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The propagators for time-dependent mass harmonic oscillators

Abstract

In this paper, the propagator for a harmonic oscillator with mass obeying the function of $m(t) = m\tan^2 \nu t$ is derived by the Feynman path integral method. The wave function of this oscillator is calculated by expanding the obtained propagator. The propagator for a harmonic oscillator with strongly pulsating mass is evaluated by the Schwinger method. The propagator for a harmonic oscillator with mass rapidly growing with time is calculated by applying the integrals of the motion of quantum systems. The comparison between these methods are also discussed.

Keywords: Feynman path integral, Schwinger method, integrals of the motion, propagator

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Introduction

The research in deriving quantum solutions for a harmonic oscillator with time-dependent frequencies or with time-dependent masses (or both simultaneously) have intensive studied in the recent years.[1–7] The main reasons for time-dependent harmonic oscillator very interesting is the application in many areas of physics such as quantum chemistry, plasma physics, and quantum optics. For example, Colegrave and Abdalla[8,9] presented that electromagnetic field intensities in a Fabry-Perot cavity can be described by a harmonic oscillator with time-dependent mass and constant frequency. The standard tool in solving wave function of Schrodinger's equation for time-dependent harmonic oscillator is the Lewis-Riesenfeld in variant operator method.[10] The another method to solve the time-dependent harmonic oscillator problems is Feynman path integral.[11,12] The Feynman path integral is the formulation which is invented for calculating the propagator. The propagator represents the transition probability amplitude of the system or Green's function of the Schrodinger's equation. The Feynman path integral was applied to derive the propagators for a harmonic oscillator with strongly pulsating mass $\left(m(t) = m\cos^2 \nu t\right)$ and a harmonic oscillator with mass growing with time $\left(m(t) = m(1+\alpha t)^2\right)$ by M Sabir and S Raja gopalan in 1991.[3] The one aims of this paper is applying the Feynman path integral to calculate the propagator for a harmonic oscillator with mass obeying the function of $m(t) = m\tan^2 \nu t$ and then expanding the obtained propagator to evaluate the wave function. The another method in calculating the propagator is the Schwinger method.[12] This method was first introduced by Schwinger in 1951 in solving the gauge invariance and vacuum polarization in QED. In 2015 S Pepore and B Sukbot applied the Schwinger method to evaluate the propagator for a harmonic oscillator with mass growing with time.[13]The another purposes of this article is employing the Schwinger method to derive the propagator for a harmonic oscillator with strongly pulsating mass. The alternative techniques in deriving the propagator is applying the integrals of the motion of quantum systems. This method was first presented by VV Dodonov, IA Malk in, and V. I. Man'ko in 1975.[14] In 2018, S Pepore applied the integrals of the motion of quantum systems to calculate the Green function for time-dependent mass harmonic oscillators,[15] dual damped oscillators, and coupled harmonic oscillators.[16] The final aims of this paper is employing this method to calculate the propagator for a harmonic oscillator with mass rapidly growing with time $\left(m(t) = m(1+\alpha t)^4\right)$. The organizations of this paper are as follows. In Section 2, the propagator for a harmonic oscillator with mass having the function of $m(t) = m\tan^2 \nu t$ is derived by Feynman path integral method. In Section 3, the propagator for a harmonic oscillator with strongly pulsating mass is calculated by the Schwinger method. In Section 4, the propagator for a harmonic oscillator with mass rapidly growing with time is evaluated by the application of the integrals of the motion of quantum systems. Finally, the conclusion is presented in Section 5.

The Feynman path integral for a harmonic oscillator with mass obeying the law of $m(t) = m\tan^2 \nu t$

Colegrave and Abdalla[9] demonstrated that the electromagnetic field intensities in a Fabry-Perot cavity can be described by a harmonic oscillator with strongly pulsating mass written by the Hamiltonian of

$$H(t) = \frac{p^2}{2m}\sec^2 \nu t + \frac{1}{2}m\cos^2 \nu t\omega^2 x^2 \quad (1)$$

Where $m(t) = m\cos^2 \nu t$ and v is the frequency of a pulsating mass. This paper we will modify the time-dependent mass by imposing $m(t) = m\tan^2 \nu t$. The Hamiltonian of this system can be described by

$$H(t) = \frac{p^2}{2m}\cot^2 \nu t + \frac{1}{2}m\tan^2 \nu t\omega^2 x^2 \quad (2)$$

Where ω is the frequency of oscillator.

The Lagrangian corresponding with the Hamiltonian in Eq.(2) can be expressed as

$$L(x,\dot{x},t) = \frac{1}{2}m\tan^2 \nu t\dot{x}^2 - \frac{1}{2}m\tan^2 \nu t\omega^2 x^2 \quad (3)$$

By using the Euler-Lagrange equation,[17] the equation of motion can be written as

$$\ddot{x} + 2\nu cot\nu t\sec^2 \nu t\dot{x} + \omega^2 x = 0 \quad (4)$$

The Eq. (4) has the solution in the form of

$$x(t) = cot\nu t\left[Acos\Omega t + Bsin\Omega t\right], \quad (5)$$

Where A and B are constants and $\Omega^2 = \omega^2 + \nu^2$. By imposing $x(t') = x'$ and $x(t'') = x''$, the classical path that connects the points (x', t') and (x'', t'') can be written as

$$x_{cl}(t) = \frac{cot\nu t}{\sin \Omega T}\left\{tan\nu t'' \sin \Omega(t - t')x'' + tan\nu t' \sin \Omega(t'' - t)x'\right\} \quad (6)$$

Where $T = t'' - t'$.

The action of the classical systems can be calculated from

$$S(x, \dot{x}, t) = \int L(x, \dot{x}, t)dt. \quad (7)$$

Substituting the Lagrangian in Eq. (3) into Eq. (7), the classical action becomes

$$S_{cl}(t) = \frac{m}{2}\tan^2 \nu t'' x''_{cl}\dot{x}''_{cl} - \frac{m}{2}\tan^2 \nu t' x'_{cl}\dot{x}'_{cl}. \quad (8)$$

Inserting the classical path in Eq. (6) into Eq. (8), the classical action can be obtained as

$$S_{cl}(t) = \frac{m\Omega}{2}\cot \Omega T\left(\tan^2 \nu t'' x''^2 + \tan^2 \nu t' x'^2\right)$$
$$-\frac{m\nu}{2}\left(\tan \nu t'' \sec^2 \nu t'' x''^2 - \tan \nu t' \sec^2 \nu t' x'^2\right) - \frac{m\Omega}{sin\Omega T}tan\nu t' tan\nu t'' x' x'' \quad .(9)$$

As suggested by Feynman [10], the Green function for a quadratic Lagrangian can be written as

$$K(x'', t''; x', t') = F(t'', t')e^{\frac{iS_{cl}(x'', t''; x', t')}{\hbar}} \quad (10)$$

where the pre-exponential function $F(t'', t')$ can be calculated from Pauli-Van Vleck[18,19] as

$$F(t'', t') = \left[\frac{i}{2\pi\hbar}\frac{\partial^2 S_{cl}(x'', t''; x', t')}{\partial x' \partial x''}\right]^{\frac{1}{2}}. \quad (11)$$

By applying Eq. (11), the pre-exponential function $F(t'', t')$ becomes

$$F(t'', t') = \left[\frac{m\Omega tan\nu t' tan\nu t''}{2\pi\hbar sin\,\Omega T}\right]^{\frac{1}{2}}. \quad (12)$$

Substituting Eqs. (9) and (12) into Eq. (10), the propagator of this oscillator can be written as

$$K(x'', t''; x', t') = \left[\frac{m\Omega tan\nu t' tan\nu t''}{2\pi\hbar sin\,\Omega T}\right]^{\frac{1}{2}}$$
$$\times \exp\left(-\frac{im\nu}{2\hbar}\left[tan\nu t'' \sec^2 \nu t'' x''^2 - tan\nu t' \sec^2 \nu t' x'^2\right]\right)$$
$$\times \exp\left(\frac{im\Omega}{2\hbar sin\Omega T}\left(\left(\tan^2 \nu t'' x''^2 + \tan^2 \nu t' x'^2\right)cos\Omega T - 2tan\nu t' tan\nu t'' x' x''\right)\right) \quad (13)$$

The next task is calculating the wave function. Beginning by defining

$$z = e^{-i\varphi}, \varphi = \Omega(t'' - t'), \quad (14)$$

$$sin\varphi = \frac{1 - z^2}{2iz}, cos\varphi = \frac{1 + z^2}{2z}, \quad (15)$$

$$\alpha = \sqrt{\frac{m\Omega}{\hbar}}tan\nu t'' x'', \beta = \sqrt{\frac{m\Omega}{\hbar}}tan\nu t' x' \quad (16)$$

the propagator in Eq. (13) can be rewritten as

$$K(x'', t''; x', t') = \left[\frac{m\Omega tan\nu t' tan\nu t'' z}{\pi\hbar}\right]^{\frac{1}{2}}\left(1 - z^2\right)^{-\frac{1}{2}}$$
$$\times \exp\left(-\frac{im\nu}{2\hbar}\left[tan\nu t'' \sec^2 \nu t'' x''^2 - tan\nu t' \sec^2 \nu t' x'^2\right]\right)$$
$$\times \exp\left\{\frac{1}{1 - z^2}\left[2\alpha\beta z - \left(\alpha^2 + \beta^2\right)\left(\frac{1 + z^2}{2}\right)\right]\right\}. \quad (17)$$

By using the formula

$$\frac{1 + z^2}{2\left(1 - z^2\right)} = \frac{1}{2} + \frac{z^2}{1 - z^2}, \quad (18)$$

The propagator in Eq. (17) can be modified to

$$K(x'', t''; x', t') = \left[\frac{m\Omega tan\nu t' tan\nu t'' z}{\pi\hbar}\right]^{\frac{1}{2}}\left(1 - z^2\right)^{-\frac{1}{2}}$$
$$\times \exp\left(-\frac{im\nu}{2\hbar}\left[tan\nu t'' \sec^2 \nu t'' x''^2 - tan\nu t' \sec^2 \nu t' x'^2\right]\right)$$
$$\times \exp\left[-\frac{1}{2}\left(\alpha^2 + \beta^2\right)\right] \times \exp\left[\frac{2\alpha\beta z - \left(\alpha^2 + \beta^2\right)z^2}{1 - z^2}\right]. \quad (19)$$

The next step is applying the Mehler's formula [20]

$$\left(1 - z^2\right)^{-\frac{1}{2}}\exp\left[\frac{2\alpha\beta z - \left(\alpha^2 + \beta^2\right)z^2}{1 - z^2}\right] = \sum_{n=0}^{\infty} H_n(\alpha)H_n(\beta)\frac{z^n}{2^n n!}, \quad (20)$$

Where $H_n(\alpha)$ and $H_n(\beta)$ are the Hermite polynomials. The propagator in Eq.(19) becomes

$$K(x'', t''; x', t') = \left[\frac{m\Omega tan\nu t' tan\nu t''}{\pi\hbar}\right]^{\frac{1}{2}}$$
$$\times \exp\left(-\frac{im\nu}{2\hbar}\left[tan\nu t'' \sec^2 \nu t'' x''^2 - tan\nu t' \sec^2 \nu t' x'^2\right]\right)$$
$$\times \exp\left[-\frac{m\Omega}{2\hbar}\left(\tan^2 \nu t'' x''^2 + \tan^2 \nu t' x'^2\right)\right]$$
$$\times \sum_{n=0}^{\infty} H_n\left(\sqrt{\frac{m\Omega}{\hbar}}tan\nu t'' x''\right)H_n\left(\sqrt{\frac{m\Omega}{\hbar}}tan\nu t' x'\right)\frac{e^{-i\Omega(t'' - t')\left(n + \frac{1}{2}\right)}}{2^n n!}. \quad (21)$$

By using the spectral representation of the propagator

$$K(x'', t''; x', t') = \sum_{n=0}^{\infty} \psi_n^*(x'', t'')\psi_n(x', t'), \quad (22)$$

The wave function of this system can be written as

$$\psi_n(x,t)=\left[\frac{tanvt}{2^n n!}\left(\frac{m\Omega}{\pi\hbar}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}\exp\left[-i\Omega\left(n+\frac{1}{2}\right)t\right]\exp\left[\frac{imv}{2\hbar}tanvt\sec^2 vtx^2\right]$$

$$\times\exp\left[-\frac{m\Omega}{2\hbar}\tan^2 vtx^2\right]H_n\left(\sqrt{\frac{m\Omega}{\hbar}}tanvtx\right). \quad (23)$$

The Schwinger method for a harmonic oscillator with strongly pulsating mass

This section is the evaluation of propagator for a harmonic oscillator with strongly pulsating mass by the Schwinger method.[12] The procedures of Schwinger method are presented in Ref.[13] The Hamiltonian operator for a harmonic oscillator with strongly pulsating mass can be written as[3]

$$\hat{H}(\tau)=\frac{\hat{p}^2(\tau)}{2m}\sec^2 v\tau+\frac{1}{2}m\cos^2 v\tau\omega^2\hat{x}^2(\tau). \quad (24)$$

By solving the Heisenberg equations

$$\hbar\frac{d\hat{x}(\tau)}{d\tau}=\left[\hat{x}(\tau)\hat{H}(\tau)\right]i\hbar\frac{d\hat{p}(\tau)}{d\tau}=\left[\hat{p}(\tau)\hat{H}(\tau)\right] \quad (25)$$

and imposing the initial conditions of $\hat{x}(\tau=0)=\hat{x}(0)$ and $\hat{p}(\tau=0)=\hat{p}(0)$,

the position operator can be expressed as

$$\hat{x}(\tau)=\left(secv\tau cosv\tau' cos\Omega(\tau-\tau')-\frac{v}{\Omega}secv\tau sinv\tau' sin\Omega(\tau-\tau')\right)\hat{x}(0)$$

$$+\frac{1}{m\Omega}secv\tau secv\tau' sin\Omega(\tau-\tau')\hat{p}(0). \quad (26)$$

By using Eq. (26), the momentum operator $\hat{p}(\tau)=m\cos^2 v\tau\dot{x}(\tau)$ can be written as

$$\hat{p}(\tau)=(mvsinv(\tau-\tau')cos\Omega(\tau-\tau')-m\Omega cosv\tau' cosv\tau sin\Omega(\tau-\tau')$$

$$-\frac{mv^2}{\Omega}sinv\tau' sinv\tau sin\Omega(\tau-\tau'))\hat{x}(0)+(secv\tau' cosv\tau cos\Omega(\tau-\tau')$$

$$+\frac{v}{\Omega}secv\tau' sinv\tau sin\Omega(\tau-\tau'))\hat{p}(0). \quad (27)$$

By applying Eq. (26) to eliminate $\hat{p}(0)$ in Eq. (27), the momentum operator can

be rewritten only in terms of $\hat{x}(\tau)$ and $\hat{x}(0)$ as

$$\hat{p}(\tau)=\left(m\Omega\cos^2 v\tau cot\Omega(\tau-\tau')+mvsinv\tau cosv\tau\right)\hat{x}(\tau)$$

$$-m\Omega cosv\tau' cosv\tau csc\Omega(\tau-\tau')\hat{x}(0). \quad (28)$$

Substituting Eq. (28) into Eq. (24) and rewriting each terms of $\hat{H}(\tau)$ in a time ordered form with $\hat{x}(\tau)$ to the left and $\hat{x}(0)$ to the right with the helping of the commutator

$$\left[\hat{x}(0),\hat{x}(\tau)\right]=\frac{i\hbar}{m\Omega}secv\tau secv\tau' sin\Omega(\tau-\tau'),$$ the time ordered Hamiltonian operator $\hat{H}_{ord}(\tau)$ can be written as

$$\hat{H}_{ord}(\tau)=(\frac{1}{2}m\Omega^2\cos^2 v\tau\csc^2\Omega(\tau-\tau')+\frac{1}{2}mv^2\sin^2 v\tau-\frac{1}{2}mv^2\cos^2 v\tau$$

$$+\frac{1}{2}mv\Omega sinv\tau cosv\tau cot\Omega(\tau-\tau'))\hat{x}^2(\tau)-(m\Omega^2 cosv\tau cosv\tau' csc\Omega(\tau-\tau')cot\Omega(\tau-\tau')$$

$$+mv\Omega cosv\tau' sinv\tau csc\Omega(\tau-\tau'))\hat{x}(\tau)\hat{x}(0)+\frac{1}{2}m\Omega^2\cos^2 v\tau'\csc^2\Omega(\tau-\tau')\hat{x}^2(0)$$

$$-\frac{i\hbar\Omega}{2}cot\Omega(\tau-\tau')-\frac{i\hbar v}{2}tanv\tau. \quad (29)$$

As presented in Ref.[13] the propagator can be calculated by

$$K(x,x';\tau)=C(x,x')\exp\left(-\frac{i}{\hbar}\int_0^\tau\frac{\langle x(t)|\hat{H}_{ord}(t)|x'(0)\rangle}{\langle x(t)|x'(0)\rangle}dt\right)$$

$$=C(x,x')\exp(-\frac{i}{\hbar}\int_0^\tau\{(\frac{1}{2}m\Omega^2\cos^2 vt\csc^2\Omega(t-\tau')+\frac{1}{2}mv\Omega sinvtcosvtcot\Omega(t-\tau')$$

$$\frac{1}{2}mv^2\sin^2 vt-\frac{1}{2}mv^2\cos^2 vt)x^2-(m\Omega^2 cosvtcosv\tau' csc\Omega(t-\tau')cot\Omega(t-\tau')$$

$$+mv\Omega cosv\tau' sinvtcsc\Omega(t-\tau'))xx'+\frac{1}{2}m\Omega^2\cos^2 v\tau'\csc^2\Omega(t-\tau')x'^2$$

$$-\frac{i\hbar\Omega}{2}cot\Omega(t-\tau')-\frac{i\hbar v}{2}tanvt\}dt), \quad (30)$$

Where $C(x,x')$ is the function of x and x'. The next step is integrating over time each terms of Eq. (30). The integrating of the first term in Eq. (30) can be obtained as

$$-\frac{imx^2}{2\hbar}\int_0^\tau(\Omega^2\cos^2 vt\csc^2\Omega(t-\tau')+v\Omega sinvtcosvtcot\Omega(t-\tau')+v^2\sin^2 vt$$

$$-v^2\cos^2 vt)dt=\frac{imv}{2\hbar}cosv\tau sinv\tau x^2+\frac{im\Omega}{2\hbar}\cos^2 v\tau cot\Omega(\tau-\tau')x^2. \quad (31)$$

The second term in Eq. (30) can be calculated by

$$\frac{im\Omega}{\hbar}cosv\tau' xx'\int_0^\tau\left(\Omega cosvtcsc\Omega(t-\tau')cot\Omega(t-\tau')+vsinvtcsc\Omega(t-\tau')\right)dt$$

$$=-\frac{im\Omega}{\hbar}cosv\tau' cosv\tau csc\Omega(\tau-\tau')xx'. \quad (32)$$

The third term in Eq. (30) can be integrated as

$$-\frac{im\Omega^2}{2\hbar}\cos^2 v\tau' x'^2\int_0^\tau\csc^2\Omega(t-\tau')dt=\frac{im\Omega}{2\hbar}\cos^2 v\tau' cot\Omega(\tau-\tau')x'^2. \quad (33)$$

Finally, the last term in Eq. (30) can be evaluated by

$$-\frac{\Omega}{2}\int_0^\tau cot\Omega(t-\tau')dt-\frac{v}{2}\int_0^\tau tanvtdt=\ln\left[\frac{cosv\tau}{sin\Omega(\tau-\tau')}\right]^{\frac{1}{2}}. \quad (34)$$

Substituting Eqs. (31)-(34) into Eq. (30), the propagator can be written as

$$K(x,x';\tau)=C(x,x')\left[\frac{cos\nu\tau}{\hbar sin\Omega(\tau-\tau')}\right]^{\frac{1}{2}}\exp\left(\frac{im\nu}{2}sin\nu\tau cos\nu\tau x^2\right)$$

$$\times\exp\left(\frac{im\Omega}{2\hbar sin\Omega(\tau-\tau')}\left[\left(\cos^2\nu\tau x^2+\cos^2\nu\tau' x'^2\right)cos\Omega(\tau-\tau')-2cos\nu\tau' cos\nu\tau xx'\right]\right). \quad (35)$$

Substituting the propagator in Eq. (35) into the equation of

$$\hbar\frac{\partial K(x,x';\tau)}{\partial x'}=\left\langle x(\tau)\middle|\hat{p}(0)\middle|x'(0)\right\rangle, \quad (36)$$

the result is

$$\hbar\frac{\partial C(x,x')}{\partial x'}=m\nu cos\nu\tau' sin\nu\tau' x' C(x,x'). \quad (37)$$

After solving Eq. (37), the function $C(x,x')$ can be obtained as

$$C(x,x')=C(x)\exp\left(-\frac{im\nu}{2\hbar}sin\nu\tau' cos\nu\tau' x'^2\right), \quad (38)$$

Where $C(x)$ is the function of x.

Substituting Eq. (38) into Eq. (35), the propagator becomes

$$K(x,x';\tau)=C(x)\left[\frac{cos\nu\tau}{\hbar sin\Omega(\tau-\tau')}\right]^{\frac{1}{2}}\exp\left(\frac{im\nu}{2}\left[sin\nu\tau cos\nu\tau x^2-sin\nu\tau' cos\nu\tau' x'^2\right]\right)$$

$$\times\exp\left(\frac{im\Omega}{2\hbar sin\Omega(\tau-\tau')}\left[\left(\cos^2\nu\tau x^2+\cos^2\nu\tau' x'^2\right)cos\Omega(\tau-\tau')-2cos\nu\tau' cos\nu\tau xx'\right]\right). \quad (39)$$

The final step is calculating $C(x)$ by substituting Eq. (39) into the equation of

$$-\hbar\frac{\partial K(x,x';\tau)}{\partial x}=\left\langle x(\tau)\middle|\hat{p}(\tau)\middle|x'(0)\right\rangle. \quad (40)$$

The obtaining result is

$$\frac{\partial C(x)}{\partial x}=0, \quad (41)$$

which imply that $C(x)=C=constant$. The constant C can be evaluated by

applying the initial condition of the propagator

$$\lim_{\tau\to 0^+}K(x,x';\tau)=\delta(x-x'). \quad (42)$$

The constant C can be obtained as

$$C=\sqrt{\frac{m\Omega cos\nu\tau'}{2\pi\hbar}}. \quad (43)$$

So, the propagator for a harmonic oscillator with strongly pulsating mass can be written as

$$K(x,x',\tau)=\left[\frac{m\Omega cos\nu\tau cos\nu\tau'}{2\pi\hbar sin\,\Omega(\tau-\tau')}\right]^{\frac{1}{2}}\exp\left(\frac{im\nu}{\hbar}\left[sin\nu\tau cos\nu\tau x^2-sin\nu\tau' cos\nu\tau' x'^2\right]\right)$$

$$\times\exp\left(\frac{im\Omega}{2\hbar sin\Omega(\tau-\tau')}\left[\left(\cos^2\nu\tau x^2+\cos^2\nu\tau' x'^2\right)cos\Omega(\tau-\tau')-2cos\nu\tau' cos\nu\tau xx'\right]\right). \quad (44)$$

The propagator for a harmonic oscillator with mass rapidly with time

This section has an idea from the calculation of the Green function for a harmonic oscillator with mass growing with time by S Pepore in 2018.[15] This paper will modify the Hamiltonian operator to

$$\hat{H}(t)=\frac{\hat{p}^2(t)}{2m(1+\alpha t)^4}+\frac{1}{2}m(1+\alpha t)^4\omega^2\hat{x}^2(t), \quad (45)$$

Where α is a constant. The aim of this section is calculating the propagator corresponding to the Hamiltonian operator in Eq. (45) by the application of the integrals of the motion of quantum systems. Beginning by solving Heisenberg's equation for $\hat{x}(t)$ and $\hat{p}(t)$ and imposing the initial conditions of $\hat{x}(0)=\hat{x}_0$ and $\hat{p}(0)=\hat{p}_0$, the position operator and momentum operator can be written as

$$\hat{x}(t)=\frac{1}{(1+\alpha t)^2}\left[cos\omega t+\frac{2\alpha}{\omega}sin\omega t\right]\hat{x}_0+\left(\frac{sin\omega t}{m\omega(1+\alpha t)^2}\right)\hat{p}_0, \quad (46)$$

$$\hat{p}(t)=\left[\left(2m\alpha(1+\alpha t)^2-2m\alpha(1+\alpha t)\right)cos\omega t-\left(m\omega(1+\alpha t)^2+\frac{4m\alpha^2}{\omega}(1+\alpha t)\right)sin\omega t\right]\hat{x}_0+\left[(1+\alpha t)^2cos\omega t-\frac{2\alpha}{\omega}(1+\alpha t)sin\omega t\right]\hat{p}_0 \quad (47)$$

By using Eq. (46) and Eq. (47), the integrals of the motion $\hat{x}_0$ and $\hat{p}_0$ can be expressed as

$$\hat{x}_0\left(\hat{x},\hat{p},t\right)=\left[(1+\alpha t)^2cos\omega t-\frac{2\alpha}{\omega}(1+\alpha t)sin\omega t\right]\hat{x}-\left(\frac{sin\omega t}{m\omega(1+\alpha t)^2}\right)\hat{p}, \quad (48)$$

$$\hat{p}_0\left(\hat{x},\hat{p},t\right)=\left[\left(m\omega(1+\alpha t)^2+\frac{4m\alpha^2}{\omega}(1+\alpha t)\right)sin\omega t-\left(2m\alpha(1+\alpha t)^2-2m\alpha(1+\alpha t)\right)cos\omega t\right]\hat{x}+\left[\frac{cos\omega t}{(1+\alpha t)^2}+\frac{2\alpha sin\omega t}{\omega(1+\alpha t)^2}\right]\hat{p}. \quad (49)$$

The Green function or propagator $K(x,x',t)$ is an eigen function of the integrals of the motion as

$$\hat{x}_0(x)K(x,x',t)=\hat{x}(x')K(x,x',t), \quad (50)$$

$$\hat{p}_0(x)K(x,x',t)=-\hat{p}(x')K(x,x',t). \quad (51)$$

By applying Eqs. (48)-(51), we can write

$$\left[x\left((1+\alpha t)^2\cos\omega t-\frac{2\alpha}{\omega}(1+\alpha t)\sin\omega t\right)+\frac{i\hbar\sin\omega t}{m\omega(1+\alpha t)^2}\frac{\partial}{\partial x}\right]K(x,x',t)=x'K(x,x',t), \quad (52)$$

$$[x\left(\left(m\omega(1+\alpha t)^2+\frac{4m\alpha^2}{\omega}(1+\alpha t)\right)\sin\omega t-\left(2m\alpha(1+\alpha t)^2-2m\alpha(1+\alpha t)\right)\cos\omega t\right)$$

$$-i\hbar\left(\frac{\cos\omega t}{(1+\alpha t)^2}+\frac{2\alpha\sin\omega t}{\omega(1+\alpha t)^2}\right)\frac{\partial}{\partial x}]K(x,x',t)=i\hbar\frac{\partial K(x,x',t)}{\partial x'}. \quad (53)$$

For solving the propagator, we must rewrite Eq. (52) and Eq. (53) to

$$\frac{\partial K(x,x',t)}{\partial x}=\frac{im\omega}{\hbar}\left\{\left((1+\alpha t)^4\cot\omega t-\frac{2\alpha}{\omega}(1+\alpha t)^3\right)x-(1+\alpha t)^2\csc\omega t x'\right\}K(x,x',t), \quad (54)$$

$$\frac{\partial K(x,x',t)}{\partial x'}=-\frac{im\omega}{\hbar}\left\{(1+\alpha t)^2\csc\omega t x-\left(\cot\omega t+\frac{2\alpha}{\omega}\right)x'\right\}K(x,x',t). \quad (55)$$

Solving Eq. (54), we obtain

$$K(x,x',t)=C(x',t)\exp\left\{\frac{i}{\hbar}\left[\frac{m\omega}{2}(1+\alpha t)^4\cot\omega t-m\alpha(1+\alpha t)^3\right)x^2-m\omega(1+\alpha t)^2\csc\omega t xx'\right]\right\}. \quad (56)$$

The constant of integration $C(x',t)$ can be calculated by substituting Eq. (56) into Eq. (55) to obtain

$$\frac{\partial C(x',t)}{\partial x'}=\frac{im\omega}{\hbar}\left(\cot\omega t+\frac{2\alpha}{\omega}\right)x'C(x',t). \quad (57)$$

Solving Eq. (57), the result is

$$C(x',t)=C(t)\exp\left[\frac{i}{\hbar}\left(\frac{m\omega}{2}\cot\omega t x'^2+m\alpha x'^2\right)\right]. \quad (58)$$

Substituting Eq. (58) into Eq. (56), the propagator can be written as

$$K(x,x',t)=C(t)\exp[\frac{i}{\hbar}(\frac{m\omega}{2}\cot\omega t\left((1+\alpha t)^4x^2+x'^2\right)$$

$$-m\alpha\left((1+\alpha t)^3x^2-x'^2\right)-m\omega(1+\alpha t)^2\csc\omega t xx')]. \quad (59)$$

The next step is calculating $C(t)$. Substituting the propagator in Eq. (59) into the Schrodinger equation

$$i\hbar\frac{\partial K(x,x',t)}{\partial t}=-\frac{\hbar^2}{2m(1+\alpha t)^4}\frac{\partial^2K(x,x',t)}{\partial x^2}+\frac{1}{2}m(1+\alpha t)^4\omega^2x^2K(x,x',t), \quad (60)$$

we obtain

$$\frac{dC(t)}{dt}=\left(-\frac{1}{2}\omega\cot\omega t+\frac{\alpha}{(1+\alpha t)}\right)C(t). \quad (61)$$

Solving Eq. (61), the result is

$$C(t)=C\left(\frac{(1+\alpha t)^2}{\sin\omega t}\right)^{\frac{1}{2}}. \quad (62)$$

The final step is finding the constant C by using the initial condition of the propagator

$$\lim_{t\to 0^+}K(x,x',t)=\delta(x-x'). \quad (63)$$

The constant C becomes

$$C=\sqrt{\frac{m\omega}{2\pi i\hbar}}. \quad (64)$$

So, the propagator for a harmonic oscillator with mass rapidly growing with time can be written as

$$K(x,x',t)=\left[\frac{m\omega(1+\alpha t)^2}{2\pi i\hbar\sin\omega t}\right]^{\frac{1}{2}}\exp[\frac{i}{\hbar}(\frac{m\omega}{2}\cot\omega t\left((1+\alpha t)^4x^2+x'^2\right)-m\alpha\left((1+\alpha t)^3x^2-x'^2\right)-m\omega(1+\alpha t)^2\csc\omega t xx')]. \quad (65)$$

Conclusion

We have successfully derived the propagator and wave function for a harmonic oscillator with mass obeying the law of $m(t)=m\tan^2\nu t$ by Feynman path integral, the propagator for a harmonic oscillator with strongly pulsating mass by Schwinger method, and the propagator for a harmonic oscillator with mass rapidly growing with time by the application of the integrals of the motion of quantum systems. The Feynman formulation base on functional integration. The Schwinger method concern with operator algebra. The method of VV Dodonovet.al is applying the integrals of the motion operators $\hat{x}_0$ and $\hat{p}_0$. The pre-exponential function $F(t'',t')$ in Feynman path integral comes from the summation over all fluctuation amplitudes of classical paths. In Schwinger method, the pre-exponential function $C(x,x')$ appears from the commutator of $\left[\hat{x}(\tau),\hat{x}(0)\right]$. In the method of VV Dodonov et.al, the pre-exponential function $C(t)$ comes from the fact that the propagators is the solution of Schrodinger equation. These differences may shows the nature of classical mechanics which the physical observables are real numbers and the nature of quantum mechanics which the physical quantities are represented by operators. The propagator approaches in this paper are alternative methods comparison with the Schrodinger wave mechanics which base on finding wave function and the Heisenberg formulation which requires applying the creation and annihilation operators to derive the eigen functions of Hamiltonian operator. In propagator method, the wave function can be calculated by expanding the obtained propagator. In the calculation point of views, the Feynman path integral more simply than the Schwinger and Dodonov et. al. methods which some Hamiltonian operators have difficulties in solving the Heisenberg equation. However, having several methods in calculating the propagators may be usefulness.

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Conflicts of interest

The author declares there is no conflict of interest.

References

1. AB Nassar, JMF Bassalo, PDT Alencar. Nonlinear superposition law and Feynman propagator, *Phys. Lett. A*. 1986;113:365.
2. CFD Souza, ADS Dutra. The propagator for a time–dependent mass subject to a harmonic potential with a time–dependent frequencies. *Phys Lett. A*. 1987;123:297.
3. M Sabir, S Rajagopalan. Path integral analysis of harmonic oscillators with time–dependent mass. Pramana. *J Phys.* 1991;37:253.
4. S Mandal. On the squeezing of coherent light coupled to a driven damped harmonic oscillator with time–dependent frequency. *Phys Lett A.* 2004;321:308.
5. SK Singh, S Mandal. The solutions of the generalized classical and quantum harmonic oscillators with time–dependent mass, frequency, two–photon parameter and external force: the squeezing effects, *Opt Commun*. 2010;283:4685.
6. B Berrabah. Quantum mechanical path integral in phase space and class of harmonic oscillators with varied frequencies. *Journal of Modern Physics.* 2016;7:359.
7. S Mandal. Classical and quantum harmonic oscillators with time–dependent mass and frequency: a new class of exactly solvable model. *Opt Commun.* 2017;386:37.
8. RK Colegrave, MS Abdalla. Harmonic oscillator with exponential decaying mass. *J Phys A*. 1981;14:2269.
9. RK Colegrave, MS Abdalla. Harmonic oscillator with strongly pulsating mass. *J Phys A*. 1982;15:1549.
10. RP Feynman, AR Hibbs, Quantum Mechanics and Path Integral. McGraw–Hill. New York. 1965.
11. S Pepore, P Winotai, T Osotchan, et al. Path integral for a harmonic oscillator with time–dependent mass and frequency. *Science Asia.* 2006;32:173.
12. J Schwinger. The theory of quantized field. *Phys Rev*. 1951;82:664.
13. S Pepore, B Sukbot. Schwinger method and Feynman path integral for a harmonic oscillator with mass growing with time, Chinese. *J Phys.* 2015;53:060004.
14. VV Dodonov, IA Malkin, VI Man'ko. Integrals of the motion, Green functions and coherent states of dynamical systems, Int. *J Theor Phys.* 1975;14:37.
15. S Pepore. Integrals of the motion and Green functions for time–dependent mass harmonic oscillators, *Rev Mex* Fis. 2018;64:30.
16. S Pepore. Integrals of the motion and Green functions for dual damped oscillators and coupled harmonic oscillators. *Rev Mex Fis*. 2018;64:150.
17. H Goldstein, C Poole, J Safko. Classical Mechanics. Addison– Wesley, San Francisco. 2000.
18. W Pauli. Selected chapters of the field quantization, ETH, Zurice, 1952.
19. C Morette. On the definition and approximation of Feynman's path integral. *Phys Rev.* 1951;81:848.
20. A Erdelyi. Higher Transcendental Functions. McGra–Hill, New York, 1953.

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