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# Some comments on electromagnetic oscillations in anisotropic cavities-wave equations and boundary conditions

## Abstract

We present several new studies on the mathematical methods formulation of the important problem of electromagnetic oscillations in cavities on anisotropic and axial- anisotropic media, especially on the writing of the relevant dynamical wave equations and their correct boundary conditions. This study is the physical model for applications of a path integral method presented earlier.[1]

**Keywords:** anisotropic cavities, electromagnetic oscillations, Maxwell equations, Hodge-Helmotz theorem

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## Electromagnetic oscillations dynamical wave equations

We start this not by writing the initial value problem for Maxwell equations in the presence of sources and in a compact domain $\Omega$ with boundary $\partial\Omega$ with spatially variable constitutive parameter $(\varepsilon(\vec{r}), \mu(\vec{r}), \sigma(\vec{r}))$.[1–4]

$$\vec{\nabla}\times\vec{E}+\mu\frac{\partial}{\partial t}\vec{H}=0 \qquad (1)$$

$$\vec{\nabla}\times\vec{H}-\mu\varepsilon\frac{\partial\vec{E}}{\partial t}=\mu\vec{j}+\sigma\mu\vec{E} \qquad (2)$$

$$\vec{\nabla}\cdot\vec{B}=0 \qquad (3)$$

$$\vec{\nabla}(\varepsilon\vec{E})=\rho \qquad (4)$$

$$\vec{E}(\vec{r},0)=\vec{E}_0(\vec{r}) \qquad (5)$$

$$\vec{H}(\vec{r},0)=\vec{H}_0(\vec{r}) \qquad (6)$$

Let us firstly search appropriated boundary conditions to be imposed on the electric and magnetic fields $\vec{E}(\vec{r},t)$ and $\vec{H}(\vec{r},t)$ respectively in order to lead to the problem unicity of the above written set of PDE's, eq(l)-eq(6).

On basis of eq(l) and eq(2), we have the obvious energy balance equation for the PDE's system above written

$$\left[\vec{H}\cdot\left(\vec{\nabla}\times\vec{E}+\mu\frac{\partial}{\partial t}\vec{H}\right)\right]-\left[\left(\vec{\nabla}\times\vec{H}\right)\cdot\vec{E}-\mu\varepsilon\frac{\partial\vec{E}}{\partial t}\cdot\vec{E}-\vec{j}\mu\vec{E}-\sigma\mu\vec{E}\cdot\vec{E}\right]=0 \qquad (7)$$

In other words, we have the following electromagnetic energy balance equation in $\Omega$

$$-\left(\frac{1}{2}\right)\int_\Omega\frac{\partial}{\partial t}(\varepsilon\mu\vec{E}^2+\mu\vec{H}^2)d^3r)=+\int_\Omega(\mu\vec{E}.\vec{J})d^3r+\int_\Omega\mu\sigma(\vec{E})^2d^3x+\int_{\partial\Omega}\overbrace{\left((\vec{E}\times\vec{H}).\vec{n}\right)}^{Flux\ Poynting\ vector}dA \qquad (8)$$

It one supposes that has two different set of solutions for Maxwell equations eq(l) and eq(2), namely $(E_1,H_1)$ and $(E_2,H_2)$, then theirs difference satisfy the obvious inequality

$$+\frac{1}{2}\frac{\partial}{\partial t}\left[\overbrace{\int_\Omega\left(\mu\in(\Delta\vec{E})^2+\mu(\Delta\vec{H})^2d^3r\right)}^{\varepsilon(t)}\right]\le-\int_{\partial\Omega}((\Delta\vec{E}\times\Delta\vec{H})\cdot\vec{n})dA \qquad (9)$$

since the loss of ohmic energy is positive-definite

$$\int_\Omega\mu\sigma(\Delta\vec{E})^2d^3x\ge 0 \qquad (10)$$

Now if one chooses boundary conditions that lead to the vanishing of the Poyinting flux on eq(8), one gets the problem unicity since

$$\frac{1}{2}\frac{\partial}{\partial t}\mathcal{E}(t)\le 0\Rightarrow\mathcal{E}(t)\le\mathcal{E}(0)=0\,. \qquad (11)$$

Let us analyze examples of boundary conditions that lead to vanishing of the Poynting flux on eq(9)

$$((\vec{E}_2-\vec{E}_1)\times(\vec{H}_2-\vec{H}_1)\cdot\mathrm{n}\,|_{\partial\Omega}=0 \qquad (12)$$

Namely:

$$\vec{E}\,|_{\partial\Omega}=0 \qquad (13)$$

$$\vec{H}\,|_{\partial\Omega}=0 \qquad (14)$$

$$\vec{E}\cdot\mathcal{T}_{1,2}\,|_{\partial\Omega}=0\Leftrightarrow(\vec{\nabla}\times\vec{H})\cdot\mathcal{T}_{1,2}\,|_{\partial\Omega} \qquad (15)$$

$$\vec{H}\cdot\mathcal{T}_{1,2}\,|_{\partial\Omega}=0 \qquad (16\ a)$$

$$(\vec{E}\times\vec{H})\cdot\mathrm{n}\,|_{\partial\Omega}=0 \qquad (16\ b)$$

where $\tilde{T}_{1,2}$ denote the tangent vectors on the domain boundary $\partial\Omega$.

Let us consider now a medium with constant electromagnetic parameters

$$\varepsilon=\varepsilon_0;\mu=\mu_0;\sigma=\sigma_0.$$

In this simply case we have the decoupled dynamical equations for Electric and Magnetic fields with the non-absorbing boundary conditions

$$\Delta\vec{E}=\varepsilon\mu\frac{\partial^2\vec{E}}{\partial t^2}+\sigma\mu\frac{\partial\vec{E}}{\partial t}+\left(\mu\frac{\partial\vec{j}}{\partial t}+\vec{\nabla}(\rho)\right) \qquad (17)$$

$$\vec{E}(\vec{r},0)=E_0(\vec{r}) \qquad (18)$$

$$-\vec{E}_t(\vec{r},0)=\vec{J}(\vec{r},0)+\sigma\vec{E}_0(\vec{r})-(\nabla\times\vec{H}_0(\vec{r})) \quad (19)$$

$$\vec{E}\cdot\mathcal{T}_{1,2}|_{\partial\Omega}=0 \quad (20)$$

$$\Delta\vec{H}=\varepsilon\mu\frac{\partial^2\vec{H}}{\partial t^2}+\mu\sigma\frac{\partial\vec{H}}{\partial t}-(\vec{\nabla}\times\vec{J}) \quad (21)$$

$$\vec{H}(\vec{r},0)=H_0(\vec{r}) \quad (22)$$

$$\vec{H}_t(\vec{r},0)=-\frac{1}{\mu}(\vec{\nabla}\times\vec{E}_0(\vec{r})) \quad (23)$$

$$(\vec{\nabla}\times\vec{H})\cdot\vec{\mathcal{T}}_{1,2}|_{\partial\Omega}=(\vec{J}\cdot\vec{\mathcal{T}}_{1,2})|_{\partial\Omega}=0 \quad (24)$$

Note that the divergence-less of the magnetic field eq (3) in this case can be insured straightforwardly by the equation below:

$$H^{\mathrm{Tr}}(\vec{r},t)=H(\vec{r},t)-\frac{1}{4\pi}\vec{\nabla}_{\vec{r}}\cdot\left\{\int_\Omega\left(\frac{\operatorname{div}(\vec{H}(\vec{r}',t)}{|\vec{r}-\vec{r}'|}\right)d^3r'\right)\right\} \quad (25)$$

Let us re-write the problem in terms of vector and scalar potentials with the condition of $-\infty<t<\infty\left(\vec{E}(\cdot,t)\ and\ \vec{H}(\cdot,t)\in L^2(R^3)\right)$ ).

Since in this case one has gauge invariance so redundancy on the Maxwell equations solutions one should choose the generalized radiation gauge as a natural gauge fixing to find a unique solution:

$$\frac{1}{\mu}\vec{\nabla}\cdot\vec{A}-\varepsilon\frac{\partial\phi}{\partial t}-\sigma\phi=0\ . \quad (26)$$

In this case the Maxwell dynamical equations eq(17)-eq(24) take the more invariant form below through the use of the electromagnetic potentials $(\vec{A},\phi)$ .

$$\begin{cases}\Delta\vec{A}-\varepsilon\mu\frac{\partial^2}{\partial t^2}\vec{A}=-\mu\vec{j}+\sigma\mu\frac{\partial}{\partial t}\vec{A}\\ \Delta\phi-\mu\varepsilon\frac{\partial^2}{\partial t^2}\phi-\mu\sigma\frac{\partial\phi}{\partial t}=\rho\\ \vec{E}\cdot\vec{\mathcal{T}}_{1,2}|_{\partial\Omega}=(-\frac{\partial}{\partial t}\vec{A}+\nabla\phi)\,.\,\vec{\mathcal{T}}_{1,2}|_{\partial\Omega}=0\\ (\vec{\nabla}\times\vec{H})\,.\,\vec{\mathcal{T}}_{1,2}|_{\partial\Omega}\equiv(\nabla\times(\vec{\nabla}\times\vec{A}))\,.\,\vec{\mathcal{T}}_{1,2}|_{\partial\Omega}=\mu(\vec{J}\cdot\vec{\mathcal{T}}_{1,2}|_{\partial\Omega})\end{cases} \quad (27)$$

At this point we address our readers to implement numerical approximate procedures to solve the above written set of linear boundary value problems in a computer by finite- differences or finite-elements.

Finally we write the associated wave equations associative to the full Maxwell equations in the context of an electromagnetic spatially variable medium namely:

$$\Delta\vec{H}-(\frac{\vec{\nabla}_\varepsilon}{\varepsilon})(\vec{r})\frac{\partial^2 H}{\partial t^2}=$$

$$\left\{-\vec{\nabla}\times\vec{j}+\sigma\mu\frac{\partial H}{\partial t}-(\vec{\nabla}\sigma)\times\vec{E}-\vec{\nabla}(\frac{\vec{H}\cdot\vec{\nabla}}{\mu(\vec{r})})-(\frac{\vec{\nabla}_\varepsilon}{\varepsilon})(\vec{r})\times(\vec{\nabla}\times\vec{H}--\sigma\vec{E}\right\}$$

$$(28)$$

$$\Delta\vec{E}-(\varepsilon\mu)(\vec{r})\frac{\partial^2\vec{E}}{\partial t^2}=\mu\frac{\partial\vec{j}}{\partial t}+\sigma\mu\frac{\partial\vec{E}}{\partial t}$$

$$-\vec{\nabla}(\frac{\rho-\vec{E}\cdot\overrightarrow{\nabla E}}{\varepsilon})-(\frac{\vec{\nabla}\mu}{\mu}\times\vec{\nabla}\times\vec{E}) \quad (29)$$

## + plus boundary and initial conditions

At this point we call the reader attention that solving electromagnetic problem in cavities with a non-trivial topological/homological class with potentials, one encounter the severe difficulty of the Helmholtz-Hodge non trivial decomposition of the electromagnetic fields in term of the above mentioned potentials. For instance, $\vec{\nabla}\cdot\vec{B}=0$ in $\Omega$ means that $\vec{B}=\vec{\nabla}\times\vec{A}+\vec{B}^{\mathrm{top}}$ where $\vec{B}^{\mathrm{top}}$ is an harmonic vector field configuration associated to the topological-homological characterization of $\Omega$ (with $\Omega$ being a domain with holes inside for instance, and of difficult determination from the local Maxwell PDE'systems! In this Helmholtz-Hodge context the Maxwell equations written in terms of potential $(\vec{A},\varphi)$ are of form with constant medium electric parameters for instance)

$$\Delta\vec{A}-\varepsilon\mu\frac{\partial^2}{\partial t^2}\vec{A}=\left(\overbrace{(\vec{\nabla}\times\vec{H}^{\mathrm{top}}-\varepsilon\mu\frac{\partial}{\partial t}\vec{E}^{\mathrm{top}}}^{\textit{homological current}}\right)+\vec{j}+\sigma\vec{E} \quad (30\text{-a})$$

$$\Delta\phi-\varepsilon\mu\frac{\partial^2}{\partial t^2}\phi=\frac{\rho}{\varepsilon}-\overbrace{\mu\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{H}^{\mathrm{top}})}^{\textit{homological charge density}} \quad (30\text{-b})$$

$$\overbrace{\vec{\nabla}\cdot\vec{A}-\}\mu\frac{\partial\phi}{\partial t}}^{\textit{radiation gauge fixing}}=0 \quad (30\text{-c})$$

## + Boundary conditions

Here the homological-topological electromagnetic field configurations are defined by the Hodge theorem.

$$\overbrace{\vec{H}=\frac{1}{\mu}(\vec{\nabla}\times\vec{A})+H^{\mathrm{top}}}^{\textit{Hodge–Helmholtz theorem}} \quad (31\text{-a})$$

$$0=\nabla\times E^{\mathrm{top}}\equiv-\mu\frac{\partial\vec{H}^{\mathrm{top}}}{dt} \quad (31\text{-b})$$

We note that non trivial topology on the manifold structure of $\Omega$ appear dynamically in Maxwell equations as sources of to electric charge and electric currents.

It appears thus, that considering from the beginning the dynamical equations written directly for the strength field $(\vec{E},\vec{H})$, all subtle and very difficulty topological-homological constraints imposed by the Hodge-Helmotz theorem are solved and already built on the boundary conditions imposed directly for the dynamical electromagnetic field equations.

Note that the set of second-order PDE's eq(17)-eq(18) may enlarge the original set of solutions of the first order eqs(l)-eq(2), opposite to the potential method $(\vec{A},\varphi)$ . So, further direct verifications of using unique solutions eq(17)-eq(18) for solving eq(l) and eq(2) should be done at the end of the problem solving.

Let us now consider the anisotropic electromagnetic irradiation case in R[3]. In this situation we have the tensorial-matricial constituve relationships on the medium electro-magnetic properties between the electric displacement vector $\vec{D}(\vec{r},t)$ and the associated electric field $\vec{E}(\vec{r},t)$ respectively between the magnetic flux vector $\vec{B}(\vec{r},t)$ and the mag- netic vector field $\vec{H}(\vec{r},t)$.

$$\vec{D}(\vec{r},t)=\overbrace{[\varepsilon(\vec{r}).]\vec{\mathrm{E}}(\vec{r},t)}^{\textit{anisotropic medium permitivity}}\Leftrightarrow(D_i(\vec{r},t)=\overbrace{\varepsilon_{ij}(\vec{r})E_j(\vec{r},t)(\vec{r}))}^{\varepsilon C_0(\mathrm{R}^3)}$$

$$\vec{B}(\vec{r},t)=[\mu(\vec{r})]\vec{H}(\vec{r},t)\Leftrightarrow\overbrace{\mu_{ij}(\vec{r})}^{\varepsilon^{ij}C_0(\mathrm{R}^3)}H_j(\vec{r},t)=B_i(\vec{r},t)$$

$$(31)$$

Since $\vec{\nabla}\cdot\vec{B}=0$ , we have that exists a potential $\vec{A}$ , such that

$$B_i(\vec{r},t)=\varepsilon^{ijk}\partial_jA_k(\vec{r},t) \quad (32)$$

As a consequence of the Maxwell equations we also have that

$$E_i(\vec{r},t) = -\frac{\partial}{\partial t}A_i(\vec{r},t) + \frac{\partial}{\partial x^i}\phi(\vec{r},t) \quad (33)$$

One has thus the following anisotropic wave equation as outcome:

$$\varepsilon_{irs}\frac{\partial}{\partial x^r}([\mu^{-1}]_{sq}(\vec{r})B_q(\vec{r},t)) - \frac{\partial}{\partial t}([\varepsilon]_{ij}\vec{r})E_j(\vec{r},t)) = J_i(\vec{r},t) \quad (34\text{-a})$$

which can be re-written as of as

$$\varepsilon_{irs}\frac{\partial}{\partial x^r}\left([\mu^{-1^-}]_{sg}\overbrace{\varepsilon^{gjk}\frac{\partial}{\partial x^j}A_k}^{B^g}\right) - \frac{\partial}{\partial t}\left([\varepsilon]_{ij}(-\frac{\partial}{\partial t}A_j + (\vec{\nabla}\phi)_j)\right) = J_i \quad (34\text{-b})$$

or equivalently

$$(\varepsilon_{irs}\varepsilon^{gjk})\left\{([\mu^{-1}]_{sg}\frac{\partial^2}{\partial x^r\partial x^j}A_k) + (\frac{\partial}{\partial x^r}[\mu^{-1}]_{sg})\frac{\partial}{\partial x^j}A_k\right\} + [\varepsilon]_{ij}\frac{\partial^2 A_j}{\partial t^2} = J_i + [\varepsilon]_{ij}\frac{\partial^2}{\partial t\partial x^j}\varphi \quad (34\text{-c})$$

In other words

$$(\varepsilon_{irs}\varepsilon^{gjk})[\mu^{-1}]_{sg})\frac{\partial}{\partial x^r\partial x^j}A_k + [\varepsilon]_{ij}\frac{\partial^2 A^j}{\partial t^2} = J_i + \left\{[\varepsilon]_{ij}\frac{\partial^2}{\partial t\partial x^j}\varphi - (\varepsilon_{irs}\varepsilon^{gjk}(\frac{\partial}{\partial x^r}[\mu^{-1}]_{sg})\frac{\partial}{\partial x^j}A_k)\right\} \quad (35)$$

After considering the radiation anisotropic gauge

$$[\varepsilon]_{ij}\frac{\partial^2}{\partial t\partial x^j}\varphi - \left(\varepsilon_{irs}\varepsilon^{gjk}\frac{\partial}{\partial x^j}[\mu^{-1}]_{sg}\frac{\partial}{\partial x^j}A_k\right) \equiv 0 \quad (36)$$

one has the anisotropic second order Maxwell wave equation for the vector potential $\vec{A}(\vec{r},t)$, decoupled from the scalar potential.

$$\underbrace{\overbrace{[\varepsilon]^{-1}_{mi}\left(\varepsilon_{irs}\varepsilon^{gjk}[\mu^{-1}]_{sg}\right)}^{C_{mrjk}(\vec{r})\equiv}}_{:\equiv\overline{C}_{mrkg}}\frac{\partial^2}{\partial x^r\partial x^j}A_k + \frac{\partial^2 A_m}{\partial t^2} = [\varepsilon]^{-1}_{mi}J_i \quad (37)$$

where the anisotropic fourth-order electromagnetic medium tensor is explicitly given by

$$C_{mrjk}(\vec{r}) = [\varepsilon^{-1}_{(\vec{r})}]_{mi}(\varepsilon_{irs}\varepsilon^{gjk}[\mu^{-1}]_{sg}(\vec{r})) \quad (38)$$

The equation for the $\phi$-electric potential in the choice radiation gauge eq (36) is devoid of dynamical content and given by a sort of Poisson equation through Maxwell equations once known the solution of the vector potential dynamics as given by eq (36)

$$\frac{\partial}{\partial x^i}([\varepsilon]_{ir}\frac{\partial}{\partial x^r}\phi) = (\rho + (\frac{\partial}{\partial x^i}[\varepsilon]_{ir})\frac{\partial}{\partial t}A_r) \quad (39)$$

We now show that it is possible to choose the anisotropic radiation gauge eq (36). Let $(\overline{A},\overline{\phi})$ be a given fixed electromagnetic potential configuration and $(\overline{A}+\nabla\Lambda, \varphi+\frac{\partial\Lambda}{\partial t})$ it is gauge transformed.

We now show that it is possible to determine the gauge transformation parameter $\Lambda(\vec{r},t)$ with the gauge field transformed electromagnetic potential configuration satisfying the gauge fixing analytical definition eq 34.

So let us suppose that

$$\left[\varepsilon_{gjk}\varepsilon_{irs}(\frac{\partial}{\partial x^r}([\mu^{-1}]_{sq}(\vec{r}))\frac{\partial}{\partial x^j}\overline{A}_k)\right] - \frac{\partial}{\partial t}\left[[\varepsilon]_{ij}(r)\frac{\partial}{\partial x^j}\overline{\phi}\right] \neq 0. \quad (40)$$

So we need to determine $\Lambda$ such that

$$\left[\varepsilon_{gjk}\varepsilon_{irs}([\mu^{-1}]_{sq})\frac{\partial}{\partial x^j}(A+\vec{\nabla}\Lambda)_k\right] - \left[\frac{\partial}{\partial t}\left([\varepsilon]_{ij}(\frac{\partial\phi}{\partial x_j} + \frac{\partial\Lambda}{\partial t})\right)\right] = 0 \quad (41)$$

We have thus, that the gauge parameter function $\Lambda(\vec{r},t)$ satisfies the second order PDE equation below

$$\left(\varepsilon_{gjk}\varepsilon_{irs}\frac{\partial}{\partial x^r}([\mu^{-1}]_{sg}\right)\frac{\partial}{\partial x^j}\left(\frac{\partial}{\partial x_k}\Lambda\right) - [\varepsilon]_{ij}\frac{\partial^2}{\partial t^2}\Lambda^j = \overbrace{eq(40)}^{source\ as\ function\ of\ (\overline{\mathrm{A}},\overline{\varphi})} \quad (42)$$

Finally, the reader should realize perturbative analytical calculations by considering weakly anisotropy around the isotropic case $(g_{1,2} \ll 1)$

$$[\mu(\vec{r})]_{ij} = \delta_{ij} + g_1\mu_{ij}(\vec{r})$$
$$[\varepsilon(\vec{r})]_{ij} = \delta_{ij} + g_2\varepsilon_{ij}(\vec{r}) \quad (43)$$

## Maxwell equations in an axial anisotropic conductive medium

Let us start this section 2 by considering the set of Maxwell equations in a medium with constitutive parameters depending solely on the spatial variable $z$, i.e. our anisotropic medium has a permittivity $\mu = \mu(z)$ and a media permeability $\mu = \mu(z)$. Note also the supposed $z$ dependence of the medium conductivity $\sigma(z)$.

The electric flux density $\vec{B}$ (see eq(31) and eq(30)) are also supposed to be time variant and solely depending on the spatial variable $z$

$$\vec{D} = \vec{D}(z,t) = (D_x, D_y, D_z)(z,t) \quad (44\text{-a})$$

$$\vec{B} = \vec{B}(z,t) = (B_x, B_y, B_z)(z,t) \quad (44\text{-b})$$

$$\vec{D} = \varepsilon(z)\vec{E}(z,t) \quad (44\text{-d})$$

$$\vec{B} = \mu(z)\vec{H}(z,t) \quad (44\text{-c})$$

We have thus the following set of partial differential equations describing the electro- magnetic pulse in a such electromagnetic axial anisotropic dependent medium $(\varepsilon(z), \mu(z))$ with a source also still depending on the time and the $z$-variable solely

$$\frac{1}{\varepsilon(z)}r\circ t(\vec{D}(z,t)) + \vec{\nabla}(\frac{1}{\varepsilon(z)})\times\vec{D}(z,t) = -\frac{\partial\vec{B}}{\partial t}(z,t) \quad (45\text{-a})$$

$$\frac{1}{\mu(z)}r\circ t(\vec{D}(z,t)) + \vec{\nabla}(\frac{1}{\mu(z)})\times\vec{B}(z,t) - \frac{\partial}{\partial t}\vec{D}(z,t) = \vec{j}(z,t) + \frac{\sigma(z)}{\varepsilon(z)}\vec{D}(z,t) \quad (45\text{-b})$$

$$\frac{\partial}{\partial z}B_z(z,t) = (\mathrm{div}\vec{B})(z,t) = 0 \quad (45\text{-c})$$

$$\frac{\partial}{\partial z}D_z(z,t) = (\mathrm{div}\vec{D})(z,t) = 0 \quad (45\text{-d})$$

After re-writing the set of axial anisotropic Maxwell equations eq(45-a)–eq(45-d) in components, it yields

$$-\frac{\partial}{\partial z}(\frac{1}{\varepsilon(z)}\frac{\partial D_y}{\partial z}) + \frac{\partial}{\partial z}(\frac{\varepsilon'(z)}{\varepsilon(z)}D_y) = -\frac{\partial B_x}{\partial t\partial z} \quad (46\text{-a})$$

$$\frac{1}{\mu(z)}\frac{\partial B_x}{\partial t\partial z}-\left(\frac{\mu'(z)}{\mu^2(z)}\right)\frac{\partial B_x}{\partial t}-\frac{\partial^2}{\partial t^2}D_y = \frac{\partial}{\partial t}j_y+\frac{\sigma(z)}{\varepsilon(z)}\frac{\partial}{\partial t}D_y \tag{46-b}$$

In a decoupled form after some elementary algebra, one gets the fully decoupled component wave equation

$$\left(-\frac{1}{\varepsilon(z)}\frac{\partial^2}{\partial z^2}D_y+\left(\frac{\varepsilon'(z)}{\varepsilon^2(z)}\right)\frac{\partial D_y}{\partial z}+\frac{\varepsilon'(z)}{\varepsilon^2(z)}\frac{\partial D_y}{\partial z}+\left(\frac{\varepsilon'}{\varepsilon^2}\right)'(z)D_y(z)\right)$$
$$=\left(-\left(\frac{\mu'(z)}{\mu(z)\varepsilon(z)}\right)\frac{\partial D_y}{\partial z}+\left(\frac{\mu'\varepsilon'}{\mu\varepsilon^2}\right)(z)D_y-\mu(z)\frac{\partial^2}{\partial t^2}D_y-\mu(z)\frac{\partial}{\partial t}j_y-\left(\frac{\mu\sigma}{\varepsilon}\right)(z)\frac{\partial}{\partial t}D_y)\right) \tag{47}$$

Similar algebraic procedures give decoupled equations for $D_x(z,t)$ and the magnetic flux density $\vec{D}=(D_x,D_y)$, after determining the candidate solutions for the electric flux density $\vec{D}=(D_x,D_y)$ .

$$\frac{\partial}{\partial z}\left(\frac{1}{\mu(z)}\frac{\partial B_x}{\partial z}\right)-\frac{\partial}{\partial z}\left(\frac{\mu'(z)}{\mu^2(z)}B_x\right)=$$
$$=\varepsilon(z)\frac{\partial^2 B_x}{\partial t^2}+\overbrace{\left\{\left(\frac{\varepsilon'}{\varepsilon}\right)(z)\frac{\partial D_y}{\partial t}(z,t)-\frac{\partial}{\partial z}j_y(z,t)+\frac{\partial}{\partial z}\left(\frac{\sigma(z)}{\varepsilon(z)}D_y(z,t)\right)\right\}}^{\text{Effective source term}} \tag{48}$$

Formal harmonic solutions for the decoupled eq (47) are easily found for a harmonic source with a definite frequency $(-\infty<t<\infty)$

$$D_y(z,t)=e^{iwt}\Phi(z) \tag{49-a}$$

$$j_y(z,t)=e^{iwt}j_y(z) \tag{49-b}$$

Here

$$\Phi(z)=\exp\left\{\overbrace{-\frac{1}{2}\int_{z_0}^{z}b(\bar{z})d\bar{z}}^{\alpha(z)\equiv}\right\}\varphi(z) \tag{50-a}$$

$$b(z)\equiv\left[\left(\frac{2\varepsilon'}{\varepsilon^2}+\frac{\mu'}{\mu\varepsilon}\right)\cdot\frac{1}{\mu}\right] \tag{50-b}$$

$$-\frac{d^2\varphi}{dz^2}+(iwd(z)-V(z))\varphi(z)=w^2\varphi-j_y(z)e^{-\alpha(z)} \tag{50-c}$$

$$d(z)\equiv\frac{\sigma(z)}{\varepsilon(z)} \tag{50-d}$$

$$V(z)=\alpha''(z)+(\alpha'(z))^2+b(z)\alpha'(z)+c(z) \tag{50-e}$$

$$a(z)\equiv\left(\frac{1}{\varepsilon\mu}\right)(z) \tag{50-f}$$

$$c(z)\equiv\left([-(\frac{\varepsilon'}{\varepsilon^2})'-\frac{\mu'}{\mu}\frac{\varepsilon'}{\varepsilon}]\cdot\frac{1}{\mu}\right)(z) \tag{50-g}$$

We have thus reduced the harmonic electromagnetic wave propagation described in a axial-anisotropic medium to the ordinary differential equation eq(50-c). Let us finally point out that at the limit of higher frequencies and $j_y(z,t)\equiv 0$ , one may introduce the effective spatial variable Z.[5]

$$Z=\frac{\omega}{\gamma}z$$
$$\varphi(z)=U(Z);d(z)=\tilde{d}(Z);etc. \tag{51}$$

with $\gamma$ being an expansion parameter and get the explicitly solutions at the asymptotic higher frequency limit

$$U(Z)=\frac{A}{\sqrt[4]{-\tilde{c}(\bar{Z})}}\exp\{i\int_{Z_0}^{Z}(-\tilde{c}(\bar{Z}))^{\frac{1}{2}}d\bar{Z}\}$$
$$+\frac{B}{\sqrt[4]{-\tilde{c}(Z)}}\exp\{-i\int_{Z_0}^{Z}(-\tilde{c}(\bar{Z}))^{\frac{1}{2}}d\bar{Z}\} \tag{52}$$

Useful for scattering problem of an electromagnetic pulse into a slab (or layers) (work in progress). Work on applied settings are in progress.[6]

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## Conflicts of interest

Authors declare there is no conflicts of interest.

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