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On the generalized laplace transform and applications

Abstract

In this work we present the main properties of the Generalized Laplace Transform, recently defined, and we show some applications. Laplace's transformation has been very useful in the studies of engineering, mathematics, physics, among other scientific areas. One of the main mathematical areas where it has many applications is in the topic of differential equations and their solution methods. In this paper, we study the stability and analysis of linear systems with the Generalized Laplace Transform.

Keywords: Generalized Laplace Transform, Differential equations, solution methods, Linear systems

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Prelimminars

Modern definitions of derivatives and integrals of non-integer order play a vital role in the theory of fractional calculus. From the original definitions of Riemann-Liouville, to the most current of Atangana-Baleanu, through many others, a body of theory has been formed that is beginning to have applications in various fields. On the other hand, although local differential operators have been known since the 1960s, it was not until 2014 that a complete formalization was achieved with the conformable derivative.[1] In 2018 we defined a new type of differential operator, called non conformable[2-8] and in 2020 we consolidate our ideas with a generalized derivative definition,[9-11] see also).[7,12] In this way, a new area has been formed in the Mathematical Sciences, which we call Generalized Calculus, with many applications and important theoretical results.

Adding to this, integral transforms are also ground-breaking inventions in calculus. The capability of integral transforms to manipulate several problems by altering the domain of the equation, have made it persistently important.

The role of the classic Laplace Transform in Mathematical Sciences is of high impact, from the theoretical development to the multiplicity of applications, we have countless investigations and publications in this regard. In its almost 140 years of life, we have seen its great importance in the studies of engineering, mathematics, physics, among other scientific areas, since in addition to being of great interest in the theoretical order, it provides a simple way to solve differential equations, turning them into algebraic equations. In particular, one of the main difficulties is finding methods to find analytical solutions to some classes of differential equations, within these methods are those that use different integral transformations (Laplace, Mellin and Fourier, for example) some attempts in this direction, to fractional and generalized differential equations can be found in.[13-26] One of the mathematical areas that is in constant development is that of Differential Equations (using new operators or defined on different functional spaces), and their solution methods, in particular, due to the multiplicity of applications and its own theoretical development, over time, researchers and productions related to this area have been increasing, you can consult in[27-30] different aspects of this increase and its overlaps with the development of Mathematics itself. In this paper, we return to the Generalized Laplace Transform defined below, and illustrate its strength and scope with stability analysis of linear systems.

Preliminaries

In[9] (see also[12] and[31]) a generalized fractional derivative was defined in the following way.

Definition 1 *Given a function* $f:[0,+\infty)\to\mathbb{R}$. *Then the N-derivative of f of order* α *is defined by*

$$J^{\alpha}_{F,\mathrm{b}}(f)(t)=\int_t^b\frac{f(s)}{F(s,\alpha)}ds, b>t. J^{\alpha}_{F,a}N^{\alpha}_{F}(J^{\alpha}_{F,a}(f))(t)=f(t)-f(a)$$

$$F(t-s,\alpha)=\Gamma(\alpha)(t-s)^{(1-\alpha)} \tag{1}$$

$$I\subseteq\mathbb{R}$$

$$a,t\in I,$$

$$0<\alpha\leq 1$$

$$f,g:[a,b]\to\mathbb{R}$$

$$\alpha\in(0,1]$$

$$J^{\alpha}_{F,a}((f)(N^{\alpha}_{F,a+}g(t)))=[f(t)g(t)]_a^b-J^{\alpha}_{F,a}((g)(N^{\alpha}_{F,a}f(t))).$$

for all $t>0$, $\alpha\in(0,1)$ being $F(\alpha,\mathrm{t})$ some absolutely continuous function.

If f is α-differentiable in some $(0,\alpha)$, and $\lim_{t\to 0^+} N^{\alpha}_{F}f(t)$ exists, then define $N^{\alpha}_{F}f(0)=\lim_{t\to 0^+} N^{\alpha}_{F}f(t)$, note that if f is differentiable, then $N^{\alpha}_{F}f(t)=F(t,\alpha)f'(t)$ where $f'(t)$ is the ordinary derivative.

This generalized differential operator contains many of the known local operators (for example, the conformable derivative of[1] and the non-conformable of[2])) and has shown its usefulness in various applications, as it can be consulted, for example, in.[3,4,8,10,11,32-34] One of the most required properties of a derivative operator is the Chain Rule,[9] to calculate the derivative of compound functions, which does not exist in the case of classical fractional derivatives $N^{\alpha}_{\Phi}(f\circ g)(t)=N^{\alpha}_{\Phi}f(g(t))=f('g(t))N^{\alpha}_{\Phi}g(t)$.

Between its own theoretical development and the multiplicity of applications, the field has grown rapidly in recent years, in such a way that a single definition of "fractional derivative or integral" does not exist, or at least is not unanimously accepted, in[35] suggests and justifies the idea of a fairly complete classification of the known operators in non-integer order Calculus, in addition, in the work[36] some reasons are presented why new operators linked to applications and developments theorists appear every day. These operators, both classic (global) or local, have been obtained by numerous mathematicians, some well known and others have not gone far enough (the Sonin derivative is enough as an example), if to this we add that, for some reason, local differential operators, which we prefer to call generalized, have been ignored and underestimated by numerous researchers, today they have been the source of development of new global operators based on their formulations.

In addition, Chapter 1 of[37] presents a history of differential operators, both local and global, from Newton to Caputo and presents a definition of local derivative with new parameter, providing a large number of applications, with a difference qualitative between both types of operators, local and global. Most importantly, Section 1.4 LIMITATIONS ... concludes "We can therefore conclude that both the Riemann – Liouville and Caputo operators are not derivatives, and then they are not fractional derivatives, but fractional operators. We agree with the result[38] that, the local fractional operator is not a fractional derivative" (p.24). As we said before, they are new tools that have demonstrated their usefulness and potential in the modeling of different processes and phenomena.

Now, we give the definition of a general fractional integral.[39] Throughout the work we will consider that the integral operator kernel T defined below is an absolutely continuous function (for additional details, the interested reader can consult[12] and).[31]

Definition 2 *Let I be an interval $I \subseteq \mathbb{R}$, $a,t \in I$ and $\alpha \in \mathbb{R}$. The integral operator J, is defined for every locally integrable function f on I as*

$$J_{F,a}^{\alpha}(f)(t)=\int_a^t \frac{f(s)}{F(s,\alpha)}ds, t>a. \tag{2}$$

$$J_{F,b}^{\alpha}(f)(t)=\int_t^b \frac{f(s)}{F(s,\alpha)}ds, b>t. \tag{3}$$

Remark 3 *As pointed out in,[4] many fractional integral operators can be obtained as particular cases of the previous one, under certain choices of the F kernel. For example, if $F(t-s,\alpha)=\Gamma(\alpha)(t-s)^{(1-\alpha)}$ the right Riemann-Liouville integral is obtained (similarly to the left), further details on Fractional Calculus and fractional integral operators linked to the generalized integral of the previous definition, can be found in[1,40-47]* The following property is one of the fundamental ones and links the integral operator with the generalized derivative, defined above (see also[12] and[31]).

Proposition 4 *Let I be an interval $I \subseteq \mathbb{R}$, $a \in I$, $0<\alpha \le 1$ and f a α-differentiable function on I such that f' is a locally integrable function on I. Then, we have for all $t \in I$*

$$J_{F,a}^{\alpha}(N_F^{\alpha}(f))(t)=f(t)-f(a).$$

Proposition 5 *Let I be an interval $I \subseteq \mathbb{R}$, $a \in I$, and $\alpha \in (0,1]$.*

$$N_F^{\alpha}(J_{F,a}^{\alpha}(f))(t)=f(t)$$

for every continuous function f on I and $a,t \in I$.

Theorem 6 *(Integration by parts) Let $f,g:[a,b]\to\mathbb{R}$ differentiable functions and $\alpha \in (0,1]$. Then, the following property hold*

$$J_{F,a}^{\alpha}((f)(N_{F,a}^{\alpha}+g(t)))=[f(t)g(t)]_a^b-J_{F,a}^{\alpha}((g)(N_{F,a}^{\alpha}f(t))).$$

$$N_{Fi,ti}^{\alpha}f(\bar{a})=\lim_{\varepsilon\to 0}\frac{f(a_1,..,a_i+\varepsilon F_i(a_i,\alpha),...,a_n)-f(a_1,...,a_i,...,a_n))}{\varepsilon}$$

$$\alpha \in (0,1]$$

$$a_i>0$$

$$\nabla_N^{\alpha}f(\bar{a})=(N_{t1}^{\alpha}f(\bar{a}),...,N_{tn}^{\alpha}f(\bar{a})) \tag{4}$$

Theorem 7 *If $f:[a,b]\to\mathbb{R}$ is a continuous function and $\alpha \in (0,1]$ then, the following inequality is fulfilled*

$$\left|J_{F,a}^{\alpha}+(f)(t)\right| \le J_{F,a}^{\alpha}+|f|(t). \tag{5}$$

Taking into account the ideas of[5] we can define the generalized partial derivatives as follows.

Definition 8 *Given a real valued function $f:\mathbb{R}^n\to\mathbb{R}$ and $\bar{a}=(a_1,...,a_n)\in\mathbb{R}^n$ a point whose ith component is positive. Then the generalized partial N-derivative of f of order α in the point $\bar{a}=(a_1,...,a_n)$ is defined by*

$$N_{Fi,ti}^{\alpha}f(\bar{a})=\lim_{\varepsilon\to 0}\frac{f(a_1,..,a_i+\varepsilon F_i(a_i,\alpha),...,a_n)-f(a_1,...,a_i,...,a_n))}{\varepsilon} \tag{6}$$

if it exists, is denoted $N_{Fi,ti}^{\alpha}f(\bar{a})$, and called the ith generalized partial derivative of f of the order $\alpha \in (0,1]$ at $\bar{a}$.

Remark 9 *If a real valued function f with n variables has all generalized partial derivatives of the order $\alpha \in (0,1]$ at $\bar{a}$, each $a_i>0$, then the generalized α-gradient of f of the order $\alpha \in (0,1]$ at $\bar{a}$ is*

$$\nabla_N^{\alpha}f(\bar{a})=(N_{t1}^{\alpha}f(\bar{a}),...,N_{tn}^{\alpha}f(\bar{a})) \tag{7}$$

Taking into account the above definitions, it is not difficult to prove the following result, on the equality of mixed partial derivatives.

Theorem 10 *Under assumptions of Definiton 8, assume that $f(t_1,t_2)$ it is a function for which, mixed generalized partial derivatives exist and are continuous, $N_{F_{1,2},t_1,t_2}^{\alpha+\beta}(f(t_1,t_2))$ and $N_{F_{2,1},t_2,t_1}^{\beta+\alpha}(f(t_1,t_2))$ over some domain of $\mathbb{R}^2$ then*

$$N_{F_{1,2},t_1,t_2}^{\alpha+\beta}(f(t_1,t_2))=N_{F_{2,1},t_2,t_1}^{\beta+\alpha}(f(t_1,t_2)) \tag{8}$$

Main results

The following generalized exponential order will play an important role in our work.

Definition 11 *Let $\alpha \in (0,1]$ and c a real number. We define the generalized exponential order in the following way*

$$E_{\alpha}^{N}(c,t)=exp(c\mathcal{F}(t,\alpha)).$$

with $\mathcal{F}(t,\alpha)=\int_0^t \frac{ds}{F(s,\alpha)}=J_{(F,0)}^{\alpha}(1)(t).$

From Definitions 1, 11 and the Chain Rule, we have $N_F^{\alpha}\{E_{\alpha}^{N}(c,t)\}=cE_{\alpha}^{N}(c,t).$

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Definition 12 *Let* $\alpha \in (0,1]$ *let* g *a function and* s *a real number. We define the Generalized Laplace Transform in the following way*

$$F(s) = (\mathcal{L}_N^{\alpha}\{g(t)\})(s) = J_{F,0}^{\alpha}(E_{\alpha}^{N}(-s,t)g(t))(\infty).$$

and its inverse transform

$$g(t) = (\mathcal{L}_N^{\alpha}\{G(s)\}^{(-1)})(t) = J_{F,0}^{\alpha}(F(t,\alpha)E_{\alpha}^{N}(s,t)G(s))(\infty)$$

Remark 13 *If* $F(t,\alpha)=1$ *then we have the usual Laplace Transform, and if* $F(t,\alpha)=t^{1-\alpha}$ *then we have the Conformable Laplace Transform defined in* [40] *(also see*[16,48-51]*). If we put* $F(t,\alpha)=\frac{1}{g'(t)}$ *then we obtain the generalized Laplace transform of*[17] *(more details in*[52,53]*).*

Theorem 14 *The Generalized Laplace Transform has the following properties:*

$$\mathcal{L}_N^{\alpha}\{\alpha g(t)+\beta h(t) = \alpha\mathcal{L}_N^{\alpha}\{g(t) + \beta\mathcal{L}_N^{\alpha}\{h(t)\} \qquad (9)$$

$$\mathcal{L}_N^{\alpha}\{N_F^{\alpha}g(t)\} = -g(0) - s\mathcal{L}_N^{\alpha}g(t)\} \qquad (10)$$

$$\mathcal{L}_N^{\alpha}\{J_{F,0}^{\alpha}(g(s))(t) = \frac{1}{s}\mathcal{L}_N^{\alpha}\{g(t)\} \qquad (11)$$

$$\mathcal{L}_N^{\alpha}\{N_F^{\alpha})^n g(t)\} = -\sum_{k=1}^{n}(-1)^k s^{n-k}((N_F^{\alpha})^{k-1})g(0) - s^n L_N^{\alpha}\{g(t)\} \qquad (12)$$

where $(N_F^{\alpha})^n = \underbrace{N_F^{\alpha}\circ N_F^{\alpha}\circ\ldots\circ N_F^{\alpha}}_{n\ times}$

Theorem 15 *Since the function* $\mathcal{F}(t,\alpha)$ *has the property* $\mathcal{F}'(t,\alpha)>0$ *then the following relation between the Generalized Laplace Transform and the classical one holds:*

$$(\mathcal{L}_N^{\alpha}\{g(t)\})(s) = (\mathcal{L}\{g(\mathcal{F}(t,\alpha)^{-1})\})(s) \quad (13)$$

Definition 16 *A function* $f:[0,\infty)\to\mathcal{R}$ *is said to be of* $g(t)-$ *exponential order if and only if there exists non-negative constants M, c, T such that* $|f(t)|\le Me^{cg(t)}$ *for* $t\ge T$.

Theorem 17 *If* $f:[0,\infty)\to\mathcal{R}$ *is a piecewise function of* $\mathcal{F}(t,\alpha)-$ *exponential order, then the Generalized Laplace Transform exists for* $s>c$.

The main properties of the Generalized Laplace Transform are presented in the following result.

Theorem 18 *If* $\alpha\in(0,1]$ *then we have*

a) $\mathcal{L}_N^{\alpha}\{1\} = \frac{1}{s}$

b) $\mathcal{L}_N^{\alpha}\{E_{\alpha}^{N}(c,t)\} = \frac{1}{s-c}$

c) $\mathcal{L}_N^{\alpha}\{g(t)E_{\alpha}^{N}(c,t)\} = g(s-c)$

d) $\mathcal{L}_N^{\alpha}\{sin(c\mathcal{F}(t,\alpha))\} = \frac{c}{s^2+c^2}$

e) $\mathcal{L}_N^{\alpha}\{cos(c\mathcal{F}(t,\alpha))\} = \frac{s}{s^2+c^2}$

f) $\mathcal{L}_N^{\alpha}\{\sinh(c\mathcal{F}(t,\alpha))\} = \frac{c}{s^2-c^2}$

g) $\mathcal{L}_N^{\alpha}\{\cosh(c\mathcal{F}(t,\alpha))\} = \frac{s}{s^2-c^2}$

The following result complete the theoretical body.

Definition 19 *Let f and g be two functions which are piecewise continuous at each interval* $[0,T]$ *and of generalized exponential order. We define the N-convolution of f and g by*

$$(f*g)_N(t) = \int_0^t f(\tau)g[\mathcal{F}^{(-1)}(\mathcal{F}(t,\alpha)-\mathcal{F}(\tau,\alpha))]\frac{d\tau}{F(\tau,\alpha)}, t\le T. \qquad (14)$$

The commutativity of the N-convolution is given in the following result.

Lemma 20 *Let f and g be two functions which are piecewise continuous at each interval* $[0,T]$ *and of generalized exponential order. Then*

$$(f*g)_N(t) = (g*f)_N(t). \qquad (15)$$

Below we present the N-Laplace transform of the N-convolution.

Theorem 21 *Let f and g be two functions which are piecewise continuous at each interval* $[0,T]$ *and of generalized exponential order. Then*

$$\mathcal{L}_N^{\alpha}\{(f*g)_N = \mathcal{L}_N^{\alpha}\{f\}\mathcal{L}_N^{\alpha}\{g\}. \qquad (16)$$

On the stability

Let us consider the following system

$$N_F^{\alpha}\bar{x}(t) = A\bar{x}(t) + \bar{f}(t) \qquad (17)$$

and its corresponding associated homogeneous system

$$N_F^{\alpha}\bar{x}(t) = A\bar{x}(t). \qquad (18)$$

With $\bar{x}\in\mathbb{R}^n$, $A=[a_{ij}]\in\mathbb{R}^n\times\mathbb{R}^n$, $\bar{f}(t) = [f_1(t), f_2(t),\ldots,f_n(t)]^T$.

Definition 22 *The solution* $\bar{x}(t)$ *of system (18) is called stable if, for any initial condition* x_0*, there exists* $\varepsilon>0$ *such that* $\|\bar{x}(t)\|<\varepsilon$ *for all* $t>0$*. The solution is called asymptotically stable if it is stable and* $\|\bar{x}(t)\|\to 0$ *as* $t\to\infty$.

Theorem 23 *The solution of system (18) is given by*

$$\bar{x}(t) = \bar{x}_0 E_{\alpha}^{N}(1,t) \qquad (19)$$

whenever the solution is differentiable on $[0,\infty)$ and $E_{\alpha}^{N}(1,t)$ is the generalized exponential order of Definition 11.

Proof. It is enough to use the properties of the generalized exponential order function.

Theorem 24 *The system (18) is is asymptotically stable if and only if the eigenvalues of* A *have strictly negative real parts.*

Proof. It is enough to take limit in (19).

As is known, the eigenvalues are obtained by solving the system $|A-\lambda|=0$, with A the matrix of system (18).

Now we will see how to obtain the explicit solution of the system (17).

For this, we will apply the generalized Laplace transform $\mathcal{L}_N^{\alpha}$ on both sides of the system (17), obtain

$$\begin{pmatrix} sX_1(s)-x_1(0) \\ sX_2(s)-x_2(0) \\ \ldots \\ sX_n(s)-x_n(0)\end{pmatrix} = A\begin{pmatrix} X_1(s) \\ X_2(s) \\ \ldots \\ X_n(s)\end{pmatrix} + \begin{pmatrix} F_1(s) \\ F_2(s) \\ \ldots \\ F_n(s)\end{pmatrix} \qquad (20)$$

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where $X_i(s)=\mathcal{L}_N^{\alpha}[x_i(t)],\quad F_i(s)=\mathcal{L}_N^{\alpha}[f_i(t)]$ with $i=1,2,\dots,n.$

$$\begin{pmatrix} X_1(s)\\ X_2(s)\\ \dots\\ X_n(s)\end{pmatrix}=\begin{pmatrix}\frac{1}{(s-a_{11})}(x_1(0)+F_1(s)+\sum_{j=1,j\neq 1}^{n}a_{1j}X_j(s))\\ \frac{1}{(s-a_{22})}(x_2(0)+F_2(s)+\sum_{j=1,j\neq 2}^{n}a_{2j}X_j(s))\\ \dots\\ \frac{1}{(s-a_{nn})}(x_n(0)+F_n(s)+\sum_{j=1,j\neq n}^{n}a_{nj}X_j(s))\end{pmatrix} \quad (21)$$

Solving this system and applying the Inverse Transform we obtain the desired solution.

Remark 25 *The results of this section contain as a particular case, those of,*[54] *relative to commensurable systems.*

Remark 26 *If we consider* $F(t,\alpha)=t^{1-\alpha}$, *that is, the conformable case, the results obtained in* [18,20,22,52,56] *can be generalized with our kernel.*

Conclusion

In this work we have taken up the recently defined Generalized Laplace Transform and have illustrated its usefulness with the analysis of the stability of generalized linear systems. Various results known from the literature can be obtained using this transform for the conformable case.

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