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Propagators for a particle in a time-dependent linear potential and a free particle

Abstract

In this paper, the propagators for a particle moving in a time-dependent linear potential and a free particle with linear damping are calculated by the application of the integrals of the motion of a quantum system. The propagator for a charged harmonic oscillator is derived from the Feynman path integrals and the propagator for a damped harmonic oscillator is evaluated by the Schwinger method. The relation between the integrals of the motion, Feynman path integrals, and Schwinger method are also described.

Keywords: propagators, free particle, harmonic oscillator, integrals of the motion, feynman path integrals, schwinger method

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Introduction

In quantum mechanics and quantum field theory, the propagator or Green function is represented as the transition probability amplitude for a particle to travel from initial space-time configuration to final space-time configuration. The standard method in calculating the propagator is Feynman path integral.[1] In 2006, S.Pepore and et al.[2] applied the Feynman path integral method to Calculate the propagator for a harmonic oscillator with time-dependent mass and frequency. The one aim of this paper is using the path integral method to derive the propagator for a charged harmonic oscillator in time-dependent electric field.

The another method in calculating the propagator is the Schwinger method.[3] This method was first formulated by Schwinger in 1951 for solving the gauge invariance and vacuum polarization in QED. In 2015, the Schwinger method was used to derive the propagator for time-dependent harmonics oscillators by S.Pepore and B.Sukbot.[4–6] The one purposes of this article is applying the Schwinger method to calculate the propagator for a damped harmonic oscillator.

In 1975, V.V. Dodonov, I.A. Malkin, and V.I. Man'ko[7] presented the connection between the integrals of the motion of a quantum system and its propagator that is the eigenfunction of the integrals of the motion describing initial points of the system trajectory in the phase space. In 2018, S. Pepore applied the integrals of the motion to calculate the propagators for time-dependent harmonic oscillators.[8,9] The one aim of this article is applying the integrals of the motion to derive the propagators for a particle moving in a time-dependent linear potential and a free particle with linear damping. The organization of this paper are as follows. In Sec.2, the propagator for a particle in a time-dependent linear potential is derived. In Sec.3, the propagator for a free particle with linear damping is obtained with the aid of the integrals of the motion. In Sec 4, the Feynman path integrals is applied to evaluate the propagator for a charged harmonic oscillator in time-dependent electric field. In Sec.5, the procedures of the Schwinger method are described. In Sec.6, the propagator for a damped harmonic oscillator are derived by the Schwinger method. Finally, the conclusion is presented in Sec.8.

The propagator for a particle moving in a time-dependent linear potential

In this section, we will calculate the propagator for a particle moving in a time-dependent linear potential described by the Hamiltonian operator.[6]

$$\hat{H}(t)=\frac{\hat{p}^2}{2m}-kt\hat{x} \tag{1}$$

Where k is a constant and t is time.

The classical equation of motion for this system is

$$m\ddot{x}-kt=0 \tag{2}$$

The classical paths in the phase space under the initial conditions $x(0)=x_0$ and $P(0)=P_0$ are given by

$$x(t)=x_0+\frac{t}{m}p_0+\frac{kt^3}{6m} \tag{3}$$

$$p(t)=p_0+\frac{kt^2}{2}. \tag{4}$$

Now we consider the systems of Eqs.(3) and (4) as an algebraic system for unknown initial position x_0 and initial momentum p_0. The variables $x, p,$ and t are taken as the parameters. The solution of this system can be written as the operator in Hilbert space as

$$\hat{x}_0\left(\hat{x},\hat{p},t\right)=\hat{x}-\frac{t}{m}\hat{p}+\frac{kt^3}{3m} \tag{5}$$

$$\hat{p}_0\left(\hat{x},\hat{p},t\right)=\hat{p}-\frac{kt^2}{2}. \tag{6}$$

The operators $\hat{x}_0$ and $\hat{p}_0$ are the integrals of the motion because theirs satisfy equation of

$$\frac{d\hat{I}}{dt}=\frac{\partial\hat{I}}{\partial t}+\frac{i}{\hbar}\left[\hat{H},\hat{I}\right]=0, \tag{7}$$

Where $\hat{I}$ may be $\hat{x}_0$ and $\hat{p}_0$. Then these operatos must satisfy equations for the Green function or propagator,[2,5,6]

$$\hat{x}_0(x)K(x,x',t)=\hat{x}(x')K(x,x',t) \tag{8}$$

$$\hat{p}_0(x)K(x,x',t)=-\hat{p}(x')K(x,x',t), \tag{9}$$

where the operators on the left-hand sides of the equations act on variables x, and on the right-hand sides, on x'.

Now we write Eqs.(8) and (9) explicitly,

$$\left(x+\frac{i\hbar t}{m}\frac{\partial}{\partial x}+\frac{kt^3}{3m}\right)K(x,x',t)=x'K(x,x',t), \quad (10)$$

$$\left(-i\hbar\frac{\partial}{\partial x}-\frac{kt^2}{2}\right)K(x,x',t)=i\hbar\frac{\partial K(x,x',t)}{\partial x'}. \quad (11)$$

By modifying Eqs.(10) and (11), the system of equation for deriving the propagator are

$$\frac{\partial K(x,x',t)}{\partial x}=\left[\frac{im(x-x')}{\hbar t}+\frac{ikt^2}{3\hbar}\right]K(x,x',t), \quad (12)$$

$$\frac{\partial K(x,x',t)}{\partial x'}=\left[-\frac{im(x-x')}{\hbar t}+\frac{ikt^2}{6\hbar}\right]K(x,x',t). \quad (13)$$

Now one can integrate Eq. (12) with respect to the variable x to obtain

$$K(x,x',t)=C(x',t)\exp\exp\left[\frac{i}{\hbar}\left(\frac{m(x-x')^2}{2t}+\frac{kt^2}{3}x\right)\right], \quad (14)$$

Substituting Eq.(14) into Eq.(13), we obtain the differential equation for $C(x',t)$ as

$$\frac{\partial C(x',t)}{\partial x'}=\left(\frac{ikt^2}{6\hbar}\right)C(x',t). \quad (15)$$

Solving Eq.(15), the function $C(x',t)$ can be expressed as

$$C(x',t)=C(t)\exp\exp\left[\frac{ikt^2}{6\hbar}x'\right]. \quad (16)$$

So, the propagator in Eq.(14) can be written as

$$K(x,x',t)=C(t)\exp\exp\left[\frac{i}{\hbar}\left(\frac{m(x-x')^2}{2t}+\frac{kt^2}{3}x+\frac{kt^2}{6}x'\right)\right]. \quad (17)$$

To obtain $C(t)$, we must substitute the propagator of Eq. (17) into the Schrodinger's equation

$$i\hbar\frac{\partial K(x,x',t)}{\partial t}=\left(-\frac{\hbar^2}{2m}\frac{\partial^2 K(x,x',t)}{\partial x^2}-ktxK(x,x',t)\right). \quad (18)$$

After some algebra, we obtain an equation

$$\frac{dC(t)}{dt}=-\left(\frac{1}{2t}+\frac{ik^2t^4}{18\hbar m}\right)C(t). \quad (19)$$

Equation (19) can be simply integrated with respect to time t ,and one obtains

$$C(t)=\frac{C}{\sqrt{t}}\exp\exp\left(-\frac{i}{\hbar}\frac{k^2t^5}{90m}\right), \quad (20)$$

where C is a constant. Substituting Eq.(20) into Eq.(17) and applying the initial condition

$$\lim_{t\to 0^+}K(x,x',t)=\delta(x-x'), \quad (21)$$

we obtain

$$C=\sqrt{\frac{m}{2\pi i\hbar}}. \quad (22)$$

So, the propagator for a particle moving in time-dependent linear potential is

$$K(x,x',t)=\sqrt{\frac{m}{2\pi i\hbar t}}\exp\exp\left[\frac{i}{\hbar}\left(\frac{m(x-x')^2}{2t}+\frac{kt^2}{3}x+\frac{kt^2}{6}x'-\frac{k^2t^5}{90m}\right)\right], \quad (23)$$

which is the same form as the result of S.Pepore and B.Sukbot calculated by the Schwinger method.[6]

The propagator for a free particle with linear damping

This section is the calculation of the propagator for a free particle with linear damping by the application of integrals of motion operators. Considering the motion of a free particle with constant mass m in a linear damping which has the damping coefficient β, the Hamiltonian operator of this system can be written as[10]

$$\hat{H}(t)=\frac{e^{-\gamma t}\hat{p}^2}{2m}, \quad (24)$$

Where $\gamma=\frac{\beta}{m}$.

The Hamilton equation of motion for position and momentum are[11]

$$\dot{x}=\frac{p}{m}e^{-\gamma t},\ \dot{p}=0. \quad (25)$$

The classical paths in the phase space under the initial conditions $x(0)=x_0$ and $p(0)=p_0$ are given by

$$x(t)=x_0+\frac{\left(1-e^{-\gamma t}\right)}{m\gamma}p_0, \quad (26)$$

$$p(t)=p_0. \quad (27)$$

Now we rewrite the systems of Eqs. (26) and (27) in terms of the initial position operator x_0 and initial momentum operator p_0 5as

$$\hat{x}_0\left(\hat{x},\hat{p},t\right)=\hat{x}-\left[\frac{1-e^{-\gamma t}}{m\gamma}\right]\hat{p}, \quad (28)$$

$$\hat{p}_0\left(\hat{x},\hat{p},t\right)=\hat{p}. \quad (29)$$

The operators $\hat{x}_0$ and $\hat{p}_0$ are the integrals of the motion because theirs satisfy Eq.(7). Then these operators must satisfy Eqs.(8), (9), and

$$\left(x+i\hbar\left(\frac{1-e^{-\gamma t}}{m\gamma}\right)\frac{\partial}{\partial x}\right)K(x,x',t)=x'K(x,x',t), \quad (30)$$

$$-i\hbar\frac{\partial K(x,x',t)}{\partial x}=i\hbar\frac{\partial K(x,x',t)}{\partial x'}. \quad (31)$$

By modifying Eqs.(30) and (31), the system of equation for calculating the propagator are

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$$\frac{\partial K(x,x',t)}{\partial x}=\left[\frac{im\gamma(x-x')}{\check{z}\left(1-e^{-\gamma t}\right)}\right]K(x,x',t), \qquad (32)$$

$$\frac{\partial K(x,x',t)}{\partial x'}=-\left[\frac{im\gamma(x-x')}{\check{z}\left(1-e^{-\gamma t}\right)}\right]K(x,x',t). \qquad (33)$$

Now one can integrate Eq.(32) with respect to the variable x to obtain

$$K(x,x',t)=C(x',t)\exp\exp\left[\frac{i}{\check{z}}\left(\frac{m\gamma}{1-e^{-\gamma t}}\left(\frac{x^2}{2}-xx'\right)\right)\right]. \qquad (34)$$

Substituting Eq.(34) into Eq.(33), we obtain the differential equation for $C(x',t)$ as

$$\frac{\partial C(x',t)}{\partial \check{x}'}=-\frac{i}{\check{z}}\left(\frac{m\gamma}{1-e^{-\gamma t}}\right)x'C(x',t). \qquad (35)$$

Solving Eq.(35), the function $C(x',t)$ can be express as

$$C(x',t)=C(t)\exp\left[\frac{i}{2\check{z}}\left(\frac{m\gamma}{1-e^{-\gamma t}}\right)x'^2\right]. \qquad (36)$$

So, the propagator in Eq.(34) can be written as

$$K(x,x',t)=C(t)\exp\left[\frac{im\gamma(x-x')^2}{2\check{z}\left(1-e^{-\gamma t}\right)}\right]. \qquad (37)$$

To find $C(t)$, we must substitute the propagator of Eq.(37) into the Schrodinger's equation

$$\check{z}\frac{\partial K(x,x',t)}{\partial t}=-\frac{\check{z}^2}{2m}e^{-\gamma t}\frac{\partial^2 K(x,x',t)}{\partial x^2}. \qquad (38)$$

After some algebra, we obtain an equation

$$\frac{dC(t)}{dt}=-\left[\frac{\gamma e^{-\gamma t}}{2\left(1-e^{-\gamma t}\right)}\right]C(t). \qquad (39)$$

Equation (39) can be simply integrated with respect to time t, and one obtains

$$C(t)=\frac{C}{\sqrt{1-e^{-\gamma t}}}, \qquad (40)$$

where C is a constant. Substituting Eq.(40) into Eq.(37) and applying the initial condition of Eq.(21), we obtain

$$C=\sqrt{\frac{m}{2\pi\check{z}}}. \qquad (41)$$

So, the propagator for a free particle with linear damping is

$$K(x,x',t)=\sqrt{\frac{m\gamma}{2\pi\check{z}\left(1-e^{-\gamma t}\right)}}\exp\left(\frac{im\gamma(x-x')^2}{\check{z}\left(1-e^{-\gamma t}\right)}\right). \qquad (42)$$

The propagator for a charged harmonic oscillator in time-dependent electric field

The aim of this section is to derive the propagator for a charged harmonic oscillator in time-dependent electric field by Feynman path integral method.[1] Considering the motion of a charged harmonic oscillator which has mass m and positive charge q moving in time-dependent electric field $Ecos\Omega t$, the Lagrangian of this system can be written as

$$L=\frac{1}{2}m\dot{x}^2-\frac{1}{2}m\omega^2x^2-qEcos\Omega tx. \qquad (43)$$

By using the Euler-Lagrange equation for the Lagrangian in Eq.(43), the equation of motion can be written as

$$\ddot{x}+\omega^2x+qEcos\Omega t=0. \qquad (44)$$

The general solution of Eq.(44) is

$$x(t)=Acos\omega t+Bsin\omega t+\frac{qE}{\left(\Omega^2-\omega^2\right)}cos\Omega t, \qquad (45)$$

Where A and B are constants. The constants B and B in Eq.(45) can be determined by imposing the boundary conditions of $x(t')=x'$ and $x(t'')=x''$. The classical path that connects the point of (x',t') and (x'',t'') can be written as

$$x_{cl}(t)=\left[\frac{sin\omega(t-t')}{sin\omega T}\right]x''-\left[\frac{sin\omega(t-t'')}{sin\omega T}\right]x'$$

$$-\frac{qE}{\left(\Omega^2-\omega^2\right)sin\omega T}\left[cos\Omega t''sin\omega(t+t')-cos\Omega t'sin\omega(t+t'')-sin\omega Tcos\Omega t\right]. \qquad (46)$$

The action can be calculated from the time-integration of the Lagrangian from t' to t''

$$S(x'',t'';x',t')=\int_{t'}^{t''}L(\dot{x},x,t)dt. \qquad (47)$$

For the action of our system, the Lagrangian in Eq.(43) is substituted into Eq.(47), and then integrated by parts of the first term on the right hand side of Eq.(43) and using the equation of motion in Eq.(44).

The classical action can be written as

$$S_{cl}(x'',t'';x',t')=\frac{m}{2}\left(x_{cl}''\dot{x}_{cl}''-x_{cl}'\dot{x}_{cl}'\right). \qquad (48)$$

Substituting the classical paths of Eq.(46) into Eq.(48), the classical action becomes

$$S_{cl}(x'',t'';x',t')=\frac{m\omega}{2}cot\omega T\left(x''^2+x'^2\right)-\frac{m\omega}{sin\omega T}x''x'+\frac{mqE}{2\left(\Omega^2-\omega^2\right)sin\omega T}\left[\omega cos\Omega t''\left(\sin 2\omega t'-cos\omega(t''+t')\right)+\omega cos\Omega t'\left(\cos 2\omega t'-sin\omega(t''+t')\right)+\omega cot\omega T\left(cos\Omega t'\sin 2\omega t'-cos\Omega t''sin\omega(t''+t')\right)+\omega cos\Omega Tcos\Omega t''\right]x''$$

$$+\frac{mqE}{2\left(\Omega^2-\omega^2\right)sin\omega T}[\omega cos\Omega t''\left(\cos 2\omega t'-\sin 2\omega t'cot\omega T-cos\omega(t''+t')csc\omega T\right)-\omega cos\Omega t'\left(cos\omega(t''+t')-sin\omega(t''+t')cot\omega T-\cos 2\omega t''\right)+\Omega sin\Omega t'sin\omega T+\omega cos\omega Tcos\Omega t'-\Omega sin\Omega t'']x'$$

$$+\frac{mq^2E^2}{2\left(\Omega^2-\omega^2\right)^2\sin^2\omega T}[\ \omega\cos^2\Omega t''\left(2cos\omega t''sin\omega t'cos\omega(t''+t')-\sin 2\omega t'\cos 2\omega t'\right)-\ \omega\cos^2\Omega t'\left(2sin\omega t''cos\omega t'cos\omega(t''+t')-\sin 2\omega t''\cos 2\omega t'\right)+$$

$$\Omega sin\omega T\begin{pmatrix}\left(sin\Omega t''+sin\Omega t'\right)cos\Omega t'sin\omega(t''+t')-\\ sin\Omega t''cos\Omega t'\sin 2\omega t''-sin\Omega t'cos\Omega t''\sin 2\omega t'\end{pmatrix}-\ \omega cos\Omega t'cos\Omega t''\begin{pmatrix}\left(\sin 2\omega t''-\sin 2\omega t'\right)cos\omega(t''+t')-\\ \left(\cos 2\omega t''+\cos 2\omega t'\right)sin\omega T\end{pmatrix}$$

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$$-\frac{\Omega}{2}\left(\sin 2\Omega t'' + \sin 2\Omega t'\right)\sin^2 \omega T \quad (49)$$

The quadratic Lagrangian propagator can be separated into a pure function of time $F(t'',t')$ and the exponential function of classical action $S_{cl}(x'',t'';x',t')$ as suggested in Ref.[1]

$$K(x'',t'';x',t') = F(t'',t')e^{iS_{cl}(x'',t'';x',t')/\hbar} . \quad (50)$$

Calculation of the function $F(t'',t')$ presented by Pauli,[12] Morette,[13] or Jones and Papadoupoulos[14] can be performed by the semi classical approximation of path integral formula

$$F(t'',t') = \sqrt{\frac{1}{2\pi i\hbar}\left|\frac{\partial^2 S_{cl}}{\partial x'\partial x''}\right|} . \quad (51)$$

By substituting the classical action of Eq.(49) into Eq.(51), the pre-exponential factor can be obtained as

$$F(t'',t') = \sqrt{\frac{m\omega}{2\pi i\hbar \sin \omega T}} . \quad (52)$$

From Eqs.(49),(50) and (52), the propagator for a charged harmonic oscillator in time-dependent electric field can be expressed by

$$K(x'',t'';x',t') = \sqrt{\frac{m\omega}{2\pi i\hbar \sin \omega T}} e^{iS_{cl}(x'',t'';x',t')/\hbar} . \quad (53)$$

The procedures of the Schwinger method in calculating the non-relativistic propagator

Begin by considering a time-dependent Hamiltonian operator $\hat{H}(t)$, the propagator is defined by

$$K(x,x';t) = \langle x|\hat{T}\exp\left(-\frac{i}{\hbar}\int_0^t \hat{H}(t)dt\right)|x'\rangle, \quad (54)$$

where $\hat{T}$ is the time-ordering operator and $|x\rangle,|x'\rangle$ are the eigenvectors of the position operator $\hat{x}$ (in the Schrodinger picture) with eigenvalues x and x', respectively.

The differential equation for the propagator in Eq.(54) can be written as

$$i\hbar\frac{\partial K(x,x';t)}{\partial t} = \langle x|\hat{H}\hat{T}\exp\left(-\frac{i}{\hbar}\int_0^t \hat{H}(t)dt\right)|x'\rangle. \quad (55)$$

Applying the relation between the operators in the Heisenberg and Schrodinger pictures, we obtain the equation for the propagator in the Heisenberg picture

$$i\hbar\frac{\partial K(x,x';t)}{\partial t} = \left\langle x(t)\left|\hat{H}\left(\hat{x}(t),\hat{p}(t)\right)\right|x'(0)\right\rangle, \quad (56)$$

where $|x(t)\rangle$ and $|x'(0)\rangle$ are the eigenvectors of the operators $\hat{x}(t)$ and $\hat{x}(0)$, respectively, with the corresponding eigenvalues x and x'. Besides, $\hat{x}(t)$ and $\hat{p}(t)$ satisfy the Heisenberg equations

$$i\hbar\frac{d\hat{x}(t)}{dt} = \left[\hat{x}(t),\hat{H}\right], \; i\hbar\frac{d\hat{p}(t)}{dt} = \left[\hat{p}(t),\hat{H}\right]. \quad (57)$$

The main idea of the Schwinger method consists in the following steps.

1. The first step is solving the Heisenberg equations for $\hat{x}(t)$ and $\hat{p}(t)$, and writing the solution for $\hat{p}(t)$, only in terms of the operators $\hat{x}(t)$ and $\hat{x}(0)$.
2. The next step is substituting the solutions obtained in step (1) into the expression for $\hat{H}\left(\hat{x}(t),\hat{p}(t)\right)$ in Eq.(56) and employing the commutator $\left[\hat{x}(0),\hat{x}(t)\right]$ to rewrite each term of $\hat{H}(t)$ in a time ordered form with all operators $\hat{x}(t)$ to the left and all operators $\hat{x}(0)$. to the right. The time ordered Hamiltonian can be defined as $\hat{H}_{ord}\left(\hat{x}(t),\hat{x}(0)\right)$.
3. After this ordering, Eq.(56) can be written in the form

$$i\hbar\frac{\partial K(x,x';t)}{\partial t} = H(x,x';t)K(x,x';t), \quad (58)$$

with $H(x,x';t)$ being an ordinary function defined as

$$H(x,x';t) = \frac{\left\langle x(t)\left|\hat{H}_{ord}\left(\hat{x}(t),\hat{x}(0)\right)\right|x'(0)\right\rangle}{\langle x(t)|x'(0)\rangle}. \quad (59)$$

Integrating Eq.(58) over t, the propagator takes the form

$$K(x,x';t) = C(x,x')\exp\left\{-\frac{i}{\hbar}\int_0^t H(x,x';t)dt\right\}, \quad (60)$$

where $C(x,x')$ is an integration constant.

4. The last step is the calculating of $C(x,x')$ This is obtained by using the following conditions

$$-i\hbar\frac{\partial K(x,x';t)}{\partial x} = \left\langle x(t)\left|\hat{p}(t)\right|x'(0)\right\rangle, \quad (61)$$

$$i\hbar\frac{\partial K(x,x';t)}{\partial x'} = \left\langle x(t)\left|\hat{p}(0)\right|x'(0)\right\rangle, \quad (62)$$

and the initial condition

$$\lim_{t\to 0^+} K(x,x';t) = \delta(x-x'). \quad (63)$$

The Schwinger method for a damped harmonic oscillator

The Hamiltonian for a damped harmonic oscillator is described by[8]

$$H(t) = e^{-rt}\frac{p^2}{2m} + \frac{1}{2}m\omega^2 e^{rt}x^2, \quad (64)$$

Where r is the damping constant coefficient.

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The equation of motion corresponding to the Hamiltonian in Eq.(64) is

$$\ddot{x} + r\dot{x} + \omega^2 x = 0. \quad (65)$$

The classical solution of Eq.(65) can be written in the form

$$x(t) = e^{-\frac{rt}{2}}\left(\cos\Omega t + \frac{r}{2\Omega}\sin\Omega t\right)x' + \left(\frac{e^{-\frac{rt}{2}}\sin\Omega t}{m\Omega}\right)p', \quad (66)$$

where we impose the initial conditions $x' = x(0)$ and $p' = p(0)$.

The reduced frequency Ω in Eq.(66) is defined by $\Omega = \sqrt{\omega^2 - \frac{r^2}{4}}$. The reduced frequency Ω is real when $\omega^2 - \frac{r^2}{4} > 0$. That is, we will be concerned with the under-damped case.

By solving the Heisenberg equation in Eq. (57), the position operators $\hat{x}(t)$ can be written similarly to Eq.(66) as

$$\hat{x}(t) = e^{-\frac{rt}{2}}\left(\cos\Omega t + \frac{r}{2\Omega}\sin\Omega t\right)\hat{x}(0) + \left(\frac{e^{-\frac{rt}{2}}\sin\Omega t}{m\Omega}\right)\hat{p}(0). \quad (67)$$

The momentum operator $\hat{p}(t) = m(t)e^{rt}\dot{\hat{x}}(t)$ can be written by using Eq.(67) as

$$\hat{p}(t) = -\left(\frac{m\omega^2 e^{\frac{rt}{2}}\sin\Omega t}{\Omega}\right)\hat{x}(0) + e^{\frac{rt}{2}}\left(\cos\Omega t - \frac{r\sin\Omega t}{2\Omega}\right)\hat{p}(0). \quad (68)$$

By using Eq.(67), we can eliminate $\hat{p}(0)$ from Eq.(68) by

$$\hat{p}(t) = me^{rt}\left(\Omega\cot\Omega t - \frac{r}{2}\right)\hat{x}(t) - \left(m\Omega e^{\frac{rt}{2}}\csc\Omega t\right)\hat{x}(0). \quad (69)$$

Substituting $\hat{x}(t)$ and $\hat{p}(t)$ into the Hamiltonian operator $\hat{H}(t) = e^{-rt}\frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2 e^{rt}\hat{x}^2$ with the aid of

$$\left[\hat{x}(0), \hat{x}(t)\right] = \frac{\hbar\sin\Omega}{m\Omega}e^{-rt/2}, \quad (70)$$

the ordered Hamiltonian operator can be expressed as

$$\hat{H}_{ord}(t) = \frac{me^{rt}}{2}\left(\Omega^2\csc^2\Omega t - r\Omega\cot\Omega t + \frac{r^2}{2}\right)\hat{x}^2(t)$$
$$-m\Omega e^{\frac{rt}{2}}\left(\Omega\csc\Omega t\cot\Omega t - \frac{r}{2}\csc\Omega t\right)\hat{x}(t)\hat{x}(0) + \frac{1}{2}m\Omega^2\csc^2\Omega t\,\hat{x}^2(0)$$
$$-\frac{\hbar}{2}\left(\Omega\cot\Omega t - \frac{r}{2}\right). \quad (71)$$

Applying Eqs.(58)-(60), the propagator takes the form

$$K(x,x';t) = C(x,x')\exp[-\frac{i}{\hbar}\int_0^t\{\frac{1}{2}me^{rt}\left(\Omega^2\csc^2\Omega t - r\Omega\cot\Omega t + \frac{r^2}{2}\right)$$
$$+\frac{1}{2}m\Omega^2\csc^2\Omega t x'^2 - m\Omega e^{\frac{rt}{2}}\left(\Omega\csc\Omega t\cot\Omega t - \frac{r}{2}\csc\Omega t\right)xx'$$
$$-\frac{\hbar}{2}\left(\Omega\cot\Omega t - \frac{r}{2}\right)\}dt]. \quad (72)$$

Now, we will integrate each term of Eq.(72) with respect to time. The first term of Eq.(72) can be integrated as

$$-\frac{im}{2\hbar}x^2\int_0^t e^{rt}\left(\Omega^2\csc^2\Omega t - r\Omega\cot\Omega t + \frac{r^2}{2}\right)dt = \frac{im\Omega}{2\hbar}e^{rt}\cot\Omega t x^2 - \frac{imr}{4\hbar}e^{rt}x^2. \quad (73)$$

The second term of Eq.(72) can be calculated by

$$-\frac{im\Omega^2}{2\hbar}x'^2\int_0^t\csc^2\Omega t dt = \frac{im\Omega}{2\hbar}\cot\Omega t x'^2. \quad (74)$$

The third term of Eq.(72) can be derived by

$$\frac{im\Omega}{\hbar}xx'\int_0^t e^{\frac{rt}{2}}\left(\Omega\csc\Omega t\cot\Omega t - \frac{r}{2}\csc\Omega t\right)dt = -\frac{im\Omega}{\hbar}e^{\frac{rt}{2}}\csc\Omega t xx'. \quad (75)$$

Finally, integrating the last term of Eq.(72), the result is

$$-\int_0^t\left(\frac{\Omega}{2}\cot\Omega t - \frac{r}{4}\right)dt = -\frac{1}{2}\ln\ln(\sin\Omega t) + \frac{rt}{4}. \quad (76)$$

Combining the results of Eqs.(73)-(76), the propagator can be written as

$$K(x,x';t) = C(x,x')\sqrt{\frac{e^{\frac{rt}{2}}}{\sin\Omega t}}\exp\left(-\frac{imr}{4}e^{rt}x^2\right)$$
$$\times\exp\left[\frac{im\Omega}{2\hbar\sin\Omega t}\left(e^{rt}\cos\Omega t x^2 + \cos\Omega t x'^2 - 2e^{\frac{rt}{2}}xx'\right)\right]. \quad (77)$$

The final step is deriving the function $C(x,x')$. Substituting Eq.(77) into Eq.(62), its can be obtained that

$$\hbar\frac{\partial C(x,x')}{\partial x'} = -\frac{mr}{2}x C(x,x'). \quad (78)$$

The solution of Eq.(78) can be written as

$$C(x,x') = C(x)\exp\exp\left(\frac{imr}{4\hbar}x'^2\right), \quad (79)$$

Where $C(x)$ is a position function

The propagator in Eq.(77) can be expressed as

$$K(x,x';t) = C(x)\sqrt{\frac{e^{\frac{rt}{2}}}{\sin\Omega t}}\exp\left(-\frac{imr}{4}\left(e^{rt}x^2 - x'^2\right)\right)$$
$$\times\exp\exp\left[\frac{im\Omega}{2\hbar\sin\Omega t}\left(e^{rt}\cos\Omega t x^2 + \cos\Omega t x'^2 - 2e^{\frac{rt}{2}}xx'\right)\right]. \quad (80)$$

The next step is calculating $C(x)$. Substituting Eq.(80) into Eq.(61), the result is

$$\frac{\partial C(x)}{\partial x} = 0, \quad (81)$$

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which implies that $C(x)$. is a constant independent of x .

After applying Eq.(63), it can be obtained that

$$C=\sqrt{\frac{m\Omega}{2\pi\check{z}}}. \quad (82)$$

So, the propagator for a damped harmonic oscillator can be written as

$$K(x,x';t)=\sqrt{\frac{m\Omega e^{\frac{rt}{2}}}{2\pi\check{z}\sin\Omega}}\exp\left(-\frac{imr}{4\check{z}}\left(e^{rt}x^2-x'^2\right)\right)$$

$$\times\exp\left[\frac{im\Omega}{2\check{z}\sin\Omega t}\left(e^{rt}\cos\Omega tx^2+\cos\Omega tx'^2-2e^{\frac{rt}{2}}xx'\right)\right]. \quad (83)$$

This propagator is the same as the result of S.Pepore,[8] found by applying the integrals of motion of a quantum systems.

Conclusion

In this paper we have successfully calculated the exact propagators for time-dependent Hamiltonian systems. The method for deriving the propagators with the helping of integrals of motion of quantum systems presented in this paper can be successfully applied in solving a time-dependent linear potential and a free particle with linear damping problems. This method has the important steps in finding the constant of motion x_0 and p_0 and implying that the propagator $K(x,x',t)$ is the eigen functions of the operators $\hat{x}_0(x)$ and $\hat{p}_0(x)$. The exact propagator for a charged harmonic oscillator in time-dependent electric field was calculated by the Feynman path integral method. The crucial result in our calculation is to derive the classical action as mentioned in E.(49). The propagator for a damped harmonic oscillator has calculated by the Schwinger method. The important step in the Schwinger formalism is to find the solution of the Heisenberg equation in Eq.(67) and to express the Hamiltonian operator in an appropriate order with the aid of the commutator in Eq.(70). The advantage of the Schwinger method in this paper is that it requires only fundamental operator algebra and some basic integration. In fact, the application of the integrals of the motion method has many common features with the Schwinger method, but the Schwinger method requires the operators $\hat{x}(t)$ and $\hat{p}(t)$ in deriving the matrix element of Hamiltonian operator in calculating the propagator in Eq.(72). In the Feynman path integrals, the pre-exponential function $C(t)$ comes from sum over all fluctuating paths that depend on calculation of the functional integration while in the integrals of the motion method this term appears from solving the Schrodinger equation of propagator. In the Schwinger formalism, the pre-exponential function $C(t)$ arises from the commutation relation of $\left[\hat{x}(t),\hat{x}(0)\right]$. These different points of view may show the connection between classical mechanics and quantum mechanics.

Finally, we have presented simple techniques in calculating the propagator. It is preferable to have many methods in deriving the propagators in the field of time-dependent Hamiltonian systems and the Feynman path integrals, Schwinger method, and integrals of the motion method are effective and appropriate techniques.

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Conflicts of interest

The author declares there is no conflict of interest.

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