

# Lie algebra classification, conservation laws and invariant solutions for modification of the generalization of the Emden–Fowler equation

## Abstract

We obtain the optimal system's generating operators associated to a modification of the generalization of the Emden–Fowler Equation. Using those operators we characterize all invariant solutions associated to a generalized. Moreover, we present the variational symmetries and the corresponding conservation laws, using Noether's theorem and Ibragimov's method. Finally, we classify the Lie algebra associated to the given equation.

**Keywords:** Invariant solutions, Lie symmetry group, Optimal system, Lie algebra classification, Variational symmetries, Conservation laws, Ibragimov's method, Noether's theorem.

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## Introduction

In,<sup>1</sup> Ibragimov presents the following equation

$$y_x x = -y^{(-1)} y_x^2 - 3x^{(-1)} y_x, \quad (1)$$

with its respective solution

$$y(x) = \pm \sqrt{C_1 - \frac{C_2}{x^2}}, \text{ where } C_1, C_2 \text{ are constants.} \quad (2)$$

This solution is obtained using the integrating factor method. In,<sup>2</sup> Muriel and Romero, calculate the  $\lambda$ -Symmetries associated to integrating factors of (1). In,<sup>3</sup> Polyanin and Zaitsev present a solution of (1) of the form

$$y(x) = C_2 \exp(C_1 |x|^4), \text{ where } C_1, C_2 \text{ are constants.} \quad (3)$$

The purpose of this work is: *i*) to calculate the Lie symmetry group, *ii*) to present the optimal algebra (optimal system) for (1), *iii*) making use of all elements of the optimal algebra, to propose invariant solutions for (1), then *iv*) to construct the Lagrangian with which we could determine the variational symmetries using Noether's theorem, and thus to present conservation laws associated, and *iv*) also using Ibragimov's method build some non-trivial conservation laws, and finally *v*) to classify the Lie algebra associated to (1), corresponding to the symmetry group. we note that equation (1) can be considered as a modification of the generalization of the Emden–Fowler Equation.

## Continuous group of Lie symmetries

In this section we study the Lie symmetry group for (1). The main result of this section can be presented as follows:

**Proposition 1** *The Lie symmetry group for the equation (1) is generated by the following vector fields:*

$$\Pi_1 = x \frac{\partial}{\partial x}, \Pi_2 = x^3 \frac{\partial}{\partial x}, \Pi_3 = xy^2 \frac{\partial}{\partial x} + (-y^3) \frac{\partial}{\partial y}, \Pi_4 = x^3 y^2 \frac{\partial}{\partial x} \quad (4)$$

$$\Pi_5 = y \frac{\partial}{\partial y}$$

*Proof.* A general form of the one-parameter Lie group admitted by (1) is given by

$$x \rightarrow x + \epsilon \xi(x, y) + O(\epsilon^2) \quad \text{and} \quad y \rightarrow y + \eta(x, y) + O(\epsilon^2)$$

where  $\epsilon$  is the group parameter. The vector field associated with the group of transformations shown above can be written as

$$\Gamma = \xi(x, y) \frac{\partial}{\partial x} + \eta(x, y) \frac{\partial}{\partial y}, \text{ where } \xi, \eta \text{ are differentiable functions}$$

in  $\mathbb{R}^2$ . Applying its second prolongation

$$\Gamma^{(2)} = \Gamma + \eta_{[x]} \frac{\partial}{\partial y_x} + \eta_{[xx]} \frac{\partial}{\partial y_{xx}}, \quad (5)$$

to eq.(1), we must find the infinitesimals  $\xi, \eta$  satisfying the symmetry condition

$$\xi(-3x^{-2} y_x) + \eta(-y_x^2 y^{-2}) + \eta_{[x]}(2y^{-1} y_x + 3x^{-1}) + \eta_{[xx]} = 0, \quad (6)$$

associated with (1). Here  $\eta_{[x]}, \eta_{[xx]}$  are the coefficients in  $\Gamma^{(2)}$  given by:

$$\begin{aligned} \eta_{[x]} &= D_x[\eta] - (D_x[\xi])y_x = \eta_x + (\eta_y - \xi_x)y_x - \xi_y y_x^2. \\ \eta_{[xx]} &= D_x[\eta_{[x]}] - (D_x[\xi])y_{xx}, \\ &= \eta_{xx} + (2\eta_{xy} - \xi_{xx})y_x + (\eta_{yy} - 2\xi_{xy})y_x^2 - \xi_y y_x y_{xx}^3 \\ &\quad + (\eta_y - 2\xi_x)y_{xx} - 3\xi_y y_x y_{xx}. \end{aligned} \quad (7)$$

Being  $D_x$  is the total derivative operator:

$D_x = \partial_x + y_x \partial_y + y_{xx} \partial_{y_{xx}} + \dots$ . Replacing (7) into (6) and using (1) we obtain:

$$\begin{aligned} &(5y^{(-1)} \xi_y - \xi_{yy})y_x^3 + (y^{(-1)} \eta_y - \eta y^{(-2)} - 2\xi_{xy} + 6x^{-1} \xi_y + \eta_y y)y_x^2 \\ &+ (-3x^{-2} \xi + 2y^{-1} \eta_x + 3x^{-1} \xi_x + 2\eta_{xy} - \xi_{xx})y_x + (\eta_{xx} + 3x^{-1} \eta_x) = 0. \end{aligned}$$

From (8), canceling the coefficients of the monomials variables in derivatives  $1, y_x^3, y_x^2$  and  $y_x$  we obtain the determining equations for the symmetry group of (1), with  $x, y \neq 0$ . That is:

$$5\xi_y - y\xi_{yy} = 0 \quad (8a)$$

$$xy\eta_y - x\eta - 2xy^2\xi_{xy} + 6y^2\xi_y + xy^2\eta_{yy} = 0, \quad (8b)$$

$$-3y\xi + 2x^2\eta_x + 3xy\xi_x + 2x^2y\eta_{xy} - x^2y\xi_{xxx} = 0. \quad (8c)$$

$$x\eta_{xx} + 3\xi_x = 0. \quad (8d)$$

Solving the system of equations (8a)–(8d) for  $\xi$  and  $\eta$  we get

$$\xi = c_1x + c_2x^3 + c_3xy^2 + c_4x^3y^2,$$

$$\eta = -c_3y^3 + c_5y.$$

Thus, the infinitesimal generators of the group of symmetries of (1) are the operators  $\Pi_1$ – $\Pi_5$  described in the statement of the Proposition 1; thus having the proposed result.

## Optimal algebra

Taking into account,<sup>1,4–6</sup> we present in this section the optimal algebra associated to the symmetry group of (1), that shows a systematic way to classify the invariant solutions. To obtain the optimal algebra, we should first calculate the corresponding commutator table, which can be obtained from the operator

$$[\Pi_\alpha, \Pi_\beta] = \Pi_\alpha \Pi_\beta - \Pi_\beta \Pi_\alpha = \sum_{i=1}^n (\Pi_\alpha(\xi_\beta^i) - \Pi_\beta(\xi_\alpha^i)) \frac{\partial}{\partial x^i}, \quad (9)$$

where  $i = 1, 2$ , with  $\alpha, \beta = 1, \dots, 5$  and  $\xi_\alpha^i, \xi_\beta^i$  are the corresponding coefficients of the infinitesimal operators  $\Pi_\alpha, \Pi_\beta$ . After applying the operator (12) to the symmetry group of (1), we obtain the operators that are shown in the following table !

**Table 1** Commutators table associated to the symmetry group of (1)

	$\Pi_1$	$\Pi_2$	$\Pi_3$	$\Pi_4$	$\Pi_5$
$\Pi_1$	0	$2\Pi_2$	0	$2\Pi_4$	0
$\Pi_2$	$-2\Pi_1$	0	$-2\Pi_4$	0	0
$\Pi_3$	0	$2\Pi_4$	0	0	$-2\Pi_3$
$\Pi_4$	$-2\Pi_1$	0	0	0	$-2\Pi_4$
$\Pi_5$	0	0	$2\Pi_3$	$2\Pi_4$	0

Now, the next thing is to calculate the adjoint action representation of the symmetries of (1) and to do that, we use Table 1 and the operator.

$$Ad(\exp(\lambda\Pi))H = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} (ad(\Pi))^n G \text{ for the symmetries}$$

$\Pi$  and  $G$

Making use of this operator, we can construct the Table 2, which shows the adjoint representation for each  $\Pi_i$  !

**Table 2** Adjoint representation of the symmetry group of (1)

$\text{adj}[\cdot]$	$\Pi_1$	$\Pi_2$	$\Pi_3$	$\Pi_4$	$\Pi_5$
$\Pi_1$	$\Pi_1$	$e^{-2\lambda}\Pi_2$	$\Pi_3$	$e^{-2\lambda}\Pi_4$	$\Pi_5$
$\Pi_2$	$\Pi_1 + 2\lambda\Pi_2$	$\Pi_2$	$\Pi_3 + 2\lambda\Pi_4$	$\Pi_4$	$\Pi_5$
$\Pi_3$	$\Pi_1$	$\Pi_2 - 2\lambda\Pi_4$	$\Pi_3$	$\Pi_4$	$\Pi_5 + 2\lambda\Pi_3$
$\Pi_4$	$\Pi_1 + 2\lambda\Pi_4$	$\Pi_2$	$\Pi_3$	$\Pi_4$	$\Pi_5 + 2\lambda\Pi_4$
$\Pi_5$	$\Pi_1$	$\Pi_2$	$e^{-2\lambda}\Pi_3$	$e^{-2\lambda}\Pi_4$	$\Pi_5$

**Proposition 2** The optimal algebra associated to the equation (1) is given by the vector fields

$$\Pi_4, a_2\Pi_2, a_3\Pi_3, a_1\Pi_1 + a_3\Pi_3, a_2\Pi_2 + \Pi_3\Pi_3 + b_5\Pi_4, a_1\Pi_1 + b_6\Pi_4, \Pi_2 + b_7\Pi_4,$$

$$b_3\Pi_3 + \Pi_5, a_2\Pi_2 + \Pi_5, a_1\Pi_1 + \Pi_5, -2\Pi_1 + b_1\Pi_4 + \Pi_5, a_2\Pi_2 + a_3\Pi_3 + b_4\Pi_4,$$

$$\Pi_1 + b_8\Pi_2 + b_9\Pi_4, a_2\Pi_2 + \frac{a_4}{a_2}\Pi_3 + b_2\Pi_4 + \Pi_5.$$

*Proof.* To calculate the optimal algebra system, we start with the generators of symmetries (4) and a generic nonzero vector. Let

$$G = a_1\Pi_1 + a_2\Pi_2 + a_3\Pi_3 + a_4\Pi_4 + a_5\Pi_5. \quad (13)$$

The objective is to simplify as many coefficients  $a_i$  as possible, through maps adjoint to  $G$ , using Table (2).

1. Assuming  $a_5 = 1$  in (10) we have that  $G = a_1\Pi_1 + a_2\Pi_2 + a_3\Pi_3 + a_4\Pi_4 + \Pi_5$ . Applying the adjoint operator to  $\Pi_1, G$  and  $(\Pi_5, G)$  we don't have any reducción, on the other hand applying the adjoint operator to  $(\Pi_2, G)$  we get

$$G_1 = Ad(\exp(\lambda_1\Pi_2))G = a_1\Pi_1 + (a_2 + 2a_1\lambda_1)\Pi_2 + a_3\Pi_3 + (a_4 + 2a_3\lambda_1)\Pi_4 + \Pi_5 \quad (11)$$

**1.1.1**) Case  $a_1 \neq 0$ . Using  $\lambda_1 = \frac{-a_2}{2a_1}$ , with  $a_1 \neq 0$ , in (11),  $\Pi_2$

is eliminated, therefore  $G_1 = a_1\Pi_1 + a_3\Pi_3 + b_1\Pi_4 + \Pi_5$ , where  $b_1 = a_4 + \frac{a_3}{a_1}$ . Now, applying the adjoint operator to  $(\Pi_3, G_1)$ , we get

$$G_2 = Ad(\exp(\lambda_2\Pi_3))G_1 = a_1\Pi_1 + (a_3 + 2\lambda_2)\Pi_3 + b_1\Pi_4 + \Pi_5.$$

Using  $\lambda_2 = \frac{-a_3}{2}$ , is eliminated  $\Pi_3$ , then  $G_2 = a_1\Pi_1 + b_1\Pi_4 + \Pi_5$ . Applying the adjoint operator to  $(\Pi_4, G_2)$ , we get  $G_3 = Ad(\exp \lambda_3\Pi_4))G_2 = a_1\Pi_1 + (b_1 + 2\lambda_3(a_1 + 2))\Pi_4 + \Pi_5$ . (12)

**1.1.1.A**) Case  $a_1 + 2 \neq 0$ . Using  $\lambda_3 = \frac{1}{2(a_1 + 2)}$ , with  $a_1 + 2 \neq 0$ , in

(12),  $\Pi_4$  is eliminated, therefore  $G_3 = a_1\Pi_1 + \Pi_5$ . Then, we have the first element of the optimal system.

$$G_3 = a_1\Pi_1 + \Pi_5, \text{ with } a_1 \neq 0 \text{ and } a_1 + 2 \neq 0. \quad (13)$$

This is how the first reduction of the generic element (10) ends.

**1.1.1.B**) Case  $a_1 + 2 \neq 0$ . We get  $G_3 = -2\Pi_1 + b_1\Pi_4 + \Pi_5$ . Then, we have other element of the optimal system.

$$G_3 = -2\Pi_1 + b_1\Pi_4 + \Pi_5. \quad (14)$$

This is how other reduction of the generic element (13) ends.

**1.2.1**) Case  $a_1 = 0$ . We get  $G_1 = a_2\Pi_2 + a_3\Pi_3 + (a_4 + 2a_3\lambda_1)\Pi_5$ .

**1.2.1.A**) Case  $a_3 \neq 0$ . Using  $\lambda_1 = \frac{-a_4}{2a_3}$ , with  $a_3 \neq 0$ , is eliminated  $\Pi_4$ , then  $G_1 = a_2\Pi_2 + a_3\Pi_3 + \Pi_5$ . Applying the adjoint operator to  $(\Pi_3, G_1)$ , we get

$$G_4 = Ad(\exp \lambda_4\Pi_3))G_1 = a_2\Pi_2 + (a_3 + 2\lambda_4)\Pi_3 - 2\lambda_4\Pi_4 + \Pi_5. \quad (15)$$

Using  $\lambda_4 = \frac{-a_3}{2}$ , is eliminated  $\Pi_3$ , then  $G_4 = a_2\Pi_2 + a_3\Pi_4 + \Pi_5$ .

Now applying the adjoint operator to  $(\Pi_4, G_4)$ , we have

$$G_5 = Ad(\exp \lambda_5\Pi_4))G_4 = a_2\Pi_2 + (a_3 + 2\lambda_5)\Pi_4 + \Pi_5. \quad (16)$$

Using  $\lambda_5 = \frac{-a_3}{2}$ , is eliminated  $\Pi_4$ , then we have other element of the optimal system.

$$G_5 = a_2\Pi_2 + \Pi_5. \quad (17)$$

This is how other reduction of the generic element (10) ends.

**1.2.B**) **Case  $a_3 = 0$** . We get  $G_1 = a_2\Pi_2 + a_4\Pi_4 + \Pi_5$ . Now applying the adjoint operator to  $(\Pi_3, G_1)$ , we have

$$G_6 = Ad(\exp(\lambda_6\Pi_3))G_1 = a_2\Pi_2 + 2\lambda_6\Pi_3 + (a_4 - 2a_2\lambda_6)\Pi_4 + \Pi_5. \quad (18)$$

**1.2.B.1)** **Case  $a_2 \neq 0$** . Using  $\lambda_6 = \frac{a_4}{2a_2}$ , with  $a_2 \neq 0$ , is eliminated  $\Pi_4$ , then  $G_6 = a_2\Pi_2 + \frac{a_4}{a_2}\Pi_3 + \Pi_5$ . Now applying the adjoint operator to  $(\Pi_4, G_4)$ , we get

$$G_7 = Ad(\exp(\lambda_7\Pi_4))G_6 = a_2\Pi_2 + \frac{a_4}{a_2}\Pi_3 + 2\lambda_7\Pi_4 + \Pi_5. \quad (19)$$

It's clear that we don't have any reduction, then using  $\lambda_7 = \frac{b_2}{2}$ , then we have other element of the optimal system.

$$G_7 = a_2\Pi_2 + \frac{a_4}{a_2}\Pi_3 + b_2\Pi_4 + \Pi_5. \quad (20)$$

This is how other reduction of the generic element (10) ends.

**1.2.B.2)** **Case  $a_2 = 0$** . We get  $G_6 = 2\lambda_6\Pi_3 + a_4\Pi_4 + \Pi_5$ . It is clear that we don't have any reduction, then using  $\lambda_6 = \frac{b_3}{2}$ , we have  $G_6 = b_3\Pi_3 + a_4\Pi_4 + \Pi_5$ . Now applying the adjoint operator to  $(\Pi_4, G_6)$ , we have

$$G_8 = Ad(\exp(\lambda_8\Pi_4))G_6 = b_3\Pi_3 + (a_4 + 2\lambda_8)\Pi_4 + \Pi_5. \quad (21)$$

Using  $\lambda_8 = \frac{-a_4}{2}$ , is eliminated  $\Pi_4$ , then we have other element of the optimal system.

$$G_8 = b_3\Pi_3 + \Pi_5. \quad (22)$$

This is how other reduction of the generic element (10) ends.

2. Assuming  $a_5 = 0$  and  $a_4 = 1$  in (10), we have that  $G = a_1\Pi_1 + a_2\Pi_2 + a_3\Pi_3 + \Pi_4$ . Applying the adjoint operator to  $(\Pi_1, G)$  and  $(\Pi_5, G)$  we don't have any reduction, on the other hand applying the adjoint operator to  $(\Pi_2, G)$  we get

$$G_9 = Ad(\exp(\lambda_9\Pi_2))G = a_1\Pi_1 + (a_2 + 2a_1\lambda_9)\Pi_2 + a_3\Pi_3 + (1 + 2a_3\lambda_9)\Pi_4. \quad (23)$$

**2.1)** **Case  $a_1 \neq 0$** . Using  $\lambda_9 = \frac{-a_2}{2a_1}$ , with  $a_1 \neq 0$ , in (26),  $\Pi_2$  is eliminated, therefore  $G_9 = a_1\Pi_1 + a_3\Pi_3 + b_4\Pi_4$ , where  $b_3 = 1 - \frac{a_3a_2}{a_1}$ . Now, applying the adjoint operator to  $(\Pi_3, G_9)$ , we don't have any reduction, after applying the adjoint operator to  $(\Pi_4, G_9)$ , we get  $G_{10} = Ad(\exp(\lambda_{10}\Pi_4))G_9 = a_1\Pi_1 + a_3\Pi_3 + (b_3 + 2a_1\lambda_{10})\Pi_4$ . How  $a_1 \neq 0$ , we can use  $\lambda_{10} = \frac{-b_3}{2a_1}$ , is eliminated  $\Pi_4$ , thus we have other element of the optimal system.

$$G_{10} = a_1\Pi_1 + a_3\Pi_3. \quad (24)$$

This is how other reduction of the generic element (10) ends.

**2.2)** **Case  $a_1 = 0$** . We get  $G_9 = a_2\Pi_2 + a_3\Pi_3 + (1 + 2a_3\lambda_9)\Pi_4$ .

**2.2.A)** **Case  $a_3 \neq 0$** . Using  $\lambda_9 = \frac{-1}{2a_3}$ , with  $a_3 \neq 0$ ,  $\Pi_4$  is eliminated, therefore  $G_9 = a_2\Pi_2 + a_3\Pi_3$ . Now, applying the adjoint operator to  $(\Pi_3, G_9)$ , we get  $G_{11} = Ad(\exp(\lambda_{11}\Pi_3))G_9 = a_2\Pi_2 + a_3\Pi_3 - 2a_1\lambda_{11}\Pi_4$ .

**2.2.A.1)** **Case  $a_2 \neq 0$** . It's clear that we don't have any reduction, using  $\lambda_{11} = \frac{-b_4}{2}$ , with  $a_2 \neq 0$ , we get  $G_{11} = a_2\Pi_2 + a_3\Pi_3 + b_4\Pi_4$ . Now, applying the adjoint operator to  $(\Pi_4, G_{11})$ , we don't have any reduction, thus we have other element of the optimal system.

$$G_{11} = a_2\Pi_2 + a_3\Pi_3 + b_4\Pi_4. \quad (25)$$

This is how other reduction of the generic element (10) ends.

**2.2.A.2)** **Case  $a_2 = 0$** . We get  $G_{11} = a_3\Pi_3$ . Now, applying the adjoint operator to  $(\Pi_4, G_{11})$ , we don't have any reduction, thus we have other element of the optimal system.

$$G_{11} = a_3\Pi_3. \quad (26)$$

This is how other reduction of the generic element (10) ends.

**2.2.B)** **Case  $a_3 = 0$** . We get  $G_9 = a_2\Pi_2 + \Pi_4$ . Now, applying the adjoint operator to  $(\Pi_3, G_9)$ , we have  $G_{12} = Ad(\exp(\lambda_{12}\Pi_3))G_9 = a_2\Pi_2 + (1 - 2a_2\lambda_{12})\Pi_4$ .

**2.2.B.1)** **Case  $a_2 \neq 0$** . Using  $\lambda_{12} = \frac{1}{2a_2}$ , with  $a_2 \neq 0$ , is eliminated  $\Pi_4$ , then  $G_{12} = a_2\Pi_2$ . Now, applying the adjoint operator to  $(\Pi_4, G_{12})$ , we don't have any reduction, thus we have other element of the optimal system.

$$G_{12} = a_2\Pi_2. \quad (27)$$

This is how other reduction of the generic element (10) ends.

**2.2.B.2)** **Case  $a_2 = 0$** . We get  $G_{12} = \Pi_4$ . Now, applying the adjoint operator to  $(\Pi_4, G_{12})$ , we don't have any reduction, thus we have other element of the optimal system.

$$G_{12} = \Pi_4. \quad (28)$$

This is how other reduction of the generic element (10) ends.

3. Assuming  $a_5 = a_4 = 0$  and  $a_3 = 1$  in (10), we have that  $G = a_1\Pi_1 + a_2\Pi_2 + \Pi_3$ . Applying the adjoint operator to  $(\Pi_1, G)$  and  $(\Pi_5, G)$  we don't have any reduction, on the other hand applying the adjoint operator to  $(\Pi_2, G)$  we get

$$G_{13} = Ad(\exp(\lambda_{13}\Pi_2))G = a_1\Pi_1 + (a_2 + 2a_1\lambda_{13})\Pi_2 + \Pi_3 + 2\lambda_{13}\Pi_4. \quad (29)$$

**3.1)** **Case  $a_1 \neq 0$** . Using  $\lambda_{13} = \frac{-a_2}{2a_1}$ , with  $a_1 \neq 0$ , in (29),  $\Pi_2$  is eliminated, therefore  $G_{13} = a_1\Pi_1 + \Pi_3 + b_3\Pi_4$ , where  $b_3 = \frac{a_2}{a_1}$ . Now, applying the adjoint operator to  $(\Pi_3, G_{13})$ , we don't have any

reduction, after applying the adjoint operator to  $(\Pi_4, G_9)$ , we get

$$G_{14} = Ad(\exp(\lambda_{14}\Pi_4))G_{13} = a_1\Pi_1 + \Pi_3 + (b_3 + 2a_1\lambda_{13})\Pi_4. \text{ As } a_1 \neq 0,$$

we can use  $\lambda_{13} = \frac{-b_3}{2a_1}$ , is eliminated  $\Pi_4$ , then we have other element of the optimal system.

$$G_{14} = a_1\Pi_1 + \Pi_3. \quad (30)$$

This is how other reduction of the generic element (10) ends.

**3.2) Case  $a_1 = 0$ .** We get  $G_{13} = a_2\Pi_2 + \Pi_3 + 2\lambda_{13}\Pi_4$ ,

using  $\lambda_{13} = \frac{b_5}{2a_2}$ , then  $G_{13} = a_2\Pi_2 + \Pi_3 + b_5\Pi_4$ . Now, applying the adjoint operator to  $(\Pi_3, G_{13})$ , we get

$$G_{14} = Ad(\exp(\lambda_{14}\Pi_3))G_{13} = a_2\Pi_2 + \Pi_3 + (b_5 - 2a_2\lambda_{14})\Pi_4.$$

**3.2.A) Case  $a_2 \neq 0$ .** Using  $\lambda_{14} = \frac{b_5}{2a_2}$ , with  $a_2 \neq 0$ , is eliminated

$\Pi_4$ , then  $G_{14} = a_2\Pi_2 + \Pi_3$ . Now applying the adjoint operator to  $(\Pi_4, G_{14})$  we don't have any reduction, then we have other element of the optimal system.

$$G_{14} = a_2\Pi_2 + \Pi_3. \quad (31)$$

This is how other reduction of the generic element (13) ends.

**3.2.B) Case  $a_2 = 0$ .** We get  $G_{14} = \Pi_3 + b_5\Pi_4$ . Now applying the

adjoint operator to  $(\Pi_4, G_{14})$  we don't have any reduction, then we have other element of the optimal system.

$$G_{14} = \Pi_3 + b_5\Pi_4. \quad (32)$$

This is how other reduction of the generic element (10) ends.

4. Assuming  $a_3 = a_4 = a_3 = 0$  and  $a_2 = 1$  in (10), we have that

$G = a_1\Pi_1 + \Pi_2$ . Applying the adjoint operator to  $(\Pi_1, G)$  and  $(\Pi_5, G)$  we don't have any reduction, on the other hand applying the adjoint operator to  $(\Pi_4, G)$  we get

$$G_{15} = Ad(\exp(\lambda_{15}\Pi_2))G = a_1\Pi_1 + (1 + 2a_1\lambda_{15})\Pi_2. \quad (33)$$

**4.1) Case  $a_1 \neq 0$ .** Using  $\lambda_{15} = \frac{-1}{2a_1}$ , with  $a_1 \neq 0$ , is eliminated  $\Pi_2$ , then  $G_{15} = a_1\Pi_1$ . Now applying the adjoint operator to  $(\Pi_3, G_{15})$  we don't have any reduction, on the other hand applying the adjoint operator to  $(\Pi_4, G_{15})$  we get  $G_{16} = Ad(\exp(\lambda_{16}\Pi_4))G_{15} = a_1\Pi_1 + 2a_1\lambda_{16}\Pi_4$ . It is clear that we don't have any reduction, then using  $\lambda_{16} = \frac{b_6}{2a_1}$ , with  $a_1 \neq 0$ , we have other element of the optimal system.

$$G_{16} = a_1\Pi_1 + b_6\Pi_4. \quad (34)$$

This is how other reduction of the generic element (10) ends.

**4.2) Case  $a_1 = 0$ .** We get  $G_{15} = \Pi_2$ . Now applying the adjoint operator to  $(\Pi_3, G_{15})$  we get  $G_{17} = Ad(\exp(\lambda_{17}\Pi_3))G_{15} = \Pi_2 - 2\lambda_{17}\Pi_4$ .

It is clear that we don't have any reduction, then using  $\lambda_{17} = \frac{-b_7}{2}$ , then

$G_{17} = \Pi_2 + b_7\Pi_4$ . Now applying the adjoint operator to  $(\Pi_4, G_{17})$ , we don't have any reduction, after we have other element of the optimal system.

$$G_{17} = \Pi_2 + b_7\Pi_4. \quad (35)$$

This is how other reduction of the generic element (10) ends.

5. Assuming  $a_5 = a_4 = a_3 = a_2 = 0$  and  $a_1 = 1$  in (10), we have that  $G = \Pi_1$ . Applying the adjoint operator to  $(\Pi_1, G)$ ,  $(\Pi_3, G)$  and  $(\Pi_5, G)$  we don't have any reduction, on the other hand applying the adjoint operator to  $(\Pi_2, G)$  we get

$$G_{18} = Ad(\exp(\lambda_{18}\Pi_2))G = \Pi_1 - 2\lambda_{18}\Pi_2. \quad (36)$$

It's clear that we don't have any reduction, then using  $\lambda_{18} = \frac{b_8}{2}$ , we get  $G_{18} = \Pi_1 + b_8\Pi_2$ . Now applying the adjoint operator to  $(\Pi_4, G_{18})$ , we have

$$G_{19} = \Pi_1 + b_8\Pi_2 + 2\lambda_{19}\Pi_2. \quad (37)$$

It's clear that we don't have any reduction, then using  $\lambda_{19} = \frac{b_9}{2}$ , we have other element of the optimal system.

$$G_{19} = \Pi_1 + b_8\Pi_2 + b_8\Pi_4. \quad (38)$$

This is how other reduction of the generic element (10) ends.

## 4 Invariant solutions by the generators of the optimal algebra

In this section, we characterize the invariant solutions taking into account all operators that generate the optimal algebra presented in Proposition 2. For this purpose, we use the method of invariant curve condition<sup>5</sup> (presented in section 4.3), which is given by the following equation

$$Q(x, y, y_x) = \eta = y_x \xi = 0. \quad (39)$$

Using the element  $\Pi_4$  from Proposition 2, under the condition (42), we obtain that  $Q = \eta_4 - y_x \xi_4 = 0$  which implies  $(0) - y_x(x^3 y^2) = 0$ . After, we get  $y(x) = c$ , where  $c$  is a constant, which is an invariant solution for (1), using an analogous procedure with all of the elements of the optimal algebra (Proposition 2), we obtain both implicit and explicit invariant solutions that are shown in the Table 3, with  $c$  being a constant.

## Variational symmetries and conserved quantities

In this section, we present the variational symmetries of (1) and we are going to use them to define conservation laws via Noether's theorem.<sup>7</sup> First of all, we are going to determine the Lagrangian using the Jacobi Last Multiplier method, presented by Nucci in,<sup>8</sup> and for this reason, we are urged to calculate the inverse of the determinant  $\Delta$ ,

$$\Delta = \begin{vmatrix} x & y_x & y_{xx} \\ \Pi_{1,x} & \Pi_{1,y} & \Pi_1^{(1)} \\ \Pi_{2,x} & \Pi_{2,y} & \Pi_2^{(1)} \end{vmatrix} = \begin{vmatrix} x & y_x & y_{xx} \\ x & 0 & -y_x \\ x^3 & 0 & -3x^2 y_x \end{vmatrix},$$

where  $\Pi_{1,x}, \Pi_{1,y}, \Pi_{2,x}$ , and  $\Pi_{2,y}$  are the components of the symmetries  $\Pi_1, \Pi_2$  shown in the Proposition 4 and  $\Pi_1^{(1)}, \Pi_2^{(1)}$  as its first prolongations. Then we get  $\Delta = 2x^3 y_x$  which implies that  $M = \frac{1}{\Delta} = \frac{x^{-3}}{2y_x}$ . Now, from,<sup>8</sup> we know that  $M$  can also be written as

$M = L_{y_{xx}}$  which means that  $L_{y_{xx}} = \frac{x^{-3}}{2y_x}$ , then integrating twice with respect to  $y_x$  we obtain the Lagrangian

$$L(x, y, y_x) = \frac{x^{-3}}{2} y_x \ln(y_x) - \frac{x^{-3}}{2} y_x + y_x f_1(x, y) + f_2(x, y), \quad (40)$$

where  $f_1, f_2$  are arbitrary functions. From the preceding expression we can consider  $f_1 = f_2 = 0$ . It is possible to find more Lagrangians for (1) by considering other vector fields given in the Proposition 4. We then calculate

$$\xi(x, y) L_x + \xi_x(x, y) L + \eta(x, y) L_y + \eta_{[x]}(x, y) L_{y_x} = D_x [f(x, y)],$$

using (40) and (7). Thus we get

$$\begin{aligned} \xi \left( \frac{-3x^{-4}}{2} y_x \ln(y_x) + \frac{3x^{-4}}{2} y_x \right) + \xi_x \left( \frac{x^{-3}}{2} y_x \ln(y_x) - \frac{x^{-3}}{2} y_x \right) \\ + (\eta_x + (\eta_y - \xi_x) y_x - \xi_y y_x^2) \left( \frac{x^{-3}}{2} \ln(y_x) \right) - f_x - y_x f_y = 0. \end{aligned}$$

From the preceding expression, rearranging and associating terms with respect to  $1, y_x, y_x \ln(y_x), y_x^2 \ln(y_x)$  and  $\ln(y_x)$ , we obtain the following determinant equations

$$\xi_y = \eta_x = f_x = 0, \quad (41a)$$

$$-3\xi + x\eta_y = 0, \quad (41b)$$

$$3\xi - x\xi_x - 2x^4 f_y = 0. \quad (41c)$$

Solving the preceding system for  $\xi, \eta$  and  $f$  we obtain the infinitesimal generators of Noether's symmetries

$$\eta = a_2, \quad \xi = 0, \quad \text{and} \quad f(y) = a_4. \quad (42)$$

with  $a_2$  and  $a_4$  arbitrary constants. Then, the Noether symmetry group or variational symmetries is

$$V_1 = \frac{\partial}{\partial y}, \quad (43)$$

According to,<sup>9</sup> in order to obtain the conserved quantities or conservation laws, we should solve

$$I = (X y_x - Y) L_{y_x} - X L + f,$$

so, using (43), (47) and (48). Therefore, the conserved quantities are given by

$$I_1 = -\frac{x^{-3} \ln(y_x)}{2} + a_4, \quad (44)$$

### Nonlinear self-adjointness

In this section we present the main definitions in the N. Ibragimov's approach to nonlinear self-adjointness of differential equations adopted to our specific case. For further details the interested reader is directed to.<sup>6,10,11</sup>

Consider second order differential equation

$$\mathfrak{F}(x, y, y_{(1)}, y_2, \dots, y_s) = 0, \quad (45)$$

With independent variables  $x$  and a dependent variable  $y$ , where  $y_{(1)}, y_{(2)}, \dots, y_s$  denote the collection of  $1, 2, \dots, s$ -th order derivatives of  $y$ .

**Definition 1** Let  $\mathfrak{F}$  be a differential function and  $v = v(x)$  -the new dependent variable, known as the adjoint variable or nonlocal

variable.<sup>11</sup> The formal Lagrangian for  $\mathfrak{F} = 0$  is the differential function defined by

$$\mathfrak{L} := v \mathfrak{F}. \quad (46)$$

**Definition 2** Let  $\mathfrak{F}$  be a differential function and for the differential equation (45), denoted by  $\mathfrak{F}[y] = 0$ , we define the adjoint differential function to  $\mathfrak{F}$  by

$$\mathfrak{F}^* := \frac{\delta \mathfrak{L}}{\delta y} \quad (47)$$

and the adjoint differential equation by

$$\mathfrak{F}^*[y, v] = 0, \quad (48)$$

where the Euler operator

$$\frac{\delta}{\delta y} = \frac{\partial}{\partial y} + \sum_{m=1}^{\infty} (-1)^m D_{x_1} \dots D_{x_m} \frac{\partial}{\partial y_{x_1 x_2 \dots x_m}} \quad (49)$$

and  $D_{x_i}$  is the total derivative operator with respect to  $x_i$  defined by

$$D_{x_i} = \partial_{x_i} + y_{x_i} \partial_y + y_{x_i x_j} \partial_{y_{x_j}} + \dots + y_{x_i x_{i_1} x_{i_2} \dots x_{i_n}} \partial_{y_{x_{i_1} x_{i_2} \dots x_{i_n}}} \dots$$

**Definition 3** The differential equation (45) is said to be nonlinearly selfadjoint if there exists a substitution

$$v = \phi(x, y) \neq 0 \quad (50)$$

such that

$$\tilde{\mathfrak{F}}^*|_{v=\phi(x,y)} = \lambda \mathfrak{F} \quad (51)$$

for some undetermined coefficient  $\lambda = \lambda(x, y, \dots)$ . If  $v = \phi(y)$  in (50) and (51), the equation (45) is called quasi self-adjoint. If  $v = y$ , we say that the equation (45) is strictly self-adjoint.

Now we shall obtain the adjoint equation to the eq. (1). For this purpose we write (1) in the form (45), where

$$\mathfrak{F} := y_{xx} + y^{-1} y_x^2 + 3x^{-1} y_x = 0. \quad (52)$$

Then the corresponding formal Lagrangian (46) is given by

$$\mathfrak{L} := v(y_{xx} + y^{-1} y_x^2 + 3x^{-1} y_x) = 0 \quad (53)$$

and the Euler operator (49) assumes the following form:

$$\frac{\delta \mathfrak{L}}{\delta y} = \frac{\partial \mathfrak{L}}{\partial y} - D_x \frac{\partial \mathfrak{L}}{\partial y_x} + D_x^2 \frac{\partial \mathfrak{L}}{\partial y_{xx}}. \quad (54)$$

We calculate explicitly the Euler operator (54) applied to  $\mathfrak{L}$  determined by (58). In this way we obtain the adjoint equation (48) to (1):

$$\tilde{\mathfrak{F}}^* = v(y_x^2 y^{-2} + 3x^{-2} - 2y_{xx} y^{-1}) + v_x(-2y_x y^{-1} - 3x^{-1}) + v_{xx} = 0 \quad (55)$$

The main result in this section can be stated as follows.

**Proposition 3** The equation (1) is nonlinearly self-adjoint, with the substitution given by

$$\phi(x, y) = y(k_1 x + k_2 x^3), \quad (56)$$

where  $k_1, k_2$  are arbitrary constants.

*Proof.* Substituting in (55), and then in (52),  $v = \phi(x, y)$  and its respective derivatives, and comparing the corresponding coefficients we get five equations:

$$-\phi_y = \lambda, \quad (57a)$$

$$-y^{-1}\phi + \phi_y = 0, \quad (57b)$$

$$-y^{-1}\phi_x + \phi_{xy} = 0, \quad (57c)$$

$$3x^{-2}\phi - 3x^{-1}\phi_x + \phi_{xx} = 0, \quad (57d)$$

$$y^{-2}\phi - y^{-1}\phi_y + y\phi_{yy} = 0. \quad (57e)$$

We observe that (57c) and (57e) are obtained from (57b) by differentiation with respect to  $x$  and  $y$ . Therefore we have to study only Eqs. (57b) and (57d). Solving for  $\phi$  in (57b) we obtain

$$\phi(x, y) = c_1(x)y, \quad (58)$$

where  $c_1(x)$  is arbitrary function. Using (58) into (57d) we get  $3x^{-2}c_1(c) - 3x^{-1}c_{1x} + c_{1xx} = 0$ , thus solving for  $c_1(x)$  we have

$c_1(x) = k_1x + k_2x^3$ , then, substituting in (58) the statement in the theorem is obtained.

## 7 Conservation laws

In this section we shall establish some conservation laws for the equation (1) using the conservation theorem of N. Ibragimov in.<sup>12</sup> Since the Eq. (1) is of second order, the formal Lagrangian contains derivatives up to order two. Thus, the general formula in<sup>12</sup> for the component of the conserved vector is reduced to

$$C^x = W^j \left[ \frac{\partial \mathcal{L}}{\partial y_x} - D_x \left( \frac{\partial \mathcal{L}}{\partial y_{xx}} \right) \right] + D_x \left[ W^j \right] \left[ \frac{\partial \mathcal{L}}{\partial y_{xx}} \right], \quad (59)$$

where

$$W^j = \eta^j - \xi^j y_x$$

$j = 1, \dots, 5$  the formal Lagrangian (53)

$$\mathcal{L} := v(y_{xx} + y^{-1}y_x^2 + 3x^{-1}y_x)$$

and  $\eta^j, \xi^j$  are the infinitesimals of a Lie point symmetry admitted by Eq. (1), given in (4). Using (1), (4) and (56) into (59) we obtain the following conservation vectors for each symmetry stated in (4).

$$\begin{aligned} C_1^x &= v(xy^{-1}y_x^2 - y_x) + v_x(xy_x), \\ C_2^x &= v(-x^3y^{-1}y_x^2 - 3x^2y_x) + v_x(x^3y_x), \\ C_3^x &= v(-6y^2y_x - 3xyy_x^2 - 3x^{-1}y^3) - v_x(y^3 + xy^2y_x), \\ C_4^x &= v(-3x^3yy_x^2 - 3x^2y_x^2) + v_x(x^3y^2y_x), \\ C_5^x &= v(3y_x + 3x^{(-1)}y) - v_x(y), \end{aligned} \quad (60)$$

where  $v = y(k_1x + k_2x^3)$  and  $v_x = y(k_1 + 3k_2x^2)$ .

## Classification of Lie algebra

Generically a Finite dimensional Lie algebra in a field of characteristic 0 is classify by the Levi's theorem, which claims that any finite dimensional Lie algebra can be write as a semidirect product of a semisimple Lie algebra and a Solvable Lie algebra, the solvable Lie algebra is the Radical of that Algebra. In other words, there exist two important classes of Lie algebras, The solvable and the semisimple. In each classes mention above there are some particular classes that have other classification, for example in the solvable one, we have the nilpotent Lie algebra.

According the Lie group symmetry of generators given in the table 1. We have a five dimensional Lie algebra. First of all, we remember

some basic criteria to classify a Lie algebra, In the case of Solvable and semisimple Lie algebra. We will denote  $K(.,.)$  the Cartan-Killing form. The next propositions can be found in.<sup>3</sup>

**Proposition 4** (Cartan's theorem) A Lie algebra is semisimple if and only if its Killing form is nondegenerate.

**Proposition 5** A Lie subalgebra  $\mathfrak{g}$  is solvable if and only if  $K(X, Y) = 0$  for all  $X \in [\mathfrak{g}, \mathfrak{g}]$  and  $Y \in \mathfrak{g}$ . Other way to write that is  $K(\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]) = 0$ .

We also need the next statements to make the classification.

**Definition 4** Let  $\mathfrak{g}$  be a finite-dimensional Lie algebra over an arbitrary field  $k$ . Choose a basis  $e_j$ ,  $1 \leq i \leq n$ , in  $\mathfrak{g}$  where  $n = \dim \mathfrak{g}$  and set  $[e_i, e_j] = C_{ij}^k e_k$ . Then the coefficients  $C_{ij}^k$  are called structure constants.

**Proposition 6** Let  $\mathfrak{g}_1$  and  $\mathfrak{g}_2$  be two Lie algebras of dimension  $n$ . Suppose each has a basis with respect to which the structure constant are the same. Then  $\mathfrak{g}_1$  and  $\mathfrak{g}_2$  are isomorphic.

Let  $\mathfrak{g}$  the Lie algebra related to the symmetry group of infinitesimal generators of the equation (1) as stated by the table of the commutators, it is enough to consider the next relations:

$$\begin{aligned} [\Pi_1, \Pi_2] &= 2\Pi_2, & [\Pi_1, \Pi_4] &= 2\Pi_4, & [\Pi_2, \Pi_3] &= -2\Pi_4, \\ [\Pi_3, \Pi_5] &= -2\Pi_3, & [\Pi_3, \Pi_5] &= -2\Pi_4. \end{aligned} \text{ Using that we calculate Cartan-} \\ \text{Killing form } K \text{ as follows.}$$

$$K = \begin{bmatrix} 80004 \\ 00000 \\ 00000 \\ 00000 \\ 40008 \end{bmatrix},$$

which the determinant vanishes, and thus by Cartan criterion it is not semisimple, (see Proposition 4). Since a nilpotent Lie algebra has a Cartan-Killing form that is identically zero, we conclude, using the counter-reciprocal of the last claim, that the Lie algebra  $\mathfrak{g}$  is not nilpotent. We verify that the Lie algebra is solvable using the Cartan criteria to solvability, (Proposition 5), and then we have a solvable nonnilpotent Lie algebra. The Nilradical of the Lie algebra  $\mathfrak{g}$  is generated by  $\Pi_2, \Pi_3, \Pi_4$ , that is, we have a Solvable Lie algebra with three dimensional Nilradical. Let  $m$  the dimension of the Nilradical  $M$  of a Solvable Lie algebra, In this case, in fifth dimensional Lie algebra we have  $3 \leq m \leq 5$ . Mubarakzyanov in<sup>13</sup> classified the 5-dimensional solvable nonnilpotent Lie algebras, in particular the solvable nonnilpotent Lie algebra with three dimensional Nilradical, this Nilradical is isomorphic to  $\mathfrak{h}_3$  the Heisenberg Lie algebra. Then, by the Proposition 6, and consequently we establish a isomorphism of Lie algebras with  $\mathfrak{g}$  and the Lie algebra  $\mathfrak{g}_{5,34}$ . In summary we have the next proposition.

**Proposition 7** The 5-dimensional Lie algebra  $\mathfrak{g}$  related to the symmetry group of the equation (1) is a solvable nonnilpotent Lie algebra with three dimensional Nilradical. Besides that Lie algebra is isomorphich with  $\mathfrak{g}_{5,34}$  in the Mubarakzyanov's classification.

## Conclusion

Using the Lie symmetry group (see Proposition 1), we calculated the optimal algebra (see Proposition 2). Making use of these operators,

it was possible to characterize all invariant solutions as it was shown in [Table 3](#).

It has been shown the variational symmetries for (1), as it was shown in (43) with its corresponding conservation laws (44) and all this was using Noether's theorem, but non-trivial conservation laws were also calculated using the Ibragimov's method as it was shown in (60) using the nonlinearly self-adjoint of the equation (1) as announced in the Proposition 3.

The Lie algebra associated to the equation (1) is a solvable nonnilpotent Lie algebra with three dimensional Nilradical. Besides that Lie algebra is isomorphic with  $\mathfrak{g}_{5,34}$  in the Mubarakzyanov's classification (see Proposition 7). Therefore, the goal initially proposed was achieved. For future works, An line of work would be to use the Lie symmetry group to calculate the  $\lambda$ -symmetries of (1), and, thus, explore more conservation laws for (1) and the equivalence group theory could be also considered to obtain preliminary classifications associated to a complete classification of (1).

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## Declaration interests

The authors declare that they have no conflict of interest.

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