

Matrix solution for the inverse problem of classical dynamics of a particle

Abstract

We solve the inverse problem corresponding to the fundamental problem of the classical dynamics of a material particle through a matrix treatment: assuming knowing the mass and the position (the trajectory, in relation to an inertial reference) of a particle at all times, we impose that this corresponds to the eigenvector of a "position matrix". Subsequent development leads to a "force matrix", which has the resultant force on the particle as its eigenvector. We identified some limitations of this matrix treatment.

Keywords: Classical dynamics of a particle, Problem of eigenvalues and eigenvectors with dependence on a variable, inverse problem

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Introduction

The physical law that governs the classical dynamics of a particle can, in principle, be written in a matrix form, although this does not reveal anything new. This possibility, however, illustrates a context that may be suitable for some problems. In this matrix format, such a classical law is written as,

$$\vec{F} = \mathcal{M}\vec{a}, \quad (1)$$

where $\mathcal{M}$ would be a matrix¹ to be determined, which would necessarily have the mass m of the particle as its eigenvalue and the acceleration $\vec{a}$ as its eigenvector. Even more, the matrix $\mathcal{M}$ would have the total strength $\vec{F}$ itself as an eigenvector (linearly dependent on $\vec{a}$) associated with the same eigenvalue m ,

$$\mathcal{M}(m\vec{a}) = m(\mathcal{M}\vec{a}) \rightarrow \mathcal{M}\vec{F} = m\vec{F}.$$

We emphasize the fact that there is nothing new in this, since the introduction of the matrix $\mathcal{M}$ is nothing but a simple mathematical device that allows us to rewrite this dynamic law. It may be of some interest, however, to construct the matrix $\mathcal{M}$ given m and F , that is, to solve a simple inverse problem of matrix algebra.

In Applied Mathematics, Physics, Geophysics,¹⁻³ among others, there are discussions of direct problems and inverse problems. In Classical Mechanics, for example, a typical direct problem consists of determining the physical path taken by a particle, or by the center of mass of a system of particles, under the action of a certain field or conservative force, to which a certain Lagrangian corresponds. The corresponding inverse problem consists of the following: Given a physical trajectory, determine the Lagrangian (or the family of Lagrangians) that would generate it. In Quantum Mechanics, the problem of determining the energy spectrum for a given quantum Hamiltonian of a microscopic physical system is a kind of direct problem, the corresponding inverse problem being to determine the quantum Hamiltonian (or the potential) that would generate a certain spectrum of energies, which could have been revealed experimentally. We found in recent literature several inverse problems.⁴⁻¹⁰

Problem definition

The inverse problem corresponding to the fundamental problem of the classical dynamics of a material particle is considered here through a matrix treatment. Consider a point particle, of mass m , in motion relative to a certain terrestrial inertial reference frame (TIR), under the action of a known resultant force $\vec{f}$. With this information,

¹Having enough free parameters or degrees of freedom

and the corresponding initial conditions, it is possible to solve the (direct) problem of the dynamics of a classical particle, having as a solution the position, $\vec{r}(t)$, of the particle for every instant of time; that is, the path physically traveled by the particle. In what follows, we will contextualize our approach: we will require that the position $\vec{r}(t)$ of the considered particle is an eigenvector of a certain matrix, to be defined, here represented by R , which we will call the position matrix. Consistent with this inverse context, we write,

$$R(t)\vec{r}(t) = \lambda(t)\vec{r}(t), \quad (2)$$

where we have to consider the general situation in which the position matrix and the eigenvalue can also depend on the time variable, t , as a consequence of which it is assumed that $\vec{r}$ is known for each instant t . The parameter $\lambda(t)$, not being known initially, will be defined accordingly. The interesting thing is that, from expression (2), a "force matrix", $F(t)$, can be defined, which will have, as a consequence of the construction presented here, the total force, $\vec{f}$, acting on the particle, as its eigenvector.

We anticipate below the order in which the calculations will be presented: having built the position matrix, $R(t)$, under the requirement of having $\vec{r}(t)$ as its eigenvector, we will be able to obtain what we will call the velocity matrix by differentiating² the expression (2) in relation to the time variable, t . A subsequent differentiation, in relation to the same variable, will generate what we call the acceleration matrix which, after being multiplied by the mass of the particle, will generate the force matrix, having as an eigenvector, precisely, the force $\vec{f}$. Complementary mathematical details will be specified in the course of the corresponding calculation. It is important to note that the matrices considered must carry a sufficient number of degrees of freedom (independent parameters) so that it is possible to satisfy certain requirements consistent with the matrix context considered.

Mathematical development

In our development, we did not consider the most general matrix possible (with nine free elements) but a simple one to show the idea involved. We consider, among many possibilities, a position matrix, $R(t)$, with five free elements³,

²Of which we use indistinctly the symbols " ∂ " or " $\frac{d}{dt}$ " (less visible) according to the available space.

³It is not advisable to define the position matrix with all its non-zero elements, because in the subsequent calculations, although simple, one cannot avoid working with quite extensive expressions.

$$R(t) = \begin{pmatrix} 0 & R_{12}(t) & 0 \\ 0 & 0 & R_{23}(t) \\ R_{31}(t) & R_{32}(t) & R_{33}(t) \end{pmatrix}. \quad (3)$$

In matrix (3), the free elements will later be properly fixed. So, from (2) and (3) we have,

$$\begin{pmatrix} 0 & R_{12}(t) & 0 \\ 0 & 0 & R_{23}(t) \\ R_{31}(t) & R_{32}(t) & R_{33}(t) \end{pmatrix} \begin{pmatrix} r_1(t) \\ r_2(t) \\ r_3(t) \end{pmatrix} = \lambda(t) \begin{pmatrix} r_1(t) \\ r_2(t) \\ r_3(t) \end{pmatrix}. \quad (4)$$

Assuming that none of the components of the position vector, $\vec{r}(t)$, is zero⁴ for any instant t , we have,

$$R_{12}(t) r_2(t) = \lambda(t) r_1(t) \rightarrow R_{12}(t) = \frac{\lambda(t) r_1(t)}{r_2(t)}, \quad (5)$$

$$R_{23}(t) r_3(t) = \lambda(t) r_2(t) \rightarrow R_{23}(t) = \frac{\lambda(t) r_2(t)}{r_3(t)}, \quad (6)$$

and also,

$$R_{31}(t) r_1(t) + R_{32}(t) r_2(t) + R_{33}(t) r_3(t) = \lambda(t) r_3(t). \quad (7)$$

As, so far, $\lambda(t)$ is not defined, we can make the choice⁵: $\lambda(t) \equiv r_3(t)$, with which $\lambda(t)$ will not assume null value at any instant of time. Another consequence of this choice is that the elements of the matrix R will necessarily be units of length. With $\lambda(t) \equiv r_3(t)$, expressions (5) and (6) are written as,

$$R_{12}(t) = \frac{r_3(t) r_1(t)}{r_2(t)}, \quad (8)$$

$$R_{23}(t) = r_2(t). \quad (9)$$

Substituting (8) and (9) into (7) we have,

$$R_{31}(t) = \frac{r_3(t)^2}{r_1(t)} - \frac{r_3(t) R_{33}}{r_1(t)} - \frac{r_2(t) R_{32}}{r_1(t)}, \quad (10)$$

and so we are left with the matrix,

$$R = \begin{pmatrix} 0 & \frac{r_3(t) r_1(t)}{r_2(t)} & 0 \\ 0 & 0 & r_2(t) \\ \frac{r_3(t)^2}{r_1(t)} - \frac{r_3(t) R_{33}}{r_1(t)} - \frac{r_2(t) R_{32}}{r_1(t)} & R_{32}(t) & R_{33}(t) \end{pmatrix}. \quad (11)$$

Consequently, under the specific form given to the matrix R , only $R_{32}(t)$ and $R_{33}(t)$ remain as the only free parameters.

By a straightforward procedure, it is simple to verify that the matrix R , in (11), has $\lambda(t) = r_3(t)$ as its eigenvalue and $\vec{r}(t) = (r_1(t), r_2(t), r_3(t))^T$ as its eigenvector, independently of the expressions that can be assigned to elements $R_{32}(t)$ and $R_{33}(t)$. Recall that an inverse context was assumed in which $\vec{r}(t)$ is known at all times.

To proceed further, we derive expression (2) in relation to the variable t with the purpose of building the velocity matrix, V . This

⁴This implies, among other things, that the particle's trajectory cannot cross any plane defined by two coordinate directions of the considered RIT. A proper choice of initial conditions will help to control this.

⁵At the beginning of this section, anticipating that λ would take on some real value, we could have considered R , in (3), as Symmetric or Hermitean, since this would guarantee real eigenvalues; but, as we are looking for a position matrix with the maximum number of independent elements, this choice was not essential.

matrix, adjusting to the defined context, will have: $\partial \vec{r} = \vec{v}$ the velocity⁶ of the particle, as its corresponding eigenvector. The corresponding construction, being simple, is not straightforward, requiring an intermediate step. Let's see this.

Deriving expression (2), in relation to the variable t , we arrive at the expression,

$$(\partial R) \vec{r} + R(\partial \vec{r}) = (\partial \lambda) \vec{r} + \lambda(\partial \vec{r}). \quad (12)$$

Arranging the terms in (12) we have,

$$[(\partial R) \vec{r} - (\partial \lambda) \vec{r}] + R \vec{v} = \lambda \vec{v} \quad (13)$$

Note that to proceed we have, in principle, two options: [i] Rewrite the term in square brackets, in (13), as the product of a suitable matrix "M" by the vector $\vec{v}$; that is: $(\partial R) \vec{r} - (\partial \lambda) \vec{r} \equiv M \vec{v}$, which would be convenient, since M could be chosen with a sufficient number of free parameters, with only three conditions (requirements) to be satisfied for this equality to occur, or [ii] impose adequate conditions on the free elements of the matrix ∂R so that: $(\partial R) \vec{r} - (\partial \lambda) \vec{r} = \vec{0}$. Option [ii], however, would not be possible to be satisfied because the matrix R , as it was processed, until reaching (11) from (3), presents, in this step, only two free parameters, and there are three independent mathematical relations to be satisfied for such situation (see Appendix). Then, we must take option [i] above, ie, determine the matrix M with R , λ and $\vec{v}$ known.

The equation for M is,

$$M \vec{v} = (\partial R) \vec{r} - (\partial \lambda) \vec{r}. \quad (14)$$

An immediate consequence of (14) is that the elements of matrix M have units of length, as in the case of matrix R . Writing (14) explicitly we have,

$$\begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 & \dot{R}_{12} & 0 \\ 0 & 0 & \dot{v}_2 \\ \dot{R}_{31} & \dot{R}_{32} & \dot{R}_{33} \end{pmatrix} - \begin{pmatrix} \dot{\lambda} & 0 & 0 \\ 0 & \dot{\lambda} & 0 \\ 0 & 0 & \dot{\lambda} \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} \quad (15)$$

from which the following three independent relations result,

$$M_{11} v_1 + M_{12} v_2 + M_{13} v_3 = -(\partial \lambda) r_1 + (\partial R_{12}) r_2, \quad (16)$$

$$M_{21} v_1 + M_{22} v_2 + M_{23} v_3 = -(\partial \lambda) r_2 + v_2 r_3, \quad (17)$$

$$M_{31} v_1 + M_{32} v_2 + M_{33} v_3 = \dot{R}_{31} r_1 + \dot{R}_{32} r_2 + (\dot{R}_{33} - (\partial \lambda)) r_3, \quad (18)$$

the same ones that will take three, of the nine, elements of M to be dependent on the remaining elements (the independent ones).

Taking into account that the matrix M will be added to R , and that the matrix R , in (11), has null elements only in positions 11, 13, 21 and 22, it will be very convenient to leave (if possible) the corresponding elements in M with non-null values to have free parameters available distributed among the largest number of positions in the sum matrix and not concentrated only in a few elements. On the other hand, from expressions (16) – (18), it is possible to highlight two of the following four elements: M_{11} , M_{13} , M_{21} and M_{22} . Let these elements be M_{11} and M_{21} ,

$$M_{11} = \frac{1}{v_1} \left(-(\partial \lambda) r_1 + \partial \left(\frac{r_3 r_1}{r_2} \right) r_2 - M_{12} v_2 - M_{13} v_3 \right). \quad (19)$$

$$M_{21} = \frac{1}{v_1} \left(-(\partial \lambda) r_2 + v_2 r_3 - M_{22} v_2 - M_{23} v_3 \right) \quad (20)$$

In addition, from expression (18), we chose to highlight the element M_{32} ,

⁶Which is well defined because $\vec{r}(t)$ is known for every instant t .

$$M_{32} = \frac{1}{v_2} ((\partial R_{31})r_1 + (\partial R_{32})r_2 + (\partial R_{33} - \partial \lambda)r_3 - M_{31}v_1 - M_{33}v_3). \quad (21)$$

Where,

$$\partial R_{31} = \partial \left(\frac{r_3^2}{r_1} - \frac{r_3 R_{33}}{r_1} - \frac{r_2 R_{32}}{r_1} \right) = \frac{1}{r_1^2} (2r_1 r_3 v_3 - r_1 v_3 R_{33} - r_3^2 v_1 + r_3 v_1 R_{33} + r_2 v_1 R_{32} - r_1 r_3 (\partial R_{33}) - r_1 (\partial r_2) (\partial R_{32}) - r_1 v_2 R_{32}). \quad (22)$$

having omitted, for simplicity, the symbol that indicates the explicit temporal dependence in the corresponding terms. Next, due to freedom of choice, we fix the elements R_{33} and R_{32} as follows,

$$R_{33} = 2r_3 \quad \& \quad R_{32} = 0. \quad (23)$$

Thus, expression (22) reduces to the following,

$$\partial R_{31} = \frac{1}{r_1^2} (r_3^2 v_1 - 2r_1 r_3 v_3), \quad (24)$$

and, in turn, expression (21) takes the form,

$$M_{32} = \frac{1}{v_2} \left(\frac{r_3^2 v_1}{r_1} - v_3 r_3 - M_{31}v_1 - M_{33}v_3 \right). \quad (25)$$

Note that, with the choice made in (23), we have completely defined the matrix R , given in (11), as follows,

$$R = \begin{pmatrix} 0 & r_3 r_1 / r_2 & 0 \\ 0 & 0 & r_2 \\ -r_3^2 / r_1 & 0 & 2r_3 \end{pmatrix} \quad (26)$$

Continuing with the calculation of the elements of matrix M , from (19) and (20), we have,

$$M_{11} = r_3 - \frac{r_1 r_3 v_2}{r_2 v_1} - \frac{1}{v_1} (M_{12}v_2 + M_{13}v_3), \quad (27)$$

$$M_{21} = \frac{1}{v_1} (-r_2 v_3 + r_3 v_2 - M_{22}v_2 - M_{23}v_3). \quad (28)$$

There are six elements of the matrix M , identified as: M_{12} , M_{13} , M_{22} , M_{23} , M_{31} and M_{33} , which, until now, remain independent. Once the matrix M is formally defined, we substitute the expression (14) in (13), being able to write,

$$(M + R)\vec{v} = \lambda \vec{v} \rightarrow V\vec{v} = \lambda \vec{v}, \quad (29)$$

Where,

$$V = \begin{pmatrix} M_{11} & M_{12} + (r_3 r_1 / r_2) & M_{13} \\ M_{21} & M_{22} & M_{23} + r_2 \\ M_{31} - (r_3^2 / r_1) & M_{32} & M_{33} + 2r_3 \end{pmatrix}, \quad (30)$$

with M_{11} , M_{21} and M_{32} being the dependent elements given in expressions (27), (28) and (25), respectively. Note, for example, from (27), that when we fix M_{12} and M_{13} , the element M_{11} is defined. A similar situation is found among the elements in the other rows of the matrix V , as shown by expressions (28) and (25). Furthermore, in (30) it is observed, as in the cases of the matrices R and M , that all their elements have units of length.

To proceed further, we derive (29) with respect to the time variable, obtaining,

$$(\partial V)\vec{v} + V(\partial \vec{v}) = (\partial \lambda)\vec{v} + \lambda(\partial \vec{v}). \quad (31)$$

Arranging terms, we get,

$$[(\partial V)\vec{v} - (\partial \lambda)\vec{v}] + V\vec{a} = \lambda \vec{a}, \quad (32)$$

where: $\partial \vec{v} = \vec{a}$, is the acceleration of the particle. Proceeding as before, we write for a matrix Q , with elements that can be dependent on t , the equation,

$$Q\vec{a} = (\partial V)\vec{v} - (\partial \lambda)\vec{v}, \quad (33)$$

or explicitly,

$$\begin{pmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} \left(\frac{\partial V_{11}}{\partial V_{21}} \frac{\partial V_{12}}{\partial V_{22}} \frac{\partial V_{13}}{\partial V_{23}} \right) - \begin{pmatrix} \partial \lambda & 0 & 0 \\ 0 & \partial \lambda & 0 \\ 0 & 0 & \partial \lambda \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \end{pmatrix}, \quad (34)$$

from which the following three independent relations result,

$$Q_{11}a_1 + Q_{12}a_2 + Q_{13}a_3 = (\partial V_{11} - \partial \lambda)v_1 + (\partial V_{12})v_2 + (\partial V_{13})v_3, \quad (35)$$

$$Q_{21}a_1 + Q_{22}a_2 + Q_{23}a_3 = (\partial V_{21})v_1 + (\partial V_{22} - \partial \lambda)v_2 + (\partial V_{23})v_3, \quad (36)$$

$$Q_{31}a_1 + Q_{32}a_2 + Q_{33}a_3 = (\partial V_{31})v_1 + (\partial V_{32})v_2 + (\partial V_{33} - \partial \lambda)v_3. \quad (37)$$

From expressions (35) – (37) it is convenient to choose Q_{13} , Q_{22} and Q_{33} as the dependent elements; the same ones that we put in evidence below,

$$Q_{13} = \frac{1}{a_3} ((\partial V_{11} - v_3)v_1 + (\partial V_{12})v_2 + (\partial V_{13})v_3 - Q_{12}a_2 - Q_{11}a_1), \quad (38)$$

$$Q_{22} = \frac{1}{a_2} ((\partial V_{21})v_1 + (\partial V_{22} - v_3)v_2 + (\partial V_{23})v_3 - Q_{23}a_3 - Q_{21}a_1), \quad (39)$$

$$Q_{33} = \frac{1}{a_3} ((\partial V_{31})v_1 + (\partial V_{32})v_2 + (\partial V_{33} - v_3)v_3 - Q_{32}a_2 - Q_{31}a_1). \quad (40)$$

Leaving the elements Q_{11} , Q_{12} , Q_{21} , Q_{23} , Q_{31} , Q_{32} as the only independent elements. In expressions (38) – (40) we have that,

$$\begin{aligned} \partial V_{11} = v_3 - \frac{1}{r_2 v_1} (r_1 r_3 a_2 + r_1 v_2 v_3 + r_3 v_1 v_2) + \frac{1}{r_2^2 v_1^2} (r_1 r_3 v_1 v_2^2 + r_1 r_2 r_3 v_2 a_1) + \\ - \frac{1}{v_1} (a_2 M_{12} + v_2 (\partial M_{12}) + a_3 M_{13} + v_3 (\partial M_{13})) + \frac{1}{v_1^2} (v_2 a_1 M_{12} + v_3 a_1 M_{13}), \end{aligned} \quad (41)$$

$$\partial V_{12} = \partial M_{12} + \frac{1}{r_1} (r_2 v_3 - r_3 v_2) + \frac{1}{r_1^2} (r_2 r_3 v_1), \quad (42)$$

$$\partial V_{13} = \partial M_{13}, \quad (43)$$

$$\partial V_{21} = \frac{1}{v_1} (-r_2 a_3 + r_3 a_2 - a_2 M_{22} - v_2 (\partial M_{22}) - a_2 M_{33} - v_3 (\partial M_{23}) +$$

$$\frac{a_1}{v_1^2} (r_2 v_3 - r_3 v_2 + v_2 M_{22} + v_3 M_{23}), \quad (44)$$

$$\partial V_{22} = \partial M_{22}, \quad (45)$$

$$\partial V_{23} = \partial M_{23} + v_2, \quad (46)$$

$$\partial V_{31} = \partial M_{31} - \frac{1}{r_1} (2r_3 v_3) + \frac{1}{r_1^2} (v_1 r_3^2), \quad (47)$$

$$\partial V_{32} = \frac{1}{r_1^2 v_2^2} (r_1 r_3^2 v_2 a_1 + 2r_1 r_3 v_1 v_2 v_3 - r_1 r_3^2 v_1 a_2 - r_3^2 v_2 v_1^2) +$$

$$\frac{a_2}{v_2^2} (r_3 v_3 + v_1 M_{31} + v_3 M_{33}) - v_2 (v_3^2 + r_3 a_3 + a_1 M_{31} + v_1 (\partial M_{31}) + a_3 M_{33} + v_3 (\partial M_{33})), \quad (48)$$

$$\partial V_{33} = \partial M_{33} + 2v_3. \quad (49)$$

Note the consistency of the units between the terms that appear in each of the expressions (41) to (49).

Following the anticipated sequence of calculations, of the expressions that define the matrices Q and V , we can write, using (32) and (33), the following expression,

$$(Q + V)\vec{a} = \lambda\vec{a}, \quad (50)$$

being $\vec{a}$ an eigenvector of the matrix “ $Q + V$ ”, we have that “ $m\vec{a}$ ”, being m the numerical parameter corresponding to the mass of the particle, is also an eigenvector of this matrix,

$$(Q + V)m\vec{a} = \lambda(m\vec{a}), \quad (51)$$

The matrix in (51) we will call the force matrix, F , so that,

$$F\vec{f} = \lambda\vec{f}. \quad (52)$$

If we write the matrix F as follows,

$$F = \begin{pmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{pmatrix}, \quad (53)$$

then its elements are defined as follows,

$$F_{11} = Q_{11} + M_{11}, \quad (54)$$

(Q_{11} being an independent element and M_{11} given in (27)),

$$F_{12} = Q_{12} + M_{12} + \frac{r_1 r_3}{r_2}, \quad (55)$$

(where: Q_{12} and M_{12} are independent elements; r_1 , r_2 and r_3 are assumed to be known),

$$F_{13} = Q_{13} + M_{13}, \quad (56)$$

(Q_{11} being a dependent element given in (38) and M_{13} an independent element),

$$F_{21} = Q_{21} + M_{21}, \quad (57)$$

(Q_{21} being an independent element and M_{21} a dependent element given in (28)),

$$F_{22} = Q_{22} + M_{22}, \quad (58)$$

(Q_{22} being a dependent element given in (39) and M_{22} an independent element),

$$F_{23} = Q_{23} + M_{23} + r_2, \quad (59)$$

(Q_{23} and M_{23} being independent elements, r_2 given initially),

$$F_{31} = Q_{31} + M_{31} - \frac{r_3^2}{r_1}, \quad (60)$$

(Q_{31} and M_{31} being independent elements),

$$F_{32} = Q_{32} + M_{32}, \quad (61)$$

(Q_{32} being an independent element and M_{32} a dependent element given in (25)),

$$F_{33} = Q_{33} + M_{33} + r_3, \quad (62)$$

(Q_{33} being a dependent element given in (40) and M_{33} being an independent element), where each of the elements of the matrices Q and M has been previously found. The choice made of independent elements for the matrices R , M and Q has allowed each of the elements of F to be defined by at least one free parameter.

Concisely we say that, within a defined and particular context, the matrix given through expressions (53) to (62) and the matrix that we have called the force matrix were constructed. The name of this matrix is due to the fact that, precisely, the force (which acts on the considered particle) is its property; namely, its eigenvector. With the development presented in the previous sections, we have solved the

inverse problem corresponding to the classical dynamics of a material particle in a simplified situation (when the maximum number of free elements in the matrices is not considered).

The matrix approach and Galileo's transformations

Let two independent observers use different inertial frames, which we call O' and O , which use coordinates x', y', z' and x, y, z , respectively, and the (intervals of) times recorded by these observers are designated by t' and t , respectively. Suppose that O' is in motion relative to O with constant speed $O\vec{V} = d\vec{\xi}/dt$, where $\xi(t)$ is the position of the origin of coordinates of O' with respect to O at time t . Suppose that the observers describe, separately, the motion of the same material particle of mass m on which a resultant force acts, described by $\vec{f}'$ in the frame O' , and by $\vec{f}$ in the frame O .

From the Galileo transformations, given by the compact expressions,

$$\vec{r}' = \vec{r} - \vec{\xi} \quad \& \quad t' = t, \quad (63)$$

it is known that the forces $\vec{f}'$ and $\vec{f}$ check,

$$\vec{f}'(\vec{r}', \vec{v}', t') = \vec{f}(\vec{r}, \vec{v}, t). \quad (64)$$

On the other hand, based on the context already defined and the development presented, we can consider that $\vec{f}'$ is an eigenvector of a matrix F' that can be constructed; this is,

$$F'\vec{f}' = \eta\vec{f}', \quad (65)$$

where η is the corresponding eigenvalue. Note that F' must be distinct from F , because according to our matrix treatment, these matrices are constructed from position vectors, which are distinct in different RIT's.

Assuming that matrix F' has the same structure and the same number of free parameters as those of matrix F , we have that its elements, distinct from those of matrix F , will have the same relationships among themselves as those found in section 3. On the other hand, the matrices F and F' are expected to commute consistently with the fact that they have, according to (64), a common non-zero eigenvector. The choice of the values of the (previously free) elements of the matrices, consistent with the requirement that these matrices commute, requires the inclusion of the largest number of free parameters in each matrix and calculations similar to those presented, with the difference that these would be much more long.

Conclusion

A matrix solution was constructed for the inverse problem corresponding to the problem of determining the trajectory of a material particle, given its mass m and the total force $\vec{f}$ acting on it. Within a defined and particular context, in which it is considered that the position of the particle $\vec{r}(t)$, in relation to a given RIT, corresponds to an eigenvector of a “position matrix”, R , the matrix given through the expressions (53) to (62) is what we have called the “force matrix”. The name given to this matrix is due to the fact that, precisely, the force considered is its property; namely, its eigenvector. The matrix context presented could neither be applied, nor extended, to the component forces of the total force $\vec{f}$, as it is clear from the adopted procedure that, starting from a known $\vec{r}(t)$, one can only arrive at the total force and not to its components. Note that the solution obtained is not unique: it depends on the number of free parameters that are initially available in the position matrix, which is constructed assuming that the position $\vec{r}(t)$ is known at all times.

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Conflicts of interest

The author declares there is no conflict of interest.

Appendix

Suppose we consider a matrix R , of order three, which has no fixed elements; that is, it has the maximum number of free elements: nine. Therefore, when $\vec{r}(t)$ is required to be an eigenvector of R , that is,

$$R(t) \vec{r}(t) = \lambda(t) \vec{r}(t), \quad (66)$$

we will have that three of these elements cannot be kept free, because in expression (66) there are three independent relations that must be satisfied; thus, we are left with $(9-3)$ six free elements (or degrees of freedom). Then we differentiate (66) with respect to the time variable, and the independent⁷ expression is obtained,

$$[(\partial R)\vec{r}(t) - (\partial \lambda)\vec{r}(t)] + R\vec{v}(t) = \lambda\vec{v}(t). \quad (67)$$

Here we also have that, in order to satisfy the relation (67), three other parameters must be fixed, leaving only $(6-3)$ three free elements. Having done that, if we want to keep in (78) only the expression,

$$R\vec{v} = \lambda\vec{v}, \quad (68)$$

we must require the left term in (67) to be null, that is,

$$(\partial R)\vec{r}(t) - (\partial \lambda)\vec{r}(t) = \vec{0}, \quad (69)$$

but for that we need the last three free parameters that are left to be used to guarantee the verification of this vector equality. By differentiating the expression (68) in relation to the variable t , we will have,

$$[(\partial R)\vec{v}(t) - (\partial \lambda)\vec{v}(t)] + R\vec{a}(t) = \lambda\vec{a}(t). \quad (70)$$

but this equality, under the conditions considered, cannot be satisfied since we no longer have free parameters. The only way to get an expression consistent with this procedure is to rewrite the term in square brackets in (70) as the product of a suitable matrix (which is introduced into the problem with sufficient degrees of freedom) and the vector $\vec{a}(t)$.

References

1. Prilepko A, Orlovsky D, Vasin I. Methods for solving inverse problems in mathematical physics. Marcel Dekker, Inc., New York; 2000.
2. Lantz V, Abiri N, Carlsson G, et al. Deep learning for inverse problems in quantum mechanics. *International Journal of Quantum Chemistry*. 2021;121:e26599.
3. Vasco D. An algebraic formulation of geophysical inverse problems. *Geophysical Journal International*. 2000;142:970–990.
4. Fayziev Y, Buvaev Q, Juraev D, et al. The inverse problem for determining the source function in the equation with the Riemann–Liouville fractional derivative. *Global and Stochastic Analysis*. 2022;9(2):43–52.
5. Yuldashev TK, Karimov ET. Inverse Problem for a Mixed Type Integro–Differential Equation with Fractional Order Caputo Operators and Spectral Parameters. *Axioms*. 2020;9:121.
6. Ashurov R, Muhiddinova O. Inverse problem of determining the heat source density for the subdiffusion equation. *Differential equations*. 2020;56(12):1550–1563.
7. Hu G, Qu F, Zhang B. Direct and inverse problems for electromagnetic scattering by a doubly periodic structure with a partially coated dielectric. *Mathematical Methods in the Applied Sciences*. 2010;33:147–156.
8. Popova M, He S. Electromagnetic direct and inverse problems for a surface-breaking crack in a conductor at a high frequency. *Journal of Applied Physics*. 1999;86(7):3997–4003.
9. Dorren H, Muyzert E, Snieder R. The stability of one-dimensional inverse scattering. *Inverse Problems*. 1994;10:865–880.
10. Sivardière J. On the inverse Kepler problem. *European Journal of Physics*. 1985;6:245–248.

⁷Where, to simplify the notation, we have suppressed the symbol that indicates the temporal dependence in the corresponding terms.