Forum Article

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Rotational velocity of a relativistic heat conducting fluid configuration in non-circular axisymmetric stationary spacetime

Abstract

The present work is focused to study the rotational velocity of a heat conducting fluid configuration based on Carter's model and related consequences under the assumption that the background space time is non-circular stationary and axi symmetric. The level surfaces of constant angular velocity about rotation axis do not coincide with level surfaces of constant effective angular momentum per baryon corresponding to the matter part of fluid because of the variation of Killing twist scalars coupled with thermodynamic quantities in meridional planes. The rotation of matter part of fluid bears an intrinsic relationship with heat flow, injection energy per baryon, chemical potential of matter part of fluid, and rotational potential created by dynamic space time as an outcome of interaction between the motion of the entropy fluid and of the matter part of fluid. The meridional circulation velocity plays a key role in the creation of the entropy production besides the contributions made by other thermodynamic quantities. The entropy fluid is not co rotating with the matter part of fluid in the presence of dissipation caused by the heat flow. It is found that a linear combination of the injection energy gradient and the gradient of rotational velocity about rotation axis is constant along the matter part of fluid flow lines.

Keywords: rotational velocity, Carter's model, constant angular velocity, heat flow, injection energy per baryon, injection energy gradient, gradient of rotational velocity

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Introduction

Relativistic dissipative fluid dynamics is important to understand the irreversible thermodynamic processes of hot dense nuclear matter that is created in supernovae explosions[1-3] leading to the formation of stellar compact objects like neutron stars[4] as well as needed to explain physical phenomenon found in laboratory experiments involving relativistic heavy-ion collision.[5] A relativistic theory of dissipative fluid based on irreversible thermodynamic processes has first been formulated by Eckart.[6] But this theory encountered a difficulty in the sense that the occurrence of causality violation and instability[7,8] is inevitable due to the absence of relaxation timescales corresponding to dissipative quantities such as bulk viscous pressure, shear stress tensor, and heat flow within the theory. In order to circumvent the problems of a causality and instability in a relativistic framework Israel and Stewart (IS)[9] formulated a new theory of relativistic dissipative fluid dynamics by invoking Grad's 14-moment approximation[10] coupled with Boltzmann equations incorporating relaxation timescales corresponding to dissipative quantities. But this theory also encountered the problem of instability[11-12] and is unsatisfactory to some extent in the case of heavy-ion collision experiment.[13] IS theory[9] based on Grad's moment approximation leads to undesirable features like infinite number of equations with different transport coefficients describing dissipations.[14] Despite considerable efforts,[15-16] consequences related to the onset of dissipation are not well known.

A new direction of investigating dissipative phenomenon originating due to heat flow stems from the ground-breaking work of Carte[17] in which the entropy element is thought of as a fluid. The entropy entrainment is a basic element from which an analysis of causal property of thermal propagation is built up.[18] This idea is exploited in[18] that led to the formulation of relativistic version of Cattaneo equation describing causality preserving heat conduction. The crucial fact of Carter's model to realize is that the existence of a pair of particle vorticity 2-form and thermal vorticity 2-form builds up a pair of source-free Maxwell's like equations which describe the evolution of a heat conducting fluid.[17] The matter part of fluid's 4-velocity and the heat flow vector bear an intrinsic relationship of mutual interdependence. Consequently, prior choice of components of both fluid's 4-velocity and heat flow vector without solving them from Maxwell's like equations may not be physically consistent with the evolution equations of a heat conducting fluid.[19] The existence of meridional circulation is an inherent consequence[19] of Carter's model of a heat conducting fluid.[17] At this point, it is worth to recall the remark made by Priou[20] that IS theory[9] and Carter's variational model of heat conducting fluid[17] ceases to be equivalent in non-equilibrium situations. The reason seems to lie in the fact that the entropy fluid contribute its energy (i.e., product of entropy density and local temperature) per baryon to the matter part of fluid's energy per baryon which results in the enhancement of total energy per baryon in a Carter's model during the evolution of a heat conducting fluid. The variation of this total energy along the matter part of fluid's 4-velocity exchanges with the heat flux coupled to the effective energy per entropic associated with the entropy fluid per unit of local temperature measured in the matter part of fluid's rest frame under the assumption that the space time representing the gravitational field of such fluid configuration is non-circular stationary and axisymmetric.[21] Similar exchange law for the variation of total angular momentum per baryon of matter part of fluid with the heat flux coupled to the effective angular momentum per entropon of the entropy fluid per unit of local temperature holds under the same spacetime symmetry conditions.[21] Such physical process is not obtainable in IS theory.[9] Carter's model[17] seems to be more capable to describe the interaction between the gravitational field and the motion of a heat conducting fluid because of the existence of a pair of Maxwell's like equations and the energy-momentum tensor built up with a unique term expressing thermal stress coupled with a thermodynamic variable encoding the entropy entrainment. This stress term relates the Ricci curvature tensor via Einstein field equations in the spacelike 3-space orthogonal to the matter part of fluid's 4-velocity and therefore connects gravitational potentials characterized by the metric tensor.

As is known from the work of Lindblom[22] that the thermodynamic equilibrium of a self- gravitating dissipative fluid requires the vanishing of entropy production which in turn implies the vanishing of both the heat flow and the shear tensor associated with the fluid flow lines in Eckart's Theory.[6] Since by definition such equilibrium state of stellar object composed of a heat conducting fluid is axisymmetric and stationary,[22] it amounts to the vanishing of differential rotation and hence the stellar object rotates uniformly and the heat flow dies out. But in this conclusion, the missing link is fluid's vorticity that exists even in the case of uniform rotation.[23] The vorticity of fluid flow lines due to gravitomagnetic effect generates the coriolis force[24] that couples to heat flow.[25] Its effect is recognized in.[26] The term of coriolis force enters in the equilibrium equation if constructed from Euler's equations of motion and vortex lines are twisting.[27] Because of the presence of non-zero magnitude of fluid's vorticity, by virtue of timelike convergence condition,[28] this squared magnitude of fluid's vorticity has a strong bearing on continuous variation of temperature via Raychaudhuri equation.[28] This means that there is continuous thermal dissipation due to internal motion of the fluid under the action of coriolis force. Such situation is still not clear in the evolutionary scenario of a heat conducting fluid but expected to halt if the vorticity magnitude is bounded above.

The dissipative processes that occur in the formation and evolution of compact stellar objects involve on the one hand strong gravitational effects and on the other hand microscopic properties of hot dense matter. Some recent theoretical and numerical investigations[29,30] indicate that the study of thermal evolution of a newly born neutron star is important for understanding physical processes of observed thermal radiation from such stars on the basis of cooling theory.[31] But the way the energy-balance equation is formulated for the study of rotational effects on thermal evolution of a newly born neutron star violates the causality principle because the formulation involves Fourier's law for the description of heat conduction and an analogous construction of energy-momentum tensor that resembles with that of Eckart's relativistic version of dissipative fluids.[6] It is known that Fourier's law of heat conduction violates causality. Such construction of theoretical basis used for prediction of rotational effect suffers from causality violation and therefore seems to be inconsistent in a relativistic framework. Effort[32] towards better formulation for the description of thermal evolution in the case of a rotating star is still ongoing.

If the spacetime configuration representing the gravitational field of a self- gravitating heat conducting fluid is assumed to be axisymmetric and stationary, it must be non-circular in a Carter's model[17] because the heat flow is strongly coupled to meridional circulations.[19] The existence of meridional circulation is intrinsically related to the Killing twist scalars which build up dynamic character of the space time. This in turn says that the notion of thermal equilibrium based on Eckart's model,[6] which requires the vanishing of heat flow, ceases to hold in a Carter's model.[17] The reason is that the contribution of heat flow to momentum covector associated with both the matter part of fluid and the entropy fluid cannot be excluded until meridional circulation ceases.[19] The question that arises from asking how meridional circulation ceases at the onset of dissipation caused by the heat flow in an irreversible thermodynamic process. The thermodynamic processes that can thrust out meridional circulation during thermal evolution is yet unknown. Furthermore, it is extremely difficult to solve thermal relaxation time from relativistic version of Cattaneo equation[18] without the knowledge of components of heat flow vector and the acceleration of fluid's motion that couples to shear and rotation tensors associated with the fluid flow lines in a non-circular stationary axisymmetric spacetime. The determination of the components of heat flow vector and the matter part of fluid's 4-velocity requires the solution of a pair of Maxwell's like equations governing the evolution of a heat conducting fluid. Thus as a first step, we find solutions of a pair of Maxwell's like equations by exploiting an electrodynamic analog of the approach developed in[33] for the case of relativistic magnetohydrodynamics (RMHD) under the same spacetime symmetry assumptions and use these solutions to study the consequences related to the rotational evolution of both the matter part of fluid and the entropy fluid.

The present work is focused on the study of rotational evolution of a heat conducting fluid based on Carter's model[17] under the assumption that its background spacetime representing the gravitational field is non-circular stationary and axisymmetric. The plan of the paper is as follows. In Sec.2 we find solution of Maxwell's like equations associated with the evolution of the matter part of fluid by exploiting an electrodynamic analog of the approach developed in the case of relativistic magnetohydrodynamics (RMHD)[33] and derive the relation between differential rotation of matter part of fluid and a combination of Killing twist scalars and thermodynamic variables. Sec.3 is concerned with the rotation of matter part of fluid composed of an extra rotation caused by meridional circulation in the meridional plane in addition to the usual rotation about the rotation axis. Its connection with thermodynamic quantities is described by using the solution of Maxwell's like equations associated with the evolution of the entropy fluid. Sec.4 describes the rotational evolution of the entropy fluid and related consequences. Sec.5 is devoted to the description of the creation of injection energy. Sec.6 is focused on the differential rotation of the matter part of fluid along the thermal-fluid vorticity.

Convention: The spacetime metric is of signature $+2$. Small case Latin indices run from 0 to 3. Caiptal Latin indices are used to indicate poloidal coordinates which take values 1 and 2. Semi-colon and comma are used, respectively, to denote the covariant and partial derivatives. Constituent indices n and s are used to indicate matter and entropy part of fluid, respectively, throughout the text and not to be confused with tensor indices. Square and round bracket around indices represent, respectively, skew-symmetrization and symmetrization.

Evolution of the matter part of fluid

In this section we study the evolution of matter part of fluid described by Maxwell's like equations associated with the thermal-fluid vorticity 2-form W_{ab} and investigate various consequences related to dissipation caused by heat flow under the assumption that the space time representing the gravitational field of a self-gravitating heat conducting fluid is non-circular stationary and axi symmetric. The idea to find solutions of Maxwell's like equations is based on an analogous approach developed in[33] for the study of relativistic magneto hydrodynamics (RMHD) under the same space time symmetry assumption. This assumption implies the existence of pair of two linearly independent Killing vectors of which one is time like Killing vector $\xi^a_{(t)}$ generating a translational symmetry with open time like lines as orbits and the other one is a space like Killing vector $\xi^a_{(\varphi)}$ generating rotations about a symmetry axis.[34] There exists a family of invariant time /like 2-surfaces, called surfaces of transitivity, generated by this pair of Killing vectors that correspond to ignorable coordinates $x^0 = t$ and $x^3 = \varphi$ (i.e., $\xi^a_{(t)} = \delta^a_t$ and $\xi^a_{(\varphi)} = \delta^a_\varphi$ [34] The ignorable coordinates t and φ are called toroidal coordinates. This pair of Killing vectors constitutes the basis of tangent plane tangential to surface of transitivity. Its dual basis is of the following form:[33]

$$_{(t)a} = \frac{1}{K}\left(-g_{\varphi\varphi}\xi_{(t)a} + g_{t\varphi}\xi_{(\varphi)a}\right), \tag{2.1a}$$

$$_{(\varphi)a} = \frac{1}{K}\left(g_{t\varphi}\xi_{(t)a} - g_{tt}\xi_{(\varphi)a}\right), \tag{2.1b}$$

with the properties

$$_{(t)a}\,\xi^a_{(t)} = 1 = _{(\varphi)a}\,\xi^a_{(\varphi)} \quad \text{and} \quad _{(t)a}\,\xi^a_{(\varphi)} = 0 = _{(\varphi)a}\,\xi^a_{(t)}, \tag{2.1c}$$

where

$$K = g^2_{t\varphi} - g_{tt}g_{\varphi\varphi} > 0 \;. \tag{2.1d}$$

At every point of spacetime there is a 2-dimensional spacelike tangent plane orthogonal to the timelike 2-plane but due to non-circularity assumption a family of such spacelike 2-planes do not mesh together to form a family of spacelike 2-surfaces. Such non-integrable 2-planes are called poloidal planes (or meridional planes). We choose the poloidal coordinates $x^1 = r$ and $x^2 = z$ in cylindrical polar coordinates. Thus every vector of spacetime is decomposable into toroidal and poloidal components.

The matter part of fluid's 4-velocity u^a can be decomposed as[33]

$$u^a = \lambda\left(\xi^a_{(t)} + \Omega\xi^a_{(\varphi)}\right) + w^a \;, \tag{2.2}$$

where w^a denotes the meridional circulation velocity orthogonal to both $\xi^a_{(t)}$ and $\xi^a_{(\varphi)}$.The 4-velocity u^a obeys the normalization condition $u^a u_a = -1$. When this condition is invoked, we find from (2.2) that

$$\lambda^2 = \frac{1 + w^2}{G} \;, \tag{2.3}$$

where $G = -\left(g_{tt} + 2\Omega g_{t\varphi} + \Omega^2 g_{\varphi\varphi}\right)$ and w^2 is the squared magnitude of meridional circulation velocity.

A unit spacelike vector m^a orthogonal to u^a may be constructed such as

$$m^a = \zeta\left(\xi^a_{(\varphi)} + l\xi^a_{(t)}\right) \;, \tag{2.4}$$

It follows from (2.2) and (2.4), because of the orthogonality condition $u^a m_a = 0$, that

$$l = -\frac{u_\varphi}{u_t} = -\frac{\left(g_{t\varphi} + \Omega g_{\varphi\varphi}\right)}{\left(g_{tt} + \Omega g_{t\varphi}\right)} \;. \tag{2.5}$$

Substituting (2.4) into the normalization condition $m^a m_a = 1$ and making use of (2.3), we find that

$$\zeta^2 = \frac{u_t^2}{K\left(1 + w^2\right)} \;. \tag{2.6}$$

The source-free Maxwell's like equations associated with the thermal-fluid vorticity 2-form W_{ab} is of the following form:[17]

$$W_{ab}u^b = -_a\mathrm{E} \,, \tag{2.7}$$

where

$$W_{ab} = 2\mu_{[b;a]} \,,\; \mu_a = u_a + \alpha q_a \,, \text{and} \;\; _a = \frac{R}{n}\left(-\frac{\beta q^2}{2}\right)q_a \;. \tag{2.8}$$

Here μ_a is the conjugate momentum convector associated with the matter part of fluid corresponding to the matter current n^a and q_a is the heat flow vector. The chemical potential, entropy per baryon, and temperature measured in the matter part of fluid's rest frame u^a are, respectively, denoted by $^*\mu$, $^*s$, and $^*\theta$. The thermodynamic variables α and β are related by the relations $\beta = \left(\frac{1}{^*s} - \frac{nA^{ns}}{^*s^*\theta}\right)$ and $\alpha = \frac{A^{ns}}{\theta}$ which encode the entropy entrainment effect via A^{ns} .[18] The conservation of particle current is described by $n^a_{;a} = 0$ which is equivalent to the baryon conservation law, i.e., $\left(nu^a\right)_{;a} = 0$.

The electric part of W_{ab} is computed in the following form:[19]

$$E_a = \lambda_{,a} - \lambda\Omega j_{,a} - \left(_{,b}w^b\right)_{(t)a} + \left(j_{,b}w^b\right)_{(\varphi)a} - \frac{I}{K}\eta_{abcd}w^b\xi^c_{(t)}\xi^d_{(\varphi)}, \tag{2.9}$$

where ε and j denote, respectively, the effective energy per particle and the effective angular momentum per particle and are expressible as $-\varepsilon = {}^*\mu u_t + \alpha q_t$ and $j = {}^*\mu u_\varphi + \alpha q_\varphi$. The symbol η_{abcd} is the Levi-Civita skew-symmetric tensor and $I = {}^*W_{ab}\xi^a_{(t)}\xi^b_{(\varphi)}$ which can be explicitly determined by solving the corresponding Maxwell's like equations. Its explicit form will be derived later on in the subsequent discussions. The Hodge dual of W_{ab} is indicated by an overhead star. Contraction of (2.9) with $\xi^a_{(t)}$ and $\xi^a_{(\varphi)}$ in turn gives that

$$\mathrm{E}_t = -\varepsilon_{,b}w^b \quad \text{and} \quad \mathrm{E}_\varphi = j_{,b}w^b \;. \tag{2.10}$$

On account of (2.1a,b) and (2.10), one can reduce (2.9) to take the form

$$\mathrm{E}_a = \lambda_{,a} - \lambda\Omega j_{,a} + A_1\xi_{(t)a} - A_2\xi_{(\varphi)a} - \frac{I}{K}\eta_{abcd}w^b\xi^c_{(t)}\xi^d_{(\varphi)} \;, \tag{2.11}$$

where

$$A_1 = \frac{1}{K}\left(-g_{\varphi\varphi}\mathrm{E}_t + g_{t\varphi}\mathrm{E}_\varphi\right) \text{ and } A_2 = \frac{1}{K}\left(-g_{t\varphi}\mathrm{E}_t + g_{tt}\mathrm{E}_\varphi\right) \;. \tag{2.12}$$

Setting $\mathrm{E}_a = Aq_a$, where $A = \frac{R}{n}\left({}^*s - \frac{\beta q^2}{{}^*\theta^2}\right)$ and choosing q^a directed along m^a which is orthogonal to u^a, we can expres E_a as follows

$$\mathrm{E}_a = Aq_a = Aq\zeta\left(\xi_{(\varphi)a} + l\xi_{(t)a}\right). \tag{2.13}$$

At this point, It is important to underline that the choice for the contra variant components of the heat flow vector q^a directed along m^a imposes restriction to the vanishing of contra variant poloidal components of q^a but its none of covariant components is zero. This choice is necessary for obtaining covariant solutions of Maxwell's like equations under the spacetime symmetry assumption. Replacing left hand side of (2.11) by (2.13) and contracting the resulting equation with ${}^{*}\xi^{a}_{(t)}$ and ${}^{*}\xi^{a}_{(\varphi)}$ in turn, we find that

$$Aq\zeta l = A_1 \text{ and } Aq\zeta = -A_2 \ . \tag{2.14}$$

It follows from (2.5) and (2.14) that

$$q_t = -\Omega q_\varphi \ . \tag{2.15}$$

Making use of (2.13) with the aid of (2.14) in (2.11), we get

$$\varepsilon_{,a} - \Omega j_{,a} = \frac{I}{\lambda K}\eta_{abcd} w^b \xi^c_{(t)} \xi^d_{(\varphi)} \ . \tag{2.16}$$

The injection energy per baryon is defined by[35]

$$\Phi = \varepsilon - \Omega j \ . \tag{2.17}$$

Using (2.17) in (2.16) and inverting the resulting equation, we obtain that

$$w^a = -\frac{\lambda}{I}\eta^{abcd}\left(_{,b} + j\Omega_{,b}\right)\xi_{(t)c}\xi_{(\varphi)d} \ , \tag{2.18}$$

A result from the baryon conservation law derived in[33] is of the following form:

$$f_{,a} = -\frac{\lambda nK}{I}\left(\Phi_{,a} + j\Omega_{,a}\right) \ , \tag{2.19}$$

where f denotes Stokes stream function.[33] It follows from (2.18) and (2.19) that

$$f_{,a} = -\frac{\lambda nK}{I}\left(\Phi_{,a} + j\Omega_{,a}\right) \ , \tag{2.20}$$

which asserts that the Stokes stream function varies in accordance with the linear combination of injection energy gradient and gradient of the rotation of matter part of fluid about the rotation axis that couples the effective angular momentum per baryon. By inverting (2.19), it can be shown that $f_{,a}u^a = 0$. When this result is used in (2.20), we find that

$$\left(\Phi_{,a} + j\Omega_{,a}\right)u^a = 0 \ , \tag{2.21}$$

which exhibits the relation between the variation of the injection energy and differential rotation due to rotation about the axis of rotation along the matter part of fluid's 4-velocity. On account of (2.15), we find that

$${}^{*}\mu\left(u_t + \Omega u_\varphi\right) = -\left(\varepsilon - \Omega j\right) = -\Phi \ . \tag{2.22}$$

Inserting (2.22) in the first term of (2.21), we get

$$\left[-\left\{{}^{*}\mu\left(u_t + \Omega u_\varphi\right)\right\}_{,a} + j\Omega_{,a}\right]u^a = 0 \ . \tag{2.23}$$

The quantity I which has appeared in (2.9) needs to be determined explicitly. In order to derive its explicit expression we now use the notion of differential form and exterior calculus and employ the technical machinery developed in[33] for computational convenience. We rewrite (2.7) in the following form

$$u.d\underline{\mu} = \underline{\mathrm{E}} \ , \tag{2.24}$$

Since $u.d\underline{\mu}$ is a one-form, it can be expressed as

$$u.d\underline{\mu} = a^{*}\xi_{(t)} + b^{*}\xi_{(\varphi)} + \underline{X} \ . \tag{2.25}$$

where b and b are unspecified functions. Contracting (2.25) with $\xi_{(\varphi)}$ and $\xi_{(\varphi)}$ in turn and making use of symmetry condition,i.e. $\pounds_{\xi_{(t)}}\underline{\mu} = 0$, and $\pounds_{\xi_{(\varphi)}}\underline{\mu} = 0$ with the aid of Cartan identity, one can get

$$a = w.d\left(\underline{\mu}.\xi_{(t)}\right) \ , \tag{2.26a}$$

$$b = w.d\left(\underline{\mu}.\xi_{(\varphi)}\right) \ . \tag{2.26b}$$

An explicit expression of $\underline{\mu}$ is given by

$$\underline{\mu} = {}^{*}\mu\underline{u} + \alpha\underline{q} \tag{2.27}$$

which, on account of (2.2) and (2.4), can be cast in the following form

$$\underline{\mu} = \underline{r} + \underline{s} + {}^{*}\mu\underline{w} \tag{228}$$

where

$$\underline{r} = {}^{*}\mu\left(\lambda\underline{\xi}_{(t)} + \lambda\Omega\underline{\xi}_{(\varphi)}\right) \tag{2.29a}$$

$$\underline{s} = \alpha q\zeta\left(\underline{\xi}_{(\varphi)} + l\underline{\xi}_{(t)}\right) \tag{2.29b}$$

From (2.27), we have

$$u.d\underline{\mu} = u.d\underline{r} + u.d\underline{s} + u.d\left({}^{*}\mu\underline{w}\right) \tag{2.30}$$

In view of (2.29a,b), following computational steps given in[33], one can find that

$$u.d\underline{r} = -\lambda\left[d\left(\underline{r}.\xi_{(t)}\right) + \Omega\, d\left(\underline{r}.\xi_{(\varphi)}\right)\right] + w.d\underline{r} \ , \tag{2.31a}$$

$$u.d\underline{s} = -\lambda\left[d\left(\underline{s}.\xi_{(t)}\right) + \Omega\, d\left(\underline{s}.\xi_{(\varphi)}\right)\right] + w.d\underline{s} \ , \tag{2.31b}$$

and

$$w.d\left({}^{*}\mu w\right) = \frac{C_1}{n}df \ , \tag{2.31c}$$

where C_1 is a function which, following,[33] can be expressed as $C_1 = -\left(\frac{{}^{*}\mu}{Kn}f_{,a}\right)^{;a}$. Substitution of (2.31a-c) into (2.30) gives that

$$u.d\underline{\mu} = -\lambda\left[d\left(\underline{r} + \underline{s}\right).\xi_{(t)}\right] - \lambda\Omega\left[d\left(\underline{r} + \underline{s}\right).\xi_{(\varphi)}\right] + \frac{C_1}{n}df + w.d\underline{r} + w.d\underline{s} \ . \tag{2.32}$$

Replacing the left hand side of (2.25) by the right hand side of (2.32) and contracting the resulting equation with an arbitrary vector v lying in meridional plane orthogonal to both $\xi_{(t)}$ and $\xi_{(\varphi)}$, one may obtain

$$\underline{X}.v = -\lambda v.\left[d\left(\underline{r} + \underline{s}\right).\xi_{(t)}\right] - \lambda\Omega v.\left[d\left(\underline{r} + \underline{s}\right).\xi_{(\varphi)}\right] + \frac{C_1}{n}v.df + w.d\underline{r}.v + w.d\underline{s}.v \ . \tag{2.33}$$

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The last two terms on right hand side of (2.33), by using the results derived in,[33] can be put in the following form:

$$w,d\underline{r}.v = d\underline{r}\left(w,v\right) = \lambda^{*}\mu\left(\tau_{(t)} + \Omega\tau_{(\varphi)}\right)\frac{1}{Kn}df.v \quad (2.34a)$$

and

$$w.d\underline{s}.v = d\underline{s}\left(w,v\right) = \alpha q\zeta\left(\tau_{(\varphi)} + l\tau_{(t)}\right)\frac{1}{Kn}df.v \quad (2.34b)$$

where Killing twist scalars[34] corresponding to $\xi_{(t)}$ and $\xi_{(\varphi)}$ are, respectively, denoted by $\tau_{(t)}$ and $\tau_{(\varphi)}$ which are defined as follows :

$$\tau_{(t)} = \eta^{abcd}\xi_{(\varphi)a}\xi_{(t)b}\xi_{(t)c;d}\ ,\ \tau_{(\varphi)} = \eta^{abcd}\xi_{(t)a}\xi_{(\varphi)b}\xi_{(\varphi)d;c}\ . \quad (2.34c)$$

Substituting (2.34a-b) in the last two terms of (2.33) and simplifying, we find that

$$\underline{X} = -\lambda d\left[\left(\underline{r}+\underline{s}\right).\xi_{(t)}\right] - \lambda\Omega d\left[\left(\underline{r}+\underline{s}\right).\xi_{(\varphi)}\right] + \frac{1}{Kn}\left[KC_1 + \left(\lambda^{*}\mu + \alpha q\zeta l\right)\tau_{(t)} + \left(\lambda\Omega^{*}\mu + \alpha q\zeta\right)\tau_{(\varphi)}\right]df\ . \quad (2.35)$$

It follows from (2.25), (2.26a-b) and (2.35) that

$$u.d\underline{\mu} = \left[w.d\left(\underline{\mu}.\xi_{(t)}\right)\right]^{*}\xi_{(t)} + \left[w,d\left(\underline{\mu}.\xi_{(\varphi)}\right)\right]^{*}\xi_{(\varphi)} - \lambda d\left[\left(\underline{r}+\underline{s}\right).\xi_{(t)}\right] - \lambda\Omega d\left[\left(\underline{r}+\underline{s}\right).\xi_{(\varphi)}\right] + \frac{1}{Kn}\left[C_1K + \left(\lambda^{*}\mu + \alpha q\zeta l\right)\tau_{(t)} + \left(\lambda\Omega^{*}\mu + \alpha q\zeta\right)\tau_{(\varphi)}\right]df\ . \quad (2.36)$$

For further simplification of (2.36) we now proceed as follows.

$$\underline{\mu}.\xi_{(t)} = \underline{\mu}u_t + \alpha q_t = -\varepsilon \quad (2.37a)$$

$$\underline{\mu}.\xi_{(\varphi)} = \underline{\mu}u_\varphi + \alpha q_\varphi = j \quad (2.37b)$$

$$\left(\underline{r}+\underline{s}\right).\xi_{(t)} = \lambda^{*}\mu\left(g_{tt} + \Omega g_{t\varphi}\right) + \alpha q\zeta\left(g_{t\varphi} + lg_{tt}\right) = {}^{*}\mu u_t + \alpha q_t = -\varepsilon \quad (2.37c)$$

Similarly,

$$\left(\underline{r}+\underline{s}\right).\xi_{(\varphi)} = {}^{*}\mu u_\varphi + \alpha q_\varphi = j \quad (2.37d)$$

Substituting (2.37a-d) into (2.36) and taking (2.24) into account, we get

$$\underline{\mathrm{E}} = -\left(w.d\varepsilon\right){}^{*}\xi_{(t)} + \left(w.dj\right){}^{*}\xi_{(\varphi)} + \lambda d\varepsilon - \lambda\Omega dj + \frac{1}{Kn}\left[C_1K + \left(\lambda^{*}\mu + \alpha q\zeta\ l\right)\tau_{(t)} + \left(\lambda^{*}\mu\Omega + \alpha q\zeta\right)\tau_{(\varphi)}\right]df \quad (2.38)$$

Equating the poloidal components of E_a from (2.9) and (2.38), we find that

$$I = \left[KC_1 + \left(\lambda^{*}\mu + \alpha q\zeta\ l\right)\tau_{(t)} + \left(\lambda^{*}\mu\Omega + \alpha q\zeta\right)\tau_{(\varphi)}\right], \quad (2.39)$$

This is the required expression for I which is constituted by Killing twist scalars coupled with the chemical potential, magnitude of heat flow vector, geometrical angular momentum and rotation of the matter part of fluid about the rotation axis. The quantity I that enters in the determination of covariant poloidal components of heat flow vector illustrates the role of Killing twist scalars and their relationship with the heat flow in the meridional plane. Substituting (2.39) into (2.20), we get

$$\left[KC_1 + \left(\lambda^{*}\mu + \alpha q\zeta\ l\right)\tau_{(t)} + \left(\lambda^{*}\mu\Omega + \alpha q\zeta\right)\tau_{(\varphi)}\right]f_{,a} + \lambda Kn\ \left(\Phi_{,a} + j\Omega_{,a}\right) = 0\ , \quad (2.40)$$

which describes the evolution of the matter part of fluid in the meridional plane in terms of thermodynamic variables coupled to Killing twist scalars, the Stokes stream function, injection energy gradient, and differential rotation due to rotation about the rotation axis. Taking curl of (2.40), we find that

$$F_{,[A}f_{,B]} = \Omega_{,[A}j_{,B]}\ , \quad (2.41)$$

where $F = \frac{1}{\lambda Kn}\left[KC_1 + \left(\lambda^{*}\mu + \alpha q\zeta l\right)\tau_{(t)} + \left(\lambda^{*}\mu\Omega + \alpha q\zeta\right)\tau_{(\varphi)}\right]$. The expression of scalar function $F = \frac{I}{\lambda Kn}$ is composed of a combination of thermodynamic variables, Killing twist scalars and indicates the importance of Killing twist scalars for the dynamics of differential rotation which is usually thought to be related with thermodynamic variables that constitute the equation of state (EOS) of a hot matter. Alternatively, it suggest that the role of Killing twist scalars is inevitable in the description of differential rotation. It is evident from (2.41) that the level surfaces of constant Ω and j do not coincide because of alignment of level surfaces of constant F and f in the poloidal plane, in other words, Killing twist scalars representing the twist of dynamical spacetime endowed with meridional circulation contribute to the differential rotation at the onset of dissipation caused by the heat flow. The variation of Stokes stream function in the poloidal plane is caused by the variation of both Ω and j along the meridional circulation velocity. Thus dynamics of differential rotation seems to be complicated in the presence of meridional circulation because of its link with the Killing twist scalars. Thus we confine our attention to understand the role of meridional circulation in inducing extra rotation in addition to the usual rotation about the rotation axis and examine the relationship between the rotational velocity of the matter part of fluid and the thermodynamic variables in the following section.

Rotational velocity of matter part of fluid

This section is focused on the description of dynamic interaction between the matter part of fluid and the entropy fluid in order to understand the link between the rotational velocity of matter part of fluid and thermodynamic quantities contributed by both the matter part of fluid and the entropy fluid. Thus we need to find covariant solution of Maxwell's like equations that govern the evolution of the entropy fluid under spacetime symmetry assumption. In a non-circular stationary axisymmetric spacetime, motion of matter part of fluid is composed of rotation around the rotation axis and meridional circulation occurring in meridional plane. This gives rise to the effect that meridional circulation causes an extra rotational velocity in addition to the usual rotation Ω about the rotation axis, as is well known in a circular spacetime. The rotation of matter part of fluid is linked with the 4-velocity of the entropy fluid because of two reasons as follows. First, it is measured in the matter part of fluid's rest frame and bears a direct relation with the 4-velocity of the matter part of fluid,and second, the 4-velocity of the matter part of fluid enters in the electric part of Maxwell's like equations corresponding to the thermal vorticity 2-form Z_{ab}. The 4-velocity of the entropy fluid u_s^a measured in the matter part of fluid's frame is expressible as:[18]

$$u_s^a = ?(u^a + v_s^a) \quad , \tag{3.1}$$

where $\gamma = \left(1 - v_s^2\right)^{-\frac{1}{2}}$ and $u_a v_s^a = 0$. The relative flow of entropy measured in the matter part of fluid's rest frame is represented by v_s^a which describes the heat flow vector q^a and is given by $v_s^a = \dfrac{q^a}{{}^*s\,{}^*\theta}$. Making use of (2.2) in (3.1), we get

$$u_s^a = \gamma\left(\lambda\xi_{(t)}^a + \lambda\Omega\xi_{(\varphi)}^a + v^a\right) , \tag{3.2}$$

where

$$v^a = w^a + \frac{q^a}{{}^*s\,{}^*\theta} \quad . \tag{3.3}$$

Substituting the expression for q^a obtainable by setting $q^a = qm^a$ and using (2.4) into (3.3), we obtain

$$v^a = a_1\xi_{(t)}^a + a_2\xi_{(\varphi)}^a + w^a \ , \tag{3.4}$$

where $a_1 = \dfrac{q\zeta l}{{}^*s\,{}^*\theta}$ and $a_2 = \dfrac{q\zeta}{{}^*s\,{}^*\theta}$.

From (3.2) and (3.4), we find that

$$u_s^a = \gamma\left[\left(\lambda + a_1\right)\xi_{(t)}^a + \left(\lambda\Omega + a_2\right)\xi_{(\varphi)}^a + w^a\right] . \tag{3.5}$$

Maxwell's like equations associated with the thermal vorticity 2-form Z_{ab} are of the following form[27]

$$Z_{ab}u_s^b = \tilde{\mathrm{E}}_a \ , \tag{3.6a}$$

where

$$Z_{ab} = 2\vartheta_{\{b;a]} \ , \vartheta_a = {}^*\theta u_a + \beta q_a \text{, and } \tilde{\mathrm{E}}_a = \gamma R\left(q_a + \frac{q^2}{{}^*s\,{}^*\theta}u_a\right) . \tag{3.6b}$$

Following Gourgoulhon et al,[33] one may obtain

$$Z_{ab} = -2\bar{\varepsilon}_{,[a}{}^*\xi_{(t)b]} + 2\bar{j}_{,[a}{}^*\xi_{(\varphi)b]} + \frac{\bar{I}}{K}\eta_{abcd}\xi_{(t)}^c\xi_{(\varphi)}^d \ . \tag{3.7}$$

where $\bar{I} = {}^*Z_{ab}\xi_{(t)}^a\xi_{(\varphi)}^b$, ${}^*\theta u_t + \beta q_t = -\bar{\varepsilon}$, and ${}^*\theta u_\varphi + \beta q_\varphi = \bar{j}$. Here $\bar{\varepsilon}$ and $\bar{j}$ represents, respectively, the effective energy per entropon and effective angular momentum per entropy in the sense of Carter.[37]The explicit expression for $\bar{I}$ will be obtained later on in the subsequent discussions.

On account of (3.5) and (3.7), we find from (3.6) that

$$\tilde{\mathrm{E}}_a = \gamma\left(\bar{\varepsilon}_{,b}w^b\right){}^*\xi_{(t)a} - \gamma\left(\bar{j}_{,b}w^b\right){}^*\xi_{(\varphi)a} - \gamma\left(\lambda + a_1\right)\bar{\varepsilon}_{,a} + \gamma\left(\lambda\Omega + a_2\right)\bar{j}_{,a} + \gamma\frac{\bar{I}}{K}\eta_{abcd}w^b\xi_{(t)}^c\xi_{(\varphi)}^d \ . \tag{3.8}$$

Contracting (3.8) with $\xi_{(t)}^a$ and $\xi_{(\varphi)}^a$ in turn and using the defining expression of $\tilde{\mathrm{E}}_a$, we get

$$R\left(q_t + \frac{q^2}{{}^*s\,{}^*\theta}u_t\right) = \bar{\varepsilon}_{,a}w^a \ , \tag{3.9a}$$

$$R\left(q_\varphi + \frac{q^2}{{}^*s\,{}^*\theta}u_\varphi\right) = -\bar{j}_{,a}w^a \ . \tag{3.9b}$$

Multiplying (3.9b) by Ω and adding to (3.9a) and using (2.15), we get

$$u_t + \Omega u_\varphi = \frac{{}^*s\,{}^*\theta}{Rq^2}\left(\bar{\varepsilon}_{,a} - \Omega\bar{j}_{,a}\right)w^a \ . \tag{3.10}$$

On account (2.22), we reduce (3.10) to take form

$$\Phi = -\frac{{}^*\mu\,{}^*s\,{}^*\theta}{Rq^2}\left(\bar{\varepsilon}_{,a} - \Omega\bar{j}_{,a}\right)w^a \ , \tag{3.11}$$

which describes the injection energy of the matter part of fluid in terms of the effective energy per entropon and effective angular momentum per entropy associated with the entropy fluid that couples the rotation of matter part of fluid about the rotation axis.

Making use of (2.1a,b) in (3.8), we find contra variant toroidal components of E_a as follows :

$$\tilde{\mathrm{E}}^t = -\frac{\gamma}{K}\left[g_{\varphi\varphi}\left(\bar{\varepsilon}_{,a}w^a\right) + g_{t\varphi}\left(\bar{j}_{,a}w^a\right)\right] + \frac{\gamma Rq^2}{{}^*s\,{}^*\theta}w^t \tag{3.12a}$$

$$\tilde{\mathrm{E}}^\varphi = \frac{\gamma}{K}\left[g_{t\varphi}\left(\bar{\varepsilon}_{,a}w^a\right) + g_{tt}\left(\bar{j}_{,a}w^a\right)\right] + \frac{\gamma Rq^2}{{}^*s\,{}^*\theta}w^\varphi \tag{3.12b}$$

where third relation of (3.6b) along with (2.4) and $u_a q^a = 0$ are used. Making use of the third relation of (3.6b), we find that

$$\tilde{\Omega} = \frac{1}{l} + \frac{{}^*s\,{}^*\theta u_t}{\gamma R U q^2}\left(\tilde{E}^t - l\tilde{E}^\varphi\right) \quad (3.13)$$

where $\tilde{\Omega} = \frac{u^\varphi}{u^t}$ represents the rotational velocity of the matter part of fluid and $U = u^t u_\varphi$ is the rotational potential.[37] On account of (3.12a,b), we obtain

$$\tilde{E}^t - l\tilde{E}^\varphi = -\frac{\gamma\left(g_{\varphi\varphi} + l g_{t\varphi}\right)}{K}\left(\bar{\varepsilon}_{,a} - \Omega_{,a}\right)w^a + \frac{1}{U}\left(u_t w^t + u_\varphi w^\varphi\right) \quad (3.14)$$

It follows from (2.18) that

$$u_t w^t + u_\varphi w^\varphi = -\frac{\lambda K u_t}{\sqrt{-g}}\epsilon^{AB}\left(\Phi_{,A} + j\Omega_{,A}\right)\left({}^*\xi_{(t)B} - l\,{}^*\xi_{(\varphi)B}\right) \quad (3.15)$$

where ϵ^{AB} denotes an alternating symbol taking values 1 or -1.

It follows from (3.13), (3.14),and (3.15) that

$$\tilde{\Omega} = \frac{q^\varphi}{q^t} + \frac{{}^*s\,{}^*\theta u_t\left(g_{\varphi\varphi} + l g_{t\varphi}\right)\bar{\Phi}}{RKUq^2}\left(\ln\frac{{}^*s}{n}\right)_{,a} w^a - \frac{\lambda K u_t}{U\sqrt{-g}}\epsilon^{AB}\left(\Phi_{,A} + j\Omega_{,A}\right)\left({}^*\xi_{(t)B} - l\,{}^*\xi_{(\varphi)B}\right), \quad (3.16)$$

where the relations $q^t = l q^\varphi$ and $\left(\bar{\varepsilon}_{,a} - \Omega\bar{j}_{,a}\right)w^a = -\bar{\Phi}\left(\ln\frac{{}^*s}{n}\right)_{,a} w^a$ are used. This second relation will be derived later on and is given in Sec.5. The symbol $\bar{\Phi} = \left(\bar{\varepsilon} - \Omega\bar{j}\right)$ represents the injection energy per entropon corresponding to the entropy fluid. It is observed from (3.16) that the rotational velocity of the matter part of fluid is composed of the sum of three terms of which (i) the first term is the ratio of the toroidal components of the heat flow vector, (ii) the second term is the variation of the entropy per baryon along the meridional circulation velocity and is coupled to thermodynamic variables and metric components, and (iii) third term is the rotation contributed by the linear combination of the gradient of the injection energy corresponding to the matter part of the fluid and the differential rotation arising due to the rotation about the rotation axis and the rotational potential.

In order to understand the interpretation of (3.16) as the rotational velocity of the matter part of the fluid we turn back to (2.2) which tells us that

$$\tilde{\Omega} = \frac{\Omega + \frac{w^\varphi}{\lambda}}{1 + \frac{w^t}{\lambda}} \approx \Omega + \frac{1}{\lambda}\left(w^\varphi - \Omega w^t\right), \quad (3.17)$$

where in the expansion second and higher orders are ignored. The second term on the right hand side of (3.17) may be regarded as the rotational velocity of the matter part of fluid arising due to the meridional circulation velocity and will now be denoted by the symbol $\bar{\Omega}$ for further discussion. The first term Ω on the right hand side of (3.17) represents the rotational velocity of the matter part of fluid about the rotation axis. Making use of $w^\varphi = -\frac{1}{K}\left(g_{t\varphi}g_{tA} - g_{tt}g_{\varphi A}\right)u^A$ and $w^t = \frac{1}{K}\left(g_{\varphi\varphi}g_{tA} - g_{t\varphi}g_{\varphi A}\right)u^A$ obtainable from (2.2) and simplifying, one may obtain

$$\bar{\Omega} = \frac{1}{\lambda K}\left[\left(\tilde{\omega} - \Omega\right)g_{\varphi\varphi}g_{tA}u^A + \left(g_{tt} - \Omega\tilde{\omega}g_{\varphi\varphi}\right)g_{\varphi A}u^A\right], \quad (3.18)$$

where $\tilde{\omega} = -\frac{g_{t\varphi}}{g_{\varphi\varphi}}$ represents the frame dragging effect.[35] Thus (3.17) may be rewritten as

$$\tilde{\Omega} = \Omega + \bar{\Omega}\ . \quad (3.19)$$

It is evident from (3.16) and (3.17) that these two versions of rotational velocity of the matter part of fluid are different in the sense that (3.16) describes the rotation in terms of the geometrical angular momentum l while (3.17) split the rotation into two parts composed of rotation about the rotation axis and the rotation caused by the meridional circulations. As is known that the geometric angular momentum l is expressible as a function of rotational velocity about the rotation axis in the case of circular spacetime but due to non-circularity assumption such explicit functional relation does not seem possible because of the presence of meridional circulation velocity. Thus we need to deduce an expression for the rotational velocity of the matter part of fluid which may resemble with (3.17).We now confine our attention to derivation of such expression by invoking Maxwell's like equations describing the motion of the entropy fluid and to obtain the relation between the rotation of both the matter part and the entropy fluids in following section.

Rotational velocity of the entropy fluid

This section is devoted to study the rotation of the entropy fluid induced by the rotation of the matter part of fluid that may arise because of dynamic coupling between the matter part of fluid and the entropy fluid and to explore various consequences related with the poloidal components of the matter part of fluid's 4-velocity and the entropy production rate. The presence of Killing twist scalars due to meridional circulation of the matter part of fluid is expected to contribute to the dynamic evolution of the entropy fluid. This requires to find a covariant solution to Maxwell's like equations associated with the thermal vorticity 2-form Z_{ab}. For computational convenience, we use the exterior calculus and results developed in.[33] In the language of differential forms, (3.6) is expressible as

$$u_s . d\underline{\vartheta} = -\underline{\tilde{E}} \quad (4.1)$$

Since $u_s.d\underline{\vartheta}$ is a 1-form, it can be decomposed as

$$u_s.d\underline{\vartheta} = b_1{}^*\xi_{(t)} + b_2{}^*\xi_{(\varphi)} + \underline{Y} \tag{4.2}$$

where 1-form $\underline{Y}$ lies in the meridional plane orthogonal to both $\xi_{(t)}$ and $\xi_{(\varphi)}$. Contraction of (4.2) with $\xi_{(t)}$ and $\xi_{(\varphi)}$ in turn gives that

$$u_s.d\underline{\vartheta}.\xi_{(t)} = b_1\ , \tag{4.3a}$$

$$u_s.d\underline{\vartheta}.\xi_{(\varphi)} = b_2\ . \tag{4.3b}$$

Making use of the symmetry conditions $\pounds_{\xi_{(t)}}\underline{\vartheta} = 0 = \pounds_{\xi_{(\varphi)}}\underline{\vartheta}$ in the left hand side of (4.3a) and (4.3b), respectively, with the aid of (3.5), we get

$$b_1 = \gamma w.d\left(\underline{\vartheta}.\xi_{(t)}\right)\ ,\ b_2 = \gamma w.d\left(\underline{\vartheta}.\xi_{(\varphi)}\right)\ . \tag{4.4}$$

Substitution of (4.4) in (4.2) yields that

$$u_s.d\underline{\vartheta} = \gamma\left[w.d\left(\underline{\vartheta}.\xi_{(t)}\right)\right]{}^*\xi_{(t)} + \gamma\left[w.d\left(\underline{\vartheta}.\xi_{(\varphi)}\right)\right]{}^*\xi_{(\varphi)} + \underline{Y}\ . \tag{4.5}$$

In order to compute $\underline{Y}$, we proceed with the following construction

$$u_s.d\underline{\vartheta} = u_s.d\underline{x} + u_s.d\underline{y} + u_s.d\left({}^*\theta\underline{w}\right)\ , \tag{4.6}$$

Where

$$\underline{x} = \lambda{}^*\theta\underline{\xi}_{(t)} + \lambda\Omega{}^*\theta\underline{\xi}_{(\varphi)} \text{ and } \underline{y} = \beta q\zeta l\underline{\xi}_{(t)} + \beta q\zeta\underline{\xi}_{(\varphi)}\ . \tag{4.7}$$

Performing similar calculations as are done in Sec.3 in the case of particle vorticity 2-form, we find that

$$u_s.d\underline{\vartheta} = -\gamma\left(\lambda + a_1\right)d\left[\left(\underline{x}+\underline{y}\right).\xi_{(t)}\right] - \gamma\left(\lambda\Omega + a_2\right)d\left[\left(\underline{x}+\underline{y}\right).\xi_{(\varphi)}\right] + \frac{\gamma C_2}{n}df + \gamma w.d\underline{x} + \gamma w.d\underline{y}\ . \tag{4.8}$$

Replacing the left hand side of (4.5) by (4.8) and contracting the resulting equation with an arbitrary vector v lying in the meridional plane, we get

$$\underline{Y}.v = u_s.d\underline{\vartheta}.v = -\gamma\left(\lambda + a_1\right)v.d\left[\left(\underline{x}+\underline{y}\right).\xi_{(t)}\right] - \gamma\left(\lambda\Omega + a_2\right)v.d\left[\left(\underline{x}+\underline{y}\right).\xi_{(\varphi)}\right] + \frac{\gamma C_2}{n}v.df + \gamma w.d\underline{x}.v + \gamma w.d\underline{y}.v \tag{4.9}$$

Making use of the results $w.d\underline{x}.v = \dfrac{\lambda{}^*\theta}{Kn}\left(\tau_{(t)} + \Omega\tau_{(\varphi)}\right)df.v$ and $w.d\underline{y}.v = \dfrac{\beta q\zeta}{Kn}\left(l\tau_{(t)} + \tau_{(\varphi)}\right)df.v$

In the last two terms of (4.9) and simplifying, we find that

$$\underline{Y} = -\gamma\left(\lambda + a_1\right)d\left[\left(\underline{x}+\underline{y}\right).\xi_{(t)}\right] - \gamma\left(\lambda\Omega + a_2\right)d\left[\left(\underline{x}+\underline{y}\right).\xi_{(\varphi)}\right] + \frac{\gamma}{Kn}\left[KC_2 + \left(\lambda{}^*\theta + \beta q\zeta l\right)\tau_{(t)} + \left(\lambda\Omega{}^*\theta + \beta q\zeta\right)\tau_{(\varphi)}\right]df \tag{4.10}$$

Using $\left(\underline{x}+\underline{y}\right).\xi_{(t)} = -\bar{\varepsilon}$ and $\left(\underline{x}+\underline{y}\right).\xi_{(\varphi)} = \bar{j}$ in (4.10) and $\underline{\vartheta}.\xi_{(t)} = -\bar{\varepsilon}$, $\bar{\vartheta}.\xi_{(\varphi)} = \bar{j}$ in (4.5), respectively, and substituting the resulting expressions in (4.1), we obtain

$$\underline{\tilde{\mathrm{E}}} = \gamma\left(w.d\bar{\varepsilon}\right){}^*\xi_{(t)} - \gamma\left(w.d\bar{j}\right){}^*\xi_{(\varphi)} - \gamma\left(\lambda + a_1\right)d\bar{\varepsilon} + \gamma\left(\lambda\Omega + a_2\right)d\bar{j} - \frac{\gamma}{Kn}\left[KC_2 + \left(\lambda{}^*\theta + \beta q\zeta l\right)\tau_{(t)} + \left(\lambda\Omega{}^*\theta + \beta q\zeta\right)\tau_{(\varphi)}\right]df\ . \tag{4.11}$$

It follows from (3.8) and (4.11) that

$$\bar{I} = KC_2 + \left(\lambda{}^*\theta + \beta q\zeta l\right)\tau_{(t)} + \left(\lambda\Omega{}^*\theta + \beta q\zeta\right)\tau_{(\varphi)}, \tag{4.12}$$

This is the required expression for $\bar{I}$ which is constituted by Killing twist scalars, local temperature, the entropy entrainment, the magnitude of the heat flow vector, rotation of the matter part of fluid about the rotation axis, and gravitational potentials encoded in λ which relates meridional circulation velocity magnitude.

From (2.8) and (3.6b), we obtain that

$$u_a = \frac{{}^*s{}^*\theta}{\gamma Rq^2}\left(\tilde{\mathrm{E}}_a - \gamma\nu\mathrm{E}_a\right)\ , \tag{4.13}$$

where $\nu = \dfrac{n}{\left({}^*s - \dfrac{\beta q^2}{{}^*\theta^2}\right)}$. Making use of (2.38), (2.40), and (4.11) in (4.13), we get

$$u_a = \frac{{}^*s{}^*\theta}{Rq^2}\left[A_3{}^*\xi_{(t)a} - A_4{}^*\xi_{(\varphi)a} - \lambda\bar{\varepsilon}_{,a} + \lambda\Omega\bar{j}_{,a} + \frac{q\zeta}{{}^*s{}^*\theta}\left({}_{,a} - l\bar{\varepsilon}_{,a}\right) - \left(\frac{\bar{I}}{Kn}\right)f_{,a}\right] \tag{4.14a}$$

where

$$A_3 = \left(\bar{\varepsilon}_{,b}w^b\right) + \nu\left(\varepsilon_{,b}w^b\right)\ ,\ A_4 = \left(j_{,b}w^b\right) + \nu\left(j_{,b}w^b\right)\ . \tag{4.14b}$$

Contracting (4.14a) with $\xi^a_{(t)}$ and $\xi^a_{(\varphi)}$ in turn, respectively, we get

$$u_t = A_3\ ,\ u_\varphi = -A_4\ . \tag{4.15}$$

From the defining relations of effective energies and effective angular momenta corresponding to both the matter part and entropy fluids and the relation $+{}^*s\beta = 1$, we have

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$$nEu_t + q_t = -\left(n\varepsilon + {}^*s\bar{\varepsilon}\right) \text{ and } nEu_\varphi + q_\varphi = \left(nj + {}^*s\bar{j}\right), \quad (4.16)$$

where $E = {}^*\mu + \frac{{}^*s{}^*\theta}{n}$ is the energy of the heat conducting fluid per baryon. It follows from (2.10) that

$$Rq_t = -\nu\left(\varepsilon_{,b}w^b\right) \text{ and } Rq_\varphi = \nu\left(j_{,b}w^b\right) \quad (4.17)$$

From (4.16) and (4.17),we find that

$$l = \frac{\frac{1}{R}\left(j_{,b}w^b\right) + \left(nj + {}^*s\bar{j}\right)}{\frac{1}{R}\left(\bar{\varepsilon}_{,b}w^b\right) + \left(n\varepsilon + {}^*s\bar{\varepsilon}\right)}, \quad (4.18)$$

which determines the geometrical angular momentum per particle of the matter part of the fluid in terms of the effective energies and angular momenta corresponding to both the matter part and the entropy fluids. As is seen that the first term on the right hand side of (3.13) is the reciprocal of the geometrical angular momentum that enters in the derivation of an expression describing the rotational velocity of the matter part of fluid given by (3.18).But this derivation loses similarity with that of (3.16). Thus in order to achieve exact similarity with (3.16), we proceed as follows.

From (4.14a), with the aid of (2.1a,b), we get

$$u^t = -\frac{{}^*s{}^*\theta}{RKq^2}\left[g_{\varphi\varphi}\left(\bar{\varepsilon}_{,b}w^b\right) + g_{t\varphi}\left(\bar{j}_{,b}w^b\right) + \frac{R}{\lambda}u_\varphi q_\varphi\right] + w^t, \quad (4.19)$$

$$u^\varphi = \frac{{}^*s{}^*\theta}{RKq^2}\left[g_{t\varphi}\left(\bar{\varepsilon}_{,b}w^b\right) + g_{tt}\left(\bar{j}_{,b}w^b\right) + \frac{R}{\lambda}u_t q_\varphi\right] + w^\varphi. \quad (4.20)$$

Multiplying (4.19) by Ω and subtracting the resulting equation from (4.20) and simplifying with the aid of (2.2) and (2.22), we obtain that

$$\tilde{\Omega} = \Omega + \frac{{}^*s{}^*\theta}{\left(\lambda + u^t\right)\lambda RKq^2}\left[u_\varphi\left(\bar{\varepsilon}_{,b}w^b\right) + u_t\left(\bar{j}_{,b}w^b\right) - \frac{R\Phi}{{}^*\mu}q_\varphi\right], \quad (4.21)$$

which bears a complete resemblance with (3.16). Contraction of the third relation of (3.6b) with u^a yields that

$$\tilde{\mathrm{E}}_a u^a = -\frac{\gamma Rq^2}{{}^*s{}^*\theta}. \quad (4.22)$$

Contraction of (3.8) with u^a gives that

$$\tilde{\mathrm{E}}_a u^a = \frac{\gamma q\zeta}{{}^*s{}^*\theta}\left(\bar{j}_{,a} - l\bar{\varepsilon}_{,a}\right)w^a. \quad (4.23)$$

It follows from (4.22) and (4.23) that

$$\left(l\bar{\varepsilon}_{,a} - \bar{j}_{,a}\right)w^a = \frac{Rq}{\zeta}. \quad (4.24)$$

On account of (4.24), one can reduce (4.21) to take the following form

$$\tilde{\Omega} = \Omega - \frac{{}^*s{}^*\theta}{\left(\lambda + u^t\right)\lambda Kq^2}\left(\frac{q}{\zeta}u_t + \frac{\Phi}{{}^*\mu}q_\varphi\right) \quad (4.25)$$

which exhibits that the rotational velocity of the matter part of fluid is split into two parts: (i) rotation about the rotation axis and (ii) the rotation caused by the meridional circulation describable by the second term on the right hand side of (4.25). From (2.22) and (4.18), we get

$$\Omega = \left[\frac{1}{R}\left(\bar{j}_{,b}w^b\right) + \left(nj + {}^*s\bar{j}\right)\right]^{-1}\left[\frac{1}{R}\left(\bar{\varepsilon}_{,b}w^b\right) + \left(n\varepsilon + {}^*s\bar{\varepsilon}\right) - \frac{\Phi}{{}^*\mu}\left(nE - \frac{q^2}{{}^*s{}^*\theta}\right)\right], \quad (4.26)$$

which gives rotational velocity of the matter part of fluid about the rotation axis in terms of thermodynamic variables.

Making use of relation $u_s^a = \gamma\left(u^a + \frac{q^a}{{}^*s{}^*\theta}\right)$, we define the rotational velocity of the entropy fluid as

$${}^*\Omega = \frac{u_s^\varphi}{u_s^t} = \frac{\left(u^\varphi + \frac{q^\varphi}{{}^*s{}^*\theta}\right)}{\left(u^t + \frac{q^t}{{}^*s{}^*\theta}\right)} = \left(\tilde{\Omega} + \frac{q^\varphi}{u^t\,{}^*s{}^*\theta}\right)\left(1 + \frac{q^t}{u^t\,{}^*s{}^*\theta}\right)^{-1}, \quad (4.27)$$

which can be linearized by ignoring second and higher terms in the expansion of second small bracket on the right hand side of (4.27) to obtain

$${}^*\Omega \approx \tilde{\Omega} + \frac{q\zeta}{u^t\,{}^*s{}^*\theta}\left(1 - \tilde{\Omega}l\right), \quad (4.28)$$

where $q^t = q\zeta l$ and $q^t = q\zeta l$ are used.

Multiplying (3.9a) by l and adding the resulting equation to (3.9b), we get

$$R\left(lq_t + q_\varphi\right) = \left(l\varepsilon_{,a} - j_{,a}\right)w^a, \quad (4.29)$$

which because of (4.17) takes the form

$$lq_t + q_\varphi = \frac{q}{\zeta}. \quad (4.30)$$

Using the fact that $u^a q_a = 0$, one may find that

$$u^t\left(q_t + \tilde{\Omega}q_\varphi\right) + u^A q_A = 0. \quad (4.31)$$

It follows from (2.38) and (2.40) that

$$u^A q_A = \frac{\nu}{R}\left[-\left(\varepsilon_{,b}w^b\right){}^*\xi_{(t)A} + \left(j_{,b}w^b\right){}^*\xi_{(\varphi)A}\right]u^A, \quad (4.32)$$

which, due to (4.17) , takes the form

$$u^A q_A = q_\varphi\left({}^*\xi_{(\varphi)A} - \Omega^*\xi_{(t)A}\right)w^A. \quad (4.33)$$

Making use of (2.19) on the right hand side of (4.33), we get

$$u^A q_A = \frac{q_\varphi}{n\sqrt{-g}}\epsilon^{AB}\left({}^*\xi_{(\varphi)A} - \Omega^*\xi_{(t)A}\right)f_{,B}. \quad (4.34)$$

Using $q_t = -\Omega q_\varphi$ in (4.31) and simplifying with the aid of (4.34), we get

$$\left(1 - \tilde{\Omega}l\right) = \frac{\tilde{\Omega}q}{\Omega\zeta q_\varphi} + \frac{1}{n\Omega u^t\sqrt{-g}}\epsilon^{AB}\left({}^*\xi_{(\varphi)A} - \Omega^*\xi_{(t)A}\right)f_{,B}. \quad (4.35)$$

From (4.27) and (4.35), we get

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$$\tilde{\Omega} \approx \tilde{\Omega}\left(1+\frac{q^2}{\Omega^* s^* \theta u^t q_\varphi}\right)+\frac{q\zeta}{n\Omega^* s^* \theta\left(u^t\right)^2\sqrt{-g}}\epsilon^{AB}\left({}^*\xi_{(\varphi)A}-\Omega^*\xi_{(t)A}\right)f_{,B} \tag{4.36}$$

which exhibits that the rotation of the matter part of fluid contributes to the rotation of the entropy fluid besides contributions due to a combination of thermodynamic quantities. This in turn implies that the friction caused by the difference of rotational velocities of the entropy fluid and the matter part of fluid is directly linked with the heat flow. It is the presence of heat that causes the entropy fluid to rotate with different rotational velocity than the rotation of the matter part of fluid. Consequently, the entropy fluid is not corotating with the matter part of fluid.

Creation of injection energy per baryon

This section is concerned with the description of creation of injection energy per baryon due to interaction of heat flow with the motion of a heat conducting fluid and determination of the magnitude of meridional circulation velocity in terms of thermodynamic quantities. In order to demonstrate injection energy creation we invoke the conservation laws of energy and angular momentum currents associated with a stationary axisymmetric heat conducting fluid configuration. The energy current conservation law[21] states that there is exchange between the total energy per baryon of the matter part of fluid and of the heat flux coupled with the effective energy per entropn associated with the entropy fluid per unit of local temperature measured in the matter part of fluid's rest frame. Similar exchange law holds for the conservation of angular momentum current. These two laws are explicitly expressible as:[21]

$$nE_{,a}u^a+\left(\frac{\varepsilon}{{}^*\theta}q^a\right)_{;a}=0\ , \tag{5.1a}$$

$$nL_{,a}u^a+\left(\frac{j}{{}^*\theta}q^a\right)_{;a}=0\ , \tag{5.1b}$$

where

$$-E=\left({}^*\mu+\frac{{}^*s\,{}^*\theta}{n}\right)u_t+\frac{q_t}{n}\ ,\quad L=\left({}^*\mu+\frac{{}^*s\,{}^*\theta}{n}\right)u_\varphi+\frac{q_\varphi}{n}\ . \tag{5.1c}$$

Multiplying the relation ${}^*\mu u_t+\alpha q_t=-\varepsilon$ by n and ${}^*\theta u_t+\beta q_t=\bar{\varepsilon}$ by ${}^*s$ and adding these two resulting relations with the aid of $n\alpha+{}^*s\beta=1$, we get

$$E=\left(\varepsilon+\frac{{}^*s\bar{\varepsilon}}{n}\right)\ . \tag{5.2}$$

Similarly, we find that

$$L=\left(j+\frac{{}^*s\bar{j}}{n}\right)\ . \tag{5.3}$$

Substituting $q^a=q\zeta\left(\xi^a_{(\varphi)}+l\xi^a_{(t)}\right)$ in the second term of (5.1a) and (5.1b), respectively, and simplifying, we get

$$E_{,a}u^a=0 \quad \text{or} \quad E_{,a}w^a=0 \tag{5.4a}$$

$$L_{,a}u^a=0 \quad \text{or} \quad L_{,a}w^a=0\ . \tag{5.4b}$$

Multiplying the equation obtained by substituting (5.3) in (5.4b) by Ω and subtracting it from the resulting equation obtained by substituting (5.2) in (5.4a), we get

$$\left(\varepsilon_{,a}-\Omega j_{,a}\right)w^a+\left(\frac{{}^*s}{n}\right)\left(\bar{\varepsilon}_{,a}-\Omega\bar{j}_{,a}\right)w^a+\left(\bar{\varepsilon}-\Omega\bar{j}\right)\left(\frac{{}^*s}{n}\right)_{,a}w^a=0 \tag{5.5}$$

Contraction of (2.16) with w^a gives a relation which makes the first term zero and hence (5.5) reduces to

$$\left(\bar{\varepsilon}_{,a}-\Omega\bar{j}_{,a}\right)w^a=-\left(\bar{\varepsilon}-\Omega\bar{j}\right)\left(\frac{{}^*s}{n}\right)_{,a}w^a\ , \tag{5.6}$$

which is the required relation used in (3.16) to derive an explicit expression for the rotational velocity of matter part of fluid.

It follows from (3.11) and (5.6) that

$$\Phi=\frac{{}^*\mu\,{}^*s\,{}^*\theta}{Rq^2}\left(\bar{\varepsilon}-\Omega\bar{j}\right)\left(\frac{{}^*s}{n}\right)_{,a}w^a\ , \tag{5.7}$$

which shows that the variation of entropy per baryon along the meridional circulation velocity is responsible for the creation of the injection energy per baryon besides the contributions made by other thermodynamic quantities. If entropy per baryon is assumed to be constant along the meridional circulation velocity, then the injection energy per baryon becomes zero which is in contradiction with the very definition of injection energy.[35] Hence, we arrive at the conclusion that the variation of entropy per baryon along the meridional circulation velocity generates the injection energy per baryon.

On account of (2.15), we find from (5.1c) that

$$-E+\Omega L=\left({}^*\mu+\frac{{}^*s\,{}^*\theta}{n}\right)\left(u_t+\Omega u_\varphi\right). \tag{5.8}$$

Substituting $u_t=\lambda\left(g_{tt}+\Omega g_{t\varphi}\right)$ and $u_\varphi=\lambda\left(g_{t\varphi}+\Omega g_{\varphi\varphi}\right)$ in (5.8) and simplifying with the aid of (2.3), we get

$$\left(1+w^2\right)=\frac{\lambda}{\left({}^*\mu+\frac{{}^*s\,{}^*\theta}{n}\right)}\left(E-\Omega L\right)\ . \tag{5.9}$$

From (5.2) and (5.3), we get

$$E-\Omega L=\bar{\Phi}+\frac{{}^*s}{n}\bar{\Phi}\ , \tag{5.10}$$

where $\bar{\Phi}=\bar{\varepsilon}-\Omega\bar{j}$ represents the injection energy per entropon.

Replacing Φ from (5.7) in (5.10) and using the resulting equation in (5.9), we obtain

$$1+w^2 = \frac{\lambda^* s\bar{\Phi}}{\left(n^*\mu+\right)}\left[1+\frac{n^*\mu^*\theta}{Rq^2}\left(\ln\frac{^*s}{n}\right)_{,a} w^a\right], \quad (5.11)$$

which gives the relation between the squared magnitude of the meridional circulation velocity and the variation of entropy per baryon along the matter part of fluid's 4-velocity. This means that the flow of matter part of fluid cannot be isentropic in the presence of dissipation caused by the heat flow. The variation of entropy per baryon along the meridional circulation velocity plays the dual role in the creation of the injection energy per baryon and also contributes to the squared magnitude of meridional circulation velocity.

From (5.8) and (5.9) with the aid of (2.3) and (2.22), we get

$$\Phi = \lambda^*\mu G \ , \quad (5.12)$$

which is an alternative version of (5.7) in terms metric tensor components associated with the surface of transitivity and the rotational velocity about the rotation axis. This relation may be used to determine the conditions under which a heat conducting fluid configuration admits clockwise or anti-clockwise rotation about the rotation axis.

Link between differential rotation and thermal-fluid vorticity

In this section we find a relation describing the differential rotation of the matter part of fluid along the thermal-fluid vorticity. The key idea which motivates to explore such relation originates from most celebrated Ferraro's law of isorotation in RMHD[38] because the thermal-fluid vorticity is the magnetic part of of thermal-fluid vorticity 2-form W_{ab}. This is composed of a linear combination of matter part of fluid's vorticity vector and spacelike twist vector associated with a congruence of heat flow lines. It is defined as follows:

$$V^a = {}^*W^{ab}u_b \, , \quad (6.1a)$$

which is explicitly expressible as

$$V^a = 2\,{}^*\mu\omega^a + 2\alpha q\hat{\omega}^a \, , \quad (6.1b)$$

and obeys the following condition

$$\overset{\odot}{m}_a = \left(\ln\alpha q\right)_{,a} \, . \quad (6.1c)$$

Here ω^a is the matter part of fluid's vorticity vector defined by $\omega^a = \frac{1}{2}\eta^{abcd}u_b u_{c;d}$. The spacelike twist vector $\hat{\omega}^a = \frac{1}{2}\eta^{abcd}u_b\hat{\omega}_{cd}$ of the congruence of heat flow lines, where $\hat{\omega}_{ab}$ denotes its rotation tensor.[39] The spacelike twist vector $\hat{\omega}^a$ is directed along m^a because of the identit $\hat{\omega}^a = \left(\hat{\omega}_b m^b\right)m^a$. The curvature vector associated with the congruence of heat flow lines is $\overset{\odot}{m}_a = m_{a;b}m^b$.

From (2.4), we find that

$$\overset{\odot}{m}_a = \left(\ln\zeta\right)_{,a} + \frac{\lambda\Omega K^2\zeta^2}{u_t}l_{,a} \quad (6.2)$$

Using the defining relations of effective energy and angular momentum per particle associated with the matter part of fluid, we get

$$\alpha\left(lq_t + q_\varphi\right) = -\left(l\varepsilon - j\right) \, , \quad (6.3)$$

which because of (4.29) takes the form

$$\frac{\alpha q}{\zeta} = -\left(l\varepsilon - j\right) \, . \quad (6.4)$$

From (6.1c) and (6.2), one may find that

$$l_{,a} = \frac{u_t}{\alpha q\lambda\Omega\zeta K^2}\left(\frac{\alpha q}{\zeta}\right)_{,a} \, , \quad (6.5)$$

which shows that the entropy entrainment multiplied by the magnitude of heat flow vector contributes to the variation of geometrical angular momentum per particle.

It follows from (6.4) and (6.5) that

$$\left(l\varepsilon_{,a} - j_{,a}\right) = -\left(1+\frac{\varepsilon u_t}{\alpha\lambda q\Omega K^2}\right)\left(\frac{\alpha q}{\zeta}\right)_{,a} . \quad (6.6)$$

The magnetic part V^a of thermal-fluid vorticity 2-form W_{ab} is given by [21]

$$V^a = -\frac{1}{K}\left(Iu_\varphi + a_5\right)\xi^a_{(t)} + \frac{1}{K}\left(Iu_t + a_6\right)\xi^a_{(\varphi)} + \frac{u_t}{K}\eta^{abcd}\left(l\varepsilon_{,b} - j_{,b}\right)\xi_{(t)c}\xi_{(\varphi)d} \, , \quad (6.7)$$

where

$$a_5 = \eta^{abcd}w_a j_{,b}\xi_{(t)c}\xi_{(\varphi)d} \text{ and } a_6 = -\eta^{abcd}w\varepsilon_{a,b}\xi_{(t)c}\xi_{(\varphi)d} \, . \quad (6.8)$$

Using (6.7), we find that

$$\tilde{\Omega}_{,a}V^a = \frac{u_t}{K}\eta^{abcd}\tilde{\Omega}_{,a}\left(l\varepsilon_{,b} - j_{,b}\right)\xi_{(t)c}\xi_{(\varphi)d} \, . \quad (6.9)$$

Substituting (6.6) into (6.9) and simplifying, we obtain

$$\tilde{\Omega}_{,A}V^A = -\frac{u_t}{\sqrt{-g}}\left(1+\frac{u_t}{\alpha q\lambda\Omega K^2}\right)\epsilon^{AB}\tilde{\Omega}_{,A}\left(\frac{\alpha q}{\zeta}\right)_{,B} \, , \quad (6.10)$$

which because of (4.25) takes the form

$$\tilde{\Omega}_{,A}V^A = -\frac{u_t}{\sqrt{-g}}\left(1+\frac{\varepsilon u_t}{\alpha q\lambda\Omega K^2}\right)\epsilon^{AB}\left(\Omega_{,A} - \Lambda_{,A}\right)\left(\frac{\alpha q}{\zeta}\right)_{,B} \, , \quad (6.11)$$

where

$$\Lambda = \frac{{}^*s\,{}^*\theta}{\left(\lambda + u^t\right)\lambda K q^2}\left(\frac{q}{\zeta}u_t + \frac{\Phi}{{}^*\mu}q_\varphi\right) . \quad (6.12)$$

It is seen from (6.11) that $\tilde{\Omega}_{,A}V^A \neq 0$. This in turn implies that

$$\tilde{\Omega}_{,A}\omega^A \neq 0 \, , \quad (6.13)$$

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which exhibits that the law of gravitational isorotation ceases to hold in the sense of Glass.[40] Thus we arrive at the conclusion that the law of gravitational isorotation breaks down in the case of an axisymmetric stationary heat conducting fluid configurartion due to the entropy entrainment.

Conclusion

The present work is focused on the rotation of a heat conducting fluid configuration based on Carter's model under the assumption that the background spacetime is non-circular stationary and axisymmetry. It is found that a linear combination of the injection energy gradient and the gradient of rotational velocity about rotation axis is constant along the matter part of fluid flow lines. The level surfaces of constant angular velocity about rotation axis do not coincide with level surfaces of constant effective angular momentum per baryon corresponding to the matter part of fluid because of the variation of Killing twist scalars coupled with thermodynamic quantities in meridional planes. The rotation of matter part of fluid composed of rotation about rotation axis and an additional rotation generated by meridional circulations are completely describable in terms of thermodynamic variables such as the heat flow, injection energy per baryon, chemical potential of matter part of fluid, and the rotational potential created by dynamic space time as an outcome of interaction between the motion of the entropy fluid and of the matter part of fluid. The meridional circulation velocity contributes to the entropy production besides the contributions made by the other thermodynamic quantities. The entropy fluid is not corotating with the matter part of fluid in the presence of dissipation caused by the heat flow. It is found that the law of gravitational isorotation breaks down due to the entropy entrainment in the case of an axisymmetric stationary heat conducting fluid.

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Conflicts of interest

Author declared there is no conflicts of interest.

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