

Dimensionless formulation for the one-dimensional compressible flow of the viscous and heat-conducting micropolar fluid

Abstract

In this paper, we consider the compressible, micropolar, viscous, and heat-conducting fluid, which is in thermodynamical sense perfect and polytropic. We describe the mathematical model of the described fluid and derive its one-dimensional form. For the given set of partial differential equations we perform non-dimensionalization and introduce the corresponding relative numbers.

Keywords: compressible micropolar fluid, relative numbers

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Introduction

In micro and nanosciences, classical fluid models can rarely be adequately considered, since micro-phenomena, which cannot be covered using classical models, increasingly come to the fore. Therefore, the models which can describe the phenomena at the micro level are increasingly analyzed.¹ One example of such a model is the model of the micropolar continuum, which is the subject of this paper.

The model of the micropolar continuum was introduced by Eringen². The behavior of the continuum at the microlevel in this model is described by using one new vector field, which Eringen calls microrotation velocity, whereby microdeformations are neglected.

Although the usability of the micropolar fluid model is easy to understand and even practically proven,³ from the technical and physical point of view, we still know very little about this model, especially in the compressible case. The rheological constants are almost unknown; therefore the study of physical properties for this model is practically impossible. In such situations, the non-dimensional formulation is of great importance, which is addressed in this paper. Let us note that the micropolar fluid model could be appropriate for describing different kinds of biological fluids, smog, lubricants, gaseous stars, etc. Recently, the micropolar fluid model has been applied as the model for blood flow,⁴ for water-based nanofluid,⁵ for mimicking bacterial physical phenomena,⁶ for the behavior of epididymal material,⁷ for describing lubricants with additives, for the motion of synovial fluid in the joints,⁸ etc.

It is important to point out that the mathematical analysis of the related initial-boundary problems has progressed considerably in terms of the solution's existence analysis as well in terms of the corresponding numerical methods.⁹ This is the basis for exploring different fluid flow regimes in order to increase our understanding of the effect of micropolarity, or, in other words, the effect of microphenomena on the global fluid behaviour. The non-dimensional formulation becomes significant, precisely in this aspect of the research.

The basis for deriving non-dimensional constants was the classic fluid model and corresponding relative numbers (e.g., Mach number), which were partly redefined to accommodate the introduction of a new hydrodynamic variable (microrotation). In accordance with similar papers,¹⁰ some new constants have been introduced which are primarily used to describe the effect of micropolarity. These are the Microscopic Reynolds number, the Eringen number and the Coupling number.

This paper is organized as follows. In Section 2, we describe the model and derive its one-dimensional form, whereby we limit ourselves just to equations and not to the initial and boundary conditions. In Section 3, we introduce the relative numbers, perform the non-dimensionalization of the equations, and derive the non-dimensional form of the corresponding system.

The mathematical model

In this paper we analyze the compressible flow of an isotropic, viscous, and heat-conducting micropolar fluid, which is in the thermodynamical sense perfect and polytropic. The corresponding hydrodynamical variables are:

- ρ – mass density,
- $\mathbf{v} = (v_1, v_2, v_3)$ – velocity,
- $\mathbf{w} = (\omega_1, \omega_2, \omega_3)$ – microrotation velocity,
- E – internal energy density,
- θ – absolute temperature,
- \mathbf{T} – stress tensor,
- \mathbf{C} – couple stress tensor,
- \mathbf{q} – heat flux density vector,
- \mathbf{f} – outer body force density,
- \mathbf{g} – outer body couple density,
- p – pressure.

The mathematical model of the described flow is stated, for example, in the book of Lukaszewicz¹¹ and reads

$$\dot{\rho} = -\rho \nabla \cdot \mathbf{v}, \tag{1}$$

$$\rho \dot{\mathbf{v}} = \nabla \cdot \mathbf{T} + \rho \mathbf{f}, \tag{2}$$

$$\rho j_I \dot{\mathbf{w}} = \nabla \cdot \mathbf{C} + \mathbf{T}_x + \rho \mathbf{g}, \tag{3}$$

$$\rho \dot{E} = -\nabla \cdot \mathbf{q} + \mathbf{T} : \nabla \mathbf{v} + \mathbf{C} : \nabla \mathbf{w} - \mathbf{T}_x \cdot \mathbf{w}. \tag{4}$$

Equations (1)–(4) are respectively, local forms of the conservation laws for mass, momentum, angular momentum and energy. We assume that our fluid is isotropic, which means that the intrinsic angular momentum per unit mass \mathbf{l} can be written in the form $j_I \mathbf{w}$, where the positive constant j_I is called imicroinertia density. For the readers' convenience let us first explain the notation used in the system (1)–(18). The differential (dot) operator in equations (0.1)–(0.4) denotes material derivative defined by

$$\dot{\mathbf{a}} = \mathbf{a}_t + (\nabla \mathbf{a}) \cdot \mathbf{v}, \tag{5}$$

For vector field \mathbf{a} and

$$\dot{u} = u_t + (\nabla u) \cdot \mathbf{v}, \tag{6}$$

For scalar field u . The differential operator ∇ is classical nabla (del) operator where $\nabla \cdot \mathbf{a}$ is the divergence of corresponding vector (or tensor) field, and $\nabla \mathbf{a}$ is the gradient of the vector field \mathbf{a} (or \mathbf{u} is gradient of the scalar field u). Vector \mathbf{T}_x in the equations (3) & (4) is a vector with Cartesian components

$$\mathbf{T}_x = (T_{23} - T_{32}, T_{31} - T_{13}, T_{12} - T_{21}). \tag{7}$$

The colon operator in the equation (0.4) is the dyadic notation for the scalar product of the tensors defined by

$$\mathbf{A} : \mathbf{B} = Tr(\mathbf{A} \cdot \mathbf{B}^T), \tag{8}$$

Where Tr denotes the trace operator.

The components of the tensors \mathbf{T} and \mathbf{C} are given by

$$T_{ij} = (-p + \lambda v_{k,k}) \delta_{ij} + \mu (v_{i,j} + v_{j,i}) + \mu_r (v_{j,i} - v_{i,j}) - 2\mu_r \varepsilon_{mij} w_m, \tag{9}$$

$$C_{ij} = c_0 w_{k,k} \delta_{ij} + c_d (w_{i,j} + w_{j,i}) + c_a (w_{j,i} - w_{i,j}). \tag{10}$$

The derivatives in (9)–(10) are in the indicial notation, i.e., for

$$\mathbf{a} = (a_1, a_2, a_3) \text{ we have}$$

$$a_{i,j} = \frac{\partial a_i}{\partial x_j}. \tag{11}$$

In (9)–(10), we use Kronecker's delta symbol δ_{ij} as well as Levi-Civita's symbol ε_{ijk} , which are defined by

$$\delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j \end{cases} \tag{12}$$

And

$$\varepsilon_{ijk} = \begin{cases} +1, & \text{if } ijk \text{ is even permutation of } 123, \\ -1, & \text{if } ijk \text{ is odd permutation of } 123, \\ 0, & \text{if any index is repeated.} \end{cases} \tag{13}$$

Here, we assume the Einstein summation convention, i.e., when an index variable appears twice in a single term, it implies summation of that term over all the values of the index.

Let us note that equations (0.9)–(0.10) are constitutive equations for the micropolar continuum, whereby we have the following material parameters:

- λ – coefficient of dilatational viscosity
- μ – coefficient of dynamical viscosity
- μ_r – coefficient of rotational viscosity
- c_0 – coefficient of bulk spin (angular) viscosity
- c_d – coefficient of shear spin (angular) viscosity
- c_a – coefficient of rotational spin (angular) viscosity

The coefficients of viscosity are related through the Clausius–Duhem inequalities, as follows: (14)

$$c_d \geq 0, 3c_0 + 2c_d \geq 0, |c_d - c_a| \leq c_d + c_a. \tag{14}$$

As it is mentioned in the introduction, we assume that our fluid is perfect and polytropic in the thermodynamical sense, which we model by the following equations:

$$\mathbf{q} = -k \nabla \theta, \tag{16}$$

$$p = R \rho \theta, \tag{17}$$

$$E = c_v \theta. \tag{18}$$

Equation (16) is the Fourier law, where $k \geq 0$ is the heat conduction coefficient. Equation (17) is the ideal gas law, where $R > 0$ is the universal gas constant, while (18) presents the assumption that our fluid is polytropic. The positive constant c_v in (18) is called specific heat at a constant volume.

For simplicity reasons, we will assume that the outer impact can be neglected, i.e. we take:

$$\mathbf{f} = \mathbf{g} = 0. \tag{19}$$

To simplify the system (1)–(4), we will first substitute the (9), (10) and (16)–(18) into (1)–(4) together with (19). We get:

$$\rho_t = -(\nabla \rho) \cdot \mathbf{v} - \rho \nabla \cdot \mathbf{v}, \tag{20}$$

$$\rho \mathbf{v}_t = -\rho (\nabla \mathbf{v}) \cdot \mathbf{v} - R \nabla (\rho \theta) + (\lambda + \mu - \mu_r) \nabla (\nabla \cdot \mathbf{v}) + (\mu + \mu_r) \Delta \mathbf{v} + 2\mu_r \nabla \times \mathbf{w}, \tag{21}$$

$$j_I \rho \mathbf{w}_t = -\rho (\nabla \mathbf{w}) \cdot \mathbf{v} + 2\mu_r (\nabla \times \mathbf{v} - 2\mathbf{w}) + (c_0 + c_d - c_a) \nabla (\nabla \cdot \mathbf{w}) + (c_d + c_a) \Delta \mathbf{w}, \tag{22}$$

$$c_v \rho \theta_t = -c_v \rho (\nabla \theta) \cdot \mathbf{v} + k \Delta \theta - R \rho \theta (\nabla \cdot \mathbf{v}) + \lambda (\nabla \cdot \mathbf{v})^2 + \mu (\nabla \mathbf{v} + (\nabla \mathbf{v})^T) : (\nabla \mathbf{v} + (\nabla \mathbf{v})^T) + 4\mu_r \left(\frac{1}{2} \nabla \times \mathbf{v} - \mathbf{w} \right)^2 + c_0 (\nabla \cdot \mathbf{w})^2 + (c_d + c_a) \nabla \mathbf{w} : \nabla \mathbf{w} + (c_d - c_a) \nabla \mathbf{w} : (\nabla \mathbf{w})^T. \tag{23}$$

In this paper, we consider the model (20)–(23) for the one-dimensional flow; therefore, we assume

$$\begin{aligned} \rho(\mathbf{x}, t) &= \rho(x, t), \mathbf{v}(\mathbf{x}, t) = (v(x, t), 0, 0), \\ \omega(\mathbf{x}, t) &= (\omega(x, t), 0, 0), \theta(\mathbf{x}, t) = \theta(x, t), \end{aligned} \quad (0.24)$$

and obtain

$$\rho_t = -\rho_x v - \rho v_x, \quad (25)$$

$$\rho v_t = -\rho v v_x - R(\rho\theta)_x + (\lambda + 2\mu)v_{xx}, \quad (26)$$

$$j_I \rho \omega_t = -j_I \rho v \omega_x + (c_0 + 2c_d)\omega_{xx} - 4\mu_r \omega, \quad (27)$$

$$c_v \rho \theta_t = -c_v \rho v \theta_x + k \theta_{xx} - R \rho \theta v_x + (\lambda + 2\mu)(v_x)^2 + (c_0 + 2c_d)(\omega_x)^2 + 4\mu_r \omega^2. \quad (28)$$

Let us note that the system (25) was first analyzed by Mujaković¹² coupled with homogeneous boundary conditions for velocity, microrotation velocity and heat flux, as well as non-homogeneous initial conditions. In this work, Mujaković proved that the corresponding problem has a unique generalized solution.⁹

This model was later analyzed by other author too, whereby different mathematical properties were described, such as regularity and large time behavior of the solution.¹³ In her later works, Mujaković considered non-homogeneous boundary conditions, as well as free boundary conditions.^{14,15}

From the physical point of view, homogeneous boundary conditions for velocity and heat flux describe the solid thermo-insulated walls, with non-homogeneous boundary conditions for velocity we model the piston problem, and free boundary conditions describe the expansion of fluid into vacuum. For more details about different boundary conditions.^{9,12}

Nondimensionalization

As it is pointed out in the introduction, to get a better picture of the behaviour of the compressible micropolar flow it is essential to derive a dimensionless formulation of the problem, which is the main goal of this paper. To convert the equations (25)–(28) to their dimensionless form, we first introduce dimensionless independent variables by:

$$x^* = \frac{x}{L}, t^* = \frac{t}{\tau}, \quad (29)$$

As well as dimensionless dependent variables by:

$$\rho^* = \frac{\rho}{\rho_\infty}, v^* = \frac{v}{v_\infty}, \omega^* = \frac{\omega}{\omega_\infty}, \theta^* = \frac{\theta}{\theta_\infty}, \quad (30)$$

Where $\tau, \tau, \rho_\infty, v_\infty,$ and θ_∞ are dimension-bearing constants. According to Bayada G, et al.¹⁶ and Chen J, et al.¹⁰ we additionally take

$$\omega_\infty = \frac{L}{v_\infty}. \quad (31)$$

Now, we will convert parameters of the model into their dimensionless versions, which are called *relative numbers*. We mostly use common relative numbers, but some are slightly modified or redefined:

Strouhal number:

$$St = \frac{L}{\tau v_\infty} \quad (32)$$

which measures unsteadiness of the flow, i.e., it indicates the significance of time derivative term.

Mach number:

$$Ma = \frac{v_\infty}{\sqrt{\gamma R \theta_\infty}} \quad (33)$$

which measures compressibility of the flow. Let us note that the definition (33) is valid for perfect gas only.

Heat capacity ratio:

$$\gamma = \frac{c_p}{c_v} \quad (34)$$

which is the ratio of the heat capacity at constant pressure (c_p) to heat capacity at constant volume (c_v). The heat capacity ratio is an intrinsic property of a fluid, i.e., it contains no length scale in its definition and is dependent only on the fluid and the fluid state.

Macroscopic reynolds numbers:

$$Re_\mu = \frac{Lv_\infty \rho_\infty}{\mu}, Re_\lambda = \frac{Lv_\infty \rho_\infty}{\lambda}, Re_M = \frac{Lv_\infty \rho_\infty}{\lambda + 2\mu} \quad (35)$$

which indicate how effectively the macroscopic viscous forces compensate the inertia forces, i.e., it quantifies the importance of macroscopic viscous forces in the flow (small value of Re_M corresponds to a flow with large macroscopic viscous effects, while a large value of Re_M corresponds to a flow with small macroscopic viscous effects).

Microscopic reynolds number:

$$Re_m = \frac{j_I Lv_\infty \rho_\infty}{c_0 + 2c_d} \quad (36)$$

which quantifies the importance of microscopic viscous forces in the flow in the same manner as Re_M .

Eringen number:

$$Er = \frac{j_I}{L^2} \quad (37)$$

which governs the micropolar nature of the fluid. If Er is closer to unity, the effect of the micropolarity will be more pronounced. Square root of microinertia density is commonly used as dynamic internal characteristic length for isotropic micropolar continuum.

Coupling number:

$$N = \frac{4\mu_r}{\lambda + 2\mu} \quad (38)$$

which measures the intensity of coupling between microrotations and rotations at macrolevel, i.e. it is a measure of the degree to which a particle is constrained to rotate with the average angular velocity of the region in which it is embedded.

Prandtl number:

$$Pr = \frac{c_p \mu}{k} \tag{39}$$

which assesses the relation between momentum transport and thermal transport capacity of a fluid. For example, when Pr is small, it means that the heat diffuses quickly compared to the velocity (momentum). As well as heat capacity ratio, the Prandtl number is an intrinsic property of a fluid.

Peclet number:

$$Pe = Pr \cdot Re_\mu \tag{40}$$

which is related to both the Prandtl number, as well as the Reynolds number. It measures the relative strength of convection to diffusion. If the Péclet number is small we can neglect convection. On the other side, when Péclet is high, convection is more dominant and diffusive processes can be neglected.

Eckert number:

$$Ec = \frac{v_\infty^2}{c_p \theta_\infty} \tag{41}$$

which is used to characterize the influence of self-heating of a fluid as a consequence of heat dissipation.

We are now in the position to rewrite the equations (25)–(28) using the introduced relative numbers. To simplify the equations we omit the asterisk, and get

$$St\rho_t = -\rho_x v - \rho v_x, \tag{42}$$

$$St\rho v_t = -\rho v v_x - \frac{1}{\gamma Ma^2} (\rho\theta)_x + \frac{1}{Re_M} v_{xx}, \tag{43}$$

$$St\rho\omega_t = -\rho v\omega_x + \frac{1}{Re_m} \omega_{xx} - \frac{N}{Er \cdot Re_M} \omega, \tag{44}$$

$$St\rho\theta_t = -\rho v\theta_x + \frac{\gamma}{Pe} \theta_{xx} - \frac{Ec}{Ma} \rho\theta v_x + \frac{\gamma \cdot Ec}{Re_M} (v_x)^2 + \frac{\gamma \cdot Ec \cdot Er}{Re_m} (\omega_x)^2 + \frac{\gamma \cdot Ec \cdot N}{Re_M} \omega^2, \tag{45}$$

which is the desired form of the considered problem.

Conclusion

In this paper, for the first time, the non-dimensional form of the model for the compressible flow of an isotropic, viscous, and heat conducting micropolar fluid is given, whereby the set of redefined relative numbers is introduced. The resulting formulation, coupled with appropriate boundary and initial conditions, is of great

importance for future numerical experiments in the research of the physical properties of this fluid model.

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Conflict of interest

Author declares that there is no conflict of interest.

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