\mathfrak{F}$ be a differential function and $v = v(x)$ -the new dependent variable, known as the adjoint variable or nonlocal variable.[11] The formal Lagrangian for $\mathfrak{F} = 0$ is the differential function defined by*

$$\mathfrak{L} := v\mathfrak{F}. \tag{46}$$

Definition 2 *Let $\mathfrak{F}$ be a differential function and for the differential equation (45), denoted by $\mathfrak{F}[y] = 0$, we define the adjoint differential function to $\mathfrak{F}$ by*

$$\mathfrak{F}^* := \frac{\delta \mathfrak{L}}{\delta y} \tag{47}$$

and the adjoint differential equation by

$$\mathfrak{F}^*[y,v] = 0, \tag{48}$$

where the Euler operator

$$\frac{\delta}{\delta y} = \frac{\partial}{\partial y} + \Sigma_{m=1}^{\infty}(-1)^m D_{xi}...D_{xi,m} \frac{\partial}{\partial y_{x_{i_1} x_{i_2} ... x_{i_m}}} \tag{49}$$

and D_{xi} is the total derivative operator with respect to x_i defined by

$$D_{x_i} = \partial_{x_i} + y_{x_i}\partial_y + y_{x_i x_j}\partial_{y_{x_j}} + ... + y_{x_i x_{i_1} x_{i_2} ... x_{i_n}} \partial_{y_{x_{i_1} x_{i_2} ... x_{i_n}}} ...$$

Definition 3 *The differential equation (45) is said to be nonlinearly selfadjoint if there exists a substitution*

$$v = \phi(x,y) \neq 0 \tag{50}$$

such that

$$\tilde{\mathfrak{F}}^*\big|_{v=\phi(x,y)} = \lambda \mathfrak{F} \tag{51}$$

for some undetermined coefficient $\lambda = \lambda(x,y,...)$. If $v = \phi(y)$ in (50) and (51), the equation (45) is called quasi self-adjoint. If $v = y$, we say that the equation (45) is strictly self-adjoint.

Now we shall obtain the adjoint equation to the eq. (1). For this purpose we write (1) in the form (45), where

$$\mathfrak{F} := y_{xx} + y^{-1}y_x^2 + 3x^{-1}y_{x,} = 0. \tag{52}$$

Then the corresponding formal Lagrangian (46) is given by

$$\mathfrak{L} := v(y_{xx} + y^{-1}y_x^2 + 3x^{-1}y_x) = 0 \tag{53}$$

and the Euler operator (49) assumes the following form:

$$\frac{\delta \mathfrak{L}}{\delta y} = \frac{\partial \mathfrak{L}}{\partial y} - D_x \frac{\partial \mathfrak{L}}{\partial y_x} + D_x^2 \frac{\partial \mathfrak{L}}{\partial y_{xx}}. \tag{54}$$

We calculate explicitly the Euler operator (54) applied to $\mathfrak{L}$ determined by (58). In this way we obtain the adjoint equation (48) to (1):

$$\mathfrak{F}^* = v(y_x^2 y^{-2} + 3x^{-2} - 2y_{xx}y^{-1}) + v_x(-2y_x y^{-1} - 3x^{-1}) + v_{xx} = 0 \tag{55}$$

The main result in this section can be stated as follows.

Proposition 3 *The equation (1) is nonlinearly self-adjoint, with the substitution given by*

$$\phi(x,y) = y(k_1 x + k_2 x^3), \tag{56}$$

where k_1, k_2 are arbitrary constants.

Proof. Substituting in (55), and then in (52), $v = \phi(x,y)$ and its respective derivatives, and comparing the corresponding coefficients we get five equations:

$$-\phi_y = \lambda, \quad (57a)$$

$$-y^{-1}\phi + \phi_y = 0, \quad (57b)$$

$$-y^{-1}\phi_x + \phi_x y = 0, \quad (57c)$$

$$3x^{-2}\phi - 3x^{-1}\phi_x + \phi_{xx} = 0, \quad (57d)$$

$$y^{-2}\phi - y^{-1}\phi_y + y\phi_{yy} = 0. \quad (57e)$$

We observe that (57c) and (57e) are obtaned from (57b) by differentiation with respect to x and y. Therefore we have to study only Eqs. (57b) and (57d). Solving for ϕ in (57b) we obtain

$$\phi(x, y) = c_1(x)y, \quad (58)$$

where $c_1(x)$ is arbitrary function. Using (58) into (57d) we get $3x^{-2}c_1(c) - 3x^{-1}c_{1x} + c_{1xx} = 0$, thus solving for $c_1(x)$ we have $c_1(x) = k_1 x + k_2 x^3$, then, substituting in (58) the statement in the theorem is obtained.

7 Conservation laws

In this section we shall establish some conservation laws for the equation (1) using the conservation theorem of N. Ibragimov in.[12] Since the Eq. (1) is of second order, the formal Lagrangian contains derivatives up to order two. Thus, the general formula in[12] for the component of the conserved vector is reduced to

$$C^x = W^j\left[\frac{\partial \mathfrak{L}}{\partial_{y_x}} - D_x\left(\frac{\partial \mathfrak{L}}{\partial_{y_{xx}}}\right)\right] + D_x\left[W^j\right]\left[\frac{\partial \mathfrak{L}}{\partial_{y_{xx}}}\right], \quad (59)$$

where

$$W^j = \eta^j - \xi^j y_x$$

$j = 1,...,5$ the formal Lagrangian (53)

$$\mathfrak{L} := v(y_{xx} + y^{-1}y_x^2 + 3x^{-1}y_x)$$

and η^j, ξ^j are the infinitesimals of a Lie point symmetry admitted by Eq. (1), given in (4). Using (1), (4) and (56) into (59) we obtain the following conservation vectors for each symmetry stated in (4).

$$C_1^x = v(xy^{-1}y_x^2 - y_x) + v_x(xy_x),$$
$$C_2^x = v(-x^3y^{-1}y_x^2 - 3x^2y_x) + v_x(x^3y_x),$$
$$C_3^x = v(-6y^2y_x - 3xyy_x^2 - 3x^{-1}y^3) - v_x(y^3 + xy^2y_x), \quad (60)$$
$$C_4^x = v(-3x^3yy_x^2 - 3x^2y_xy^2)) + v_x(x^3y^2y_x),$$
$$C_5^x = v(3y_x + 3x^{(-1)}y) - v_x(y),$$

where $v = y(k_1x + k_2x^3)$ and $v_x = y(k_1 + 3k_2x^2)$.

Classification of Lie algebra

Generically a Finite dimensional Lie algebra in a field of characteristic 0 is classify by the Levi's theorem, which claims that any finite dimensional Lie algebra can be write as a semidirect product of a semisimple Lie algebra and a Solvable Lie algebra, the solvable Lie algebra is the Radical of that Algebra. In other words, there exist two important classes of Lie algebras, The solvable and the semisimple. In each classes mention above there are some particular classes that have other classification, for example in the solvable one, we have the nilpotent Lie algebra.

According the Lie group symmetry of generators given in the table 1. We have a five dimensional Lie algebra. First of all, we remember some basic criteria to classify a Lie algebra, In the case of Solvable and semisimple Lie algebra. We will denote $K(.,.)$ the Cartan-Killing form. The next propositions can be found in.[3]

Proposition 4 *(Cartan's theorem) A Lie algebra is semisimple if and only if its Killing form is nondegenerate.*

Proposition 5 *A Lie subalgebra $\mathfrak{g}$ is solvable if and only if $K(X,Y) = 0$ for all $X \in [\mathfrak{g},\mathfrak{g}]$ and $Y \in \mathfrak{g}$. Other way to write that is $K(\mathfrak{g},[\mathfrak{g},\mathfrak{g}]) = 0$.*

We also need the next statements to make the classification.

Definition 4 *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over an arbitrary field k .Choose a basis e_j, $1 \le i \le n$, , in $\mathfrak{g}$ where $n = \dim \mathfrak{g}$ and set $[e_i, e_j] = C_{ij}^k e_k$. Then the coefficients C_{ij}^k are called structure constants.*

Proposition 6 *Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be two Lie algebras of dimension n . Suppose each has a basis with respect to which the structure constant are the same. Then $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are isomorphic.*

Let $\mathfrak{g}$ the Lie algebra related to the symmetry group of infinitesimal generators of the equation (1) as stated by the table of the commutators, it is enough to consider the next relations:

$[\Pi_1, \Pi_2] = 2\Pi_2$, $[\Pi_1, \Pi_4] = 2\Pi_4$, $[\Pi_2, \Pi_3] = -2\Pi_4$, $[\Pi_3, \Pi_5] = -2\Pi_3$, $[\Pi_3, \Pi_5] = -2\Pi_4$. Using that we calculate Cartan-Killing form K as follows.

$$K = \begin{bmatrix} 80004 \\ 00000 \\ 00000 \\ 00000 \\ 40008 \end{bmatrix},$$

which the determinant vanishes, and thus by Cartan criterion it is not semisimple, (see Proposition 4). Since a nilpotent Lie álgera has a Cartan-Killing form that is identically zero, we conclude, using the counter-reciprocal of the last claim, that the Lie algebra $\mathfrak{g}$ is not nilpotent. We verify that the Lie algebra is solvable using the Cartan criteria to solvability, (Proposition 5), and then we have a solvable nonnilpotent Lie algebra. The Nilradical of the Lie algebra $\mathfrak{g}$ is generated by Π_2, Π_3, Π_4, that is, we have a Solvable Lie algebra with three dimensional Nilradical. Let m the dimension of the Nilradical M of a Solvable Lie agebra, In this case, in fith dimensional Lie algebra we have $3 \le m \le 5$. Mubarakzyanov in[13] classified the 5-dimensional solvable nonlilpotent Lie algebras, in particular the solvable nonnilpotent Lie algebra with three dimensional Nilradical, this Nilradical is isomorphic to $\mathfrak{h}_3$ the Heisenberg Lie algebra. Tnen, by the Proposition 6, and consequently we establish a isomorphism of Lie algebras with $\mathfrak{g}$ and the Lie algebra $\mathfrak{g}_{5,34}$. In summery we have the next proposition.

Proposition 7 *The 5-dimensional Lie algebra $\mathfrak{g}$ related to the symmetry group of the equation (1) is a solvable nonnilpotent Lie algebra with three dimensional Nilridical. Besides that Lie algebra is isomorphic with $\mathfrak{g}_{5,34}$ in the Mubarakzyanov's classification.*

Conclusion

Using the Lie symmetry group (see Proposition 1), we calculated the optimal algebra (see Proposition 2). Making use of these operators, it was possible to characterize all invariant solutions as it was shown in Table 3.

It has been shown the variational symmetries for (1), as it was shown in (43) with its corresponding conservation laws (44) and all this was using Noether's theorem, but non-trivial conservation laws were also calculated using the Ibragimov's method as it was shown in (60) using the nonlinearly self-adjoint of the equation (1) as announced in the Proposition 3.

The Lie algebra associated to the equation (1) is a solvable nonnilpotent Lie algebra with three dimensional Nilridical. Besides that Lie algebra is isomorphich with $\mathfrak{g}_{5,34}$ in the Mubarakzyanov's classification (see Proposition 7). Therefore, the goal initially proposed was achieved. For future works, An line of work would be to use the Lie symmetry group to calculate the λ-symmetries of (1), and, thus, explore more conservation laws for (1) and the equivalence group theory could be also considered to obtain preliminary classifications associated to a complete classification of (1).

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Declaration interests

The authors declare that they have no conflict of interest.

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