Review Article

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Metric physics

Abstract

Different axiomatics of metric spaces define different types of geometries. Standard axiomatics defines elliptic–type geometries, which are subdivided into Archimedean and non–Archimedean. The former are used in classical physics, the latter–completely discontinuous–are more correlated with quantum physics. Replacing axioms of metric spaces by other, with opposite conditions, we obtain the axioms of hyperbolic geometries underlying relativistic physics. In our work, the notion of hyperbolicity is given a certain meaning, namely, the mathematical expression of the physical principle of cause–effect through the axioms of the metric.

Keywords: metric spaces, geometries, relativistic physics, special theory of relativity, axioms, triangle, velocity, light, binary relation, planck constant, topology

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Introduction

Initially, the article was created for the presentation of the special theory of relativity (STR) on the basis of the Berwald–Moor metric. The operations of addition of unidirectional velocities and the Doppler Effect for radially moving objects in this geometry coincide with their values in the Minkowski geometry. However, there are also significant differences that have a direct relationship to the interpretation of cosmological observations. We have decided to separate (this) geometric part from the physical (exposition of STR based on the Berwald–Moore metric) because of the need for a large amount of material, so we will concentrate here only on the expression of cause–effect in terms of the axioms of the metric.

Attempts to present a special and general theory of relativity on more general, more fundamental principles have been made since the very beginning of the theory of relativity. Initially, the generalizations were based on the replacement of the Riemannian metric by Finsler one. Such an approach was used in the works of Asanov et al.,[1,2] Bogoslovsky et al.,[3–5] and others. However, the Finsler geometry in these works was used rather to describe the anisotropy of the space–time, and not as a special form of ordering (cause–effect) and signature. In some works, the causality was introduced explicitly, with the metric[6] or without it,[7–9] through special axioms of the order. The order corresponding to the cause–effect there was introduced by replacing the triangle inequality by the opposite one. Connectivity and the elimination of the complete discontinuity of space were expressed through the property of divisibility of the metric, called its Archimedean property. In mathematics, hyperbolic geometry is often understood as the geometry of Lobachevsky. In our work this concept obtained a slightly different meaning, namely, the mathematical expression of the physical principle of cause–effect through the axioms of the metric. Here is an analogy with hyperbolicity with respect to the time parameter in differential equations that determines the cause–effect in their solutions and the finite rate of diffusion of perturbations. Replacing the triangle inequality in the axioms of the metric by the opposite one also determines the order (cause–effect) and the finiteness of the perturbation velocity (the speed of light). The infinite velocity of the light, which exists in the Galileo–Newton geometry, in this sense refers to parabolic geometry. The analogue of Lobachevsky's geometry also rather refers to the geometry induced from a hyperbolic space to a sphere of imaginary radius. Non–Archimedean completely discontinuous spaces are more suitable for describing the quantum world.

It is quite possible that in the future the quantumness of space–time will also be expressed by the divisibility of the metric with certain accuracy $\varepsilon > 0$ with an additional modification of the triangle inequality (opposite inequality):

$$\rho(x,y) \le (\ge)\ \rho(x,z) + \rho(z,y) - \varepsilon,$$

With a small constant ε, similarly the Planck constant. Recall that a metric on the set X is defined by specifying pairwise distances:

$$r(x,y): X \times X \to \mathbb{R}_+,$$

Which determine the binary relation. In mathematics by a binary relation on the set A any subset of the Cartesian square $A \times A$ is usually understood. This is equivalent to specifying the function $A \times A \to \mathbb{Z}_2 = \{0,1\}$. In physics under the relation a more fuzzy notion is more often understood, when instead of a value from $\mathbb{Z}_2$ is taken a certain numerical value from $\mathbb{R}$. In particular in physics a unary relation means a function from A to $\mathbb{R}_+$ (for example, in determining a density or a fuzzy subset) or a function $A \to \mathbb{C}$ with complex values (in determining a wave function). The metric specification, from the physics point of view, is the definition of a binary relation with values in $\mathbb{R}_+$. From the mathematical point of view, the metric is not a relation, but a function defined on the Cartesian product $A \times A \to \mathbb{R}_+$, which takes non–negative values and satisfies certain conditions.

Next, we consider the basic mathematical concepts from geometry with giving them a certain physical meaning.

Metric and hyperbolic metric spaces

A metric on the set X is defined by the function $r(x,y): X \times X \to \mathbb{R}_+$, satisfying the following conditions:

1. $r(x,y) = 0 \Leftrightarrow x = y,$
2. $r(x,y) = r(y,x),$
3. $r(x,z) \le r(x,y) + r(y,z).$

The first property is the property of reflexivity, the second–of symmetry. The last property, called the triangle inequality, corresponds to the relation of nearness transitivity.

Here the defining property is the triangle inequality. Indeed, if only this one is satisfied for the function r , then we can symmetrise the distance as follows: $\rho(x,y)=\frac{1}{2}(r(x,y)+r(y,x))$ (the triangle inequality is not violated in this case) and, if necessary, to factorize by the equivalence relation $x\cong y\Leftrightarrow\rho(x,y)=0$. Further, we will assume that the condition (2) for the function r is satisfied.

Then from the condition $r(x,y)=0$ it follows that $r(x,z)=0$, and from the conditions $r(x,y)=0, r(y,z)=0$, according to the triangle inequality, it follows that $r(x,z)=0$. Therefore, we can factorize the set by the relation $x\sim y\Leftrightarrow r(x,y)=0$. Moreover, according to the triangle inequality, we get that $x\sim y\Rightarrow r(x,z)=r(y,z)$, hence the distance between the equivalent points is uniquely determined and the triangle inequality is not violated.

The triangle inequality here determines the fact that the open balls $\{y\,|\,\rho(x,y)<\varepsilon,\varepsilon>0\}$ form a basis of the induced topology. The distance function determines the closeness of points for which a property of the kind of transitivity is true: "close plus close" is "close". According to this approach, "far plus far" may be "close" , but "far plus close" remains "far".

The triangle inequality is also a nonlinear an a log of the ball convexity condition. Indeed, if the distance function is defined on a linear space, then the ball convexity condition is equivalent to the triangle inequality. Another reason for using the axiom of triangle inequality is the possibility of measuring the objective distances by integrating (summing) of small distances along some path. Let the two farmers divide the land plot and measure the appropriate distances using a certain standard, for example a common bipedal, fastened with a third stick so that the distance between the ends of the two–knife remains constant. For one of them (we denote him the first) it is advantageous for the measured value–the distance between the points A and B –to be as large as possible, and for the second–as small as possible. When the triangle inequality is satisfied, the second will never trust the measurement of the first, since that can move deviating very far from side to side and so get an arbitrarily large value of the measured distance between the points A and B . But the first can entrust the measurement to the second, since, according to the triangle inequality, the objective distance does not exceed the distance measured by the second as a sum of distances: $\sum_{i=1}^{n} r(A_{i-1},A_i)=n$, where $A_0=A, A_n=B$. Respectively, for the first is beneficial (at least not to the detriment of) any measurement by summing the distances. Thus, by measuring, they can measure the objective distances between points if the triangle inequality holds (convexity of the balls holds). In essence, this means the presence of a variational principle of measuring distances along paths as $\rho(A,B)=inf_{A=A_0,\ldots,A_n=B}\sum_i\rho(A_i,A_{i+1})$.

The variational principle also works in the case of concavity of balls, i.e. an objective measurement of distances is possible even in the case when the opposite inequality is satisfied instead of the triangle inequality. In this case, the first farmer will not trust the second, since he can significantly understate the measured value in comparison with the objective one. The second farmer can entrust the measurement to the first, since the distance measured first, which is the sum of the distances $\sum_{i=1}^{n} r(A_{i-1},A_i)$, is always not greater than the objective value, which is the upper limit of the measured distances. Here another type of variational principle is fulfilled. In this case, distances, generally speaking, do not define a topology. Apparently, therefore, this type of metrics has not been studied in detail: here the principle of transitivity of range is fulfilled–"far plus something" is "far". But here "close plus close" can also turn out to be "far". Correspondingly, in linear spaces the function $x+y$ of a sum of two arguments will not be continuous when the closeness on the product of spaces is consistent with closeness as in the categorical product of spaces. Apparently, this is the main reason why mathematicians did not pay much attention to geometry of another type, where the triangle inequality is satisfied in the opposite direction, which corresponds to the concavity of the ball (convexity of the complement). We call such geometry concave or hyperbolic (the hyperbola restricts the concave region, the ellipse–convex). In this case, we must abandon the condition (2) for the distance function, and also on the condition that distances are defined for any pairs of points. Correspondingly, we define a concave (hyperbolic) metric as a distance function on the Cartesian product taking values from $\mathbb{R}_+\cup\{*\}$, where * means that in this case the distance between points is not defined (its value is some unknown negative number). Thus, a hyperbolic metric on the space X satisfies the following conditions:

1. $r(x,x)=*$;
2. $r(x,z)\geq r(x,y)+r(y,z)$.
3. $r(x,z)\geq r(x,y)+r(y,z)$.

The last condition means that if $r(x,y)>0$ and $r(y,z)>0$ are defined, then $r(x,z)>0$ is also defined. Here the condition (1) corresponds to the anti–reflexivity (and can be obtained from condition (2)), the condition (2) corresponds to anti-symmetry. The distance function determines here not a "physical" distance, but something corresponding to a relativistic interval. The distance can be corresponded to the inverse values that are not defined for the zero value. This justifies the anti–reflexivity condition. With such axioms, all distances can be considered positive. Any way, it is more convenient to work with zero values for distances, changing the first two axioms the following way:

1. $r(x,x)=0$;
2. $r(x,y)\geq 0, x\neq y\Rightarrow r(y,x)=*$.

For relativistic events, the condition $r(x,y)>0$ can be interpreted as follows: the event y occurs after the event x (performed in any inertial frame of reference) on the "time" $r(x,y)$. This determines the order: (independent of the frame of reference) $y>x\Leftrightarrow r(x,y)>0$. The inequality of a triangle means the fulfilment of the axioms of order for such a relation "later".

In the first case, the distance between the points x, y is determined by the variational principle $inf(\sum_i\rho(x_{i-1},x_i)\,|\,x_0=x, x_n=y)$, in the second–by $sup(\sum_i\rho(x_{i-1},x_i)\,|\,x_0=x, x_n=y)$, where sup is computed over all measurable paths. If both the triangle inequality and the opposite inequality are not satisfied, in general, it would not be possible to measure the distance by summing the standard (meter). As it will be seen below, in this case it is possible that $inf=0, \sup=\infty$.

In hyperbolic space, the open balls do not define a topology.[10] In such a space, however, one can determine the interval topology by taking $U_{xy}=\{z\,|\,r(x,z)>0, r(z,y)>0, x<z<y\}$ as the base of open sets.

This can be interpreted as follows: the event x occurs later than x, but before y. However, such a topology turns out to be stronger than the continuous structure determined by the distance function: one can choose a sequence x_i of points for which $\lim_{n\to\infty} r(x, x_n) = 0$, but this sequence does not converges in the proposed interval topology. The interval topology for hyperbolic metrics is not defined by a uniform structure (as for usual metrics) and, accordingly, is not suitable for testing of completeness or for the completion of the space.

We can define a base of neighbourhoods of the point x in terms of the system of sets $U_{x,\varepsilon} = \{y \mid 0 \le r(x, y) < \varepsilon\}$. Such neighbourhoods do not induce (in general) a topology, since they can contain no neighbourhoods of other points because of the hyperbolicity of neighbourhoods (with "tails" leaving to infinity). But they form a quasi–topology[10] and are suitable for determining many characteristics defined in topological spaces.

Note that geodesic lines can be determined even without introducing the Finsler metric. By definition, the curve $x(\tau)$ is geodesic if it is covered by open subintervals, and the distance between two points coincides with the limit of sums of distances in the decomposition. For a hyperbolic geometry, a measurable (geodesic) curve has an orientation (in the other direction it is immeasurable), respectively, the subdivisions into small intervals are also oriented, and the curve in a certain "sense" is smooth.

With such definition of geodesics, non–Archimedean geometry (a subclass of convex elliptic geometries) looks in a special way, where instead of the triangle inequality the stronger inequality holds:

$$r(x, z) \le max(r(x, y), r(y, z)). \qquad (1)$$

The interval (geodesic) between two points A, B with measurable $\rho(A, B)$ is defined as the set of points C for which satisfy the condition: $\rho(A, B) = \rho(A, C) + \rho(C, B)$. The interval can consist only of the start and end points. This happens in the case of non–Archimedean spaces. Indeed, the inequality (1) implies that there is no point y between the points x and z such that $r(x, y) < r(x, z)$ and $r(y, z) < r(x, z)$. Such spaces look like discrete, indivisible (quantized). However, they can be nondiscrete in the topological sense. Here, another term is more suitable: a completely discontinuous space.

In non–Archimedean space, if Achilles has a step length of not more than 1, he can never leave a distance more than 1 from the starting position, i.e. Achilles in the non–Archimedean space will never catch up with the Turtle, if the initial distance between them is greater than the maximum of the steps of Achilles and the Turtle. In general, Achilles will not be able to move away from his initial position by a distance greater than r_0 if each step does not exceed r_0. Archimedes' principle is to deny such a statement: "For any $r_0 > 0, R > 0$, from any initial position x it is possible to attain some point y by a distance $r(x, y)$ greater than R (that is, $r(x, y) > R$) in a finite number of steps, moving only by steps that do not exceed" r_0. This means the asymptotic connection of space. However, a space with the metric defined above can be disconnected, for example, such is the set of hyperbolas $xy = n, x > 0, n = 1,2,3,\ldots$. Therefore, by analogy with,[11] under the Archimedean metric we mean a narrower class of metrics when is fulfilled the condition of uniform divisibility: "for any two points x and y, where $r(x, y) > 0$, and for any $\varepsilon > 0$ there exists a point z such that $|r(x, z) - \frac{1}{2} r(x, y)| < \varepsilon$ and" $|r(z, y) - \frac{1}{2} r(x, y)| < \varepsilon$.

From this fact that distances can be divided by 2 with any accuracy follows that it is possible to divide them into n parts with arbitrary accuracy. In addition, the condition of uniform divisibility also works for hyperbolic geometry. For a locally compact group G, even from the weak Archimedean condition (there exists a neighbourhood U of the identity such that for any element $g \in G$ which is not identity, exists a positive integer n such that $g^n \notin U$) it follows that on such a group one can introduce the Lie group structure.[11] Here the uniformity is ensured by the degrees of non–identity elements. The local compactness of a topological group corresponds to its completeness and finite dimensionality. In this case, the powers of the element A (starting from zero) correspond to direct paths from the identity point to some final point. Therefore, restricting our considering to complete spaces, by the Archimedeanness of the metric we mean the existence for any pair of points A and B an isometry φ of the interval $[0, l], l = \rho(A, B)$, from $\mathbb{R}_+$ to the given metric space, with the conditions: $\varphi(0) = A$ and $\varphi(l) = B$.

The traditional non–Archimedean metrics are p–adic metrics. Another, also common non–Archimedean metric, is the distance between the vertices of the tree (graph): the maximum distance from the vertices to the closest common for both vertices of the ancestor vertex is taken as the distance between these vertices. Note that the p–adic distance between the numbers

$$x = \sum_{i=-\infty}^{\infty} x_i p^i, \; y = \sum_{i=-\infty}^{\infty} y_i p^i$$

Is also defined as a maximum of distances to the common "ancestor". Such ancestor here is the following rational number (common prefix): $z = \sum_{i=-\infty}^{k} x_i p^i$, where k is the maximal integer for which $x_i = y_i$ for $\forall i \le k$ (remember that for p–adic numbers only a finite number of coefficients with negative indexes are different from 0).

Movings in spaces with a non–Archimedean metric have a jump type with a change in direction, like in the Markov processes. This is quite suitable for describing quantum processes. In one space, two types of metrics are not considered simultaneously. Therefore, to combine the classical approach with the quantum one, an adelic approach from the theory of numbers is usually used, which includes all compatible Archimedean and non–Archimedean completions.

In elliptic geometry, an inequality that is an immediate consequence of the triangle inequality is also true:

$$r(x, y) \ge r(x, z) - r(y, z).$$

If the points x, y in non–Archimedean geometry are at a distance $r > 0$ ($r(x, y) = r$) and the point z is closer to the point y ($r(y, z) < r$), then the distance to the point x is equal to $r(x, z) = r$, i.e., the point z locates on a sphere with center at the point x. Thus, the sphere of radius r contains all open balls of radius r with center at any point on the sphere.

When determining the distance between living species of animals through the distance on the tree, defined as the maximum distance to the nearest common ancestor, we get a non–Archimedean distance. Defining the distances between species as the natural distance between the states of the Markov process of mutations, we also obtain a non–Archimedean distance. In general, the metric is naturally defined in the state space of the Markov process, and it is non–Archimedean. It should be noted that it is difficult to clearly determine the distances here. Their values can be calculated with some accuracy by simulating the Markov process. The calculations themselves are in fact calculations of Feynman integrals over all possible paths for the evolution of the Markov process. A different approach to the use of non–Archimedean (p–adic) metrics in quantum physics develops in the works of Volovich et al.,[12] and his followers.

In a sense, non–Archimedean distances appear more often than Archimedean ones. So, in the theory of numbers, every prime ideal defines a non–Archimedean norm, and there are only a finite number of Archimedean norms (not greater than the degree of extension). An example of a non–Archimedean metric on words or texts is the following distance:

$$\rho(x,y) = \begin{cases} 0, x = y, \\ c^{-d}, x \neq y. \end{cases}$$

Here $d \geq 0$ is the length of the common prefix, $c > 1$ is a constant, for example, the number of letters in the alphabet.

Remark: In mathematics, instead of the term "non–Archimedean metric" the term "ultrametrician" is used, and the term "ultrametric relation" is used for the defining relation. At the same time, when it comes to the norming of numbers, such as the p–adic or β–adic valuation (β is a prime ideal, which is maximal by virtue of the Dedekind property), the term "non–Archimedean valuation" is used, despite the absence of order on the numbers. Some authors use the term "non–Archimedean" only for orders in ordered groups. A section of mathematics that uses instead of the field of real numbers $\mathbb{R}$ its ordered extension $\mathbb{R}$ is called a non–standard analysis. In this case, the order on $\mathbb{R}$, consistent with the order of $\{x_i\}$, is necessarily non–Archimedean. The most interesting among such extensions is the set of surreal numbers of Conway. The topology on such extensions is almost discrete, in the sense that the (countable) sequence $\{x_i\}$ converges to x if and only if all members of the sequence, starting with some, coincide with x. In this case, the distance between the points should be determined not in $\mathbb{R}_+$, but in the extension $\mathbb{R}_+$. From mathematical logic it is known that any theorem of standard analysis, proved using nonstandard analysis, can be proved without it. Therefore, we will consider the use of non–standard analysis and the term "non–archimedean" (in the sense indicated) in physical applications as non–purposeful.

As we remarked above, the principal property of the ordinary metric is the triangle inequality, and for the hyperbolic metric–the opposite triangle inequality. So, a function defined on a Cartesian product and taking real–valued values (not necessarily only positive) we'll call elliptic if

$$f(x,y) + f(y,z), \forall x,y,z;$$

call non–Archimedean if

$$f(x,z) \geq f(x,y) + f(y,z), \ \forall x,y,z.$$

and call hyperbolic if

$$f(x,z) \geq f(x,y) + f(y,z), \ \forall x,y,z.$$

We list some simple properties of such classes of functions.

i. A constant function is a metric elliptic and non–Archimedean if it is non–negative and hyperbolic–if not positive.

ii. Classes of elliptic, non–Archimedean and hyperbolic functions are closed with respect to sums, taking linear combinations with nonnegative coefficients.

iii. The classes of elliptic and non–Archimedean functions are closed with respect to taking a maximum, and hyperbolic–with respect to taking a minimum.

iv. Negative combinations (sums with negative coefficients) of elliptic functions are hyperbolic, and negative combinations of hyperbolic functions are elliptic.

v. If $\varphi(x): \mathbb{R}_+ \to \mathbb{R}_+$ is a positive continuous function, then $\varphi(f(x,z))$ is hyperbolic if $f(x,z)$ is hyperbolic and $\varphi(x)$ is convex upward ($\varphi(f(x,z))$); $\varphi(f(x,z))$ is elliptic if $f(x,z)$ is elliptic $\varphi(0) = 0$, and $\varphi(x)$ is convex downward ($\varphi''(x) < 0$).

vi. If $\{f_i(x,y)\}$ is a family of positive hyperbolic functions, then their product will also be a hyperbolic function.

Hence, as a consequence, the following remark about elliptic metrics is obtained. Let $\{f_i(x,y)\}$ be a family of elliptic functions, and all elements of the family $\{M_i - f_i(x,y)\}$ (with positive values $\{M_i\}$) satisfy the triangle inequality. Then a function $F = \prod_i M_i - \prod_i (M_i - f_i(x,y))$ satisfies the triangle inequality and, in the case of positivity of all its values, the usual metric is obtained from it. Such way, in particular, in[13] he sensitivity of Jacquard metric has been enlarged for $f_i = f, M_i = 1$.

Metrics in $\mathbb{R}^n$

Physicists in special theory of relativity (SRT) use translationally invariant metrics: here the distance from the point x to the point y is determined by the norm of the displacement vector $|y - x|$. In this case, the natural condition is that $|\lambda a| = \lambda |a|$ for any positive number λ and for any vector a. Obviously, such metrics cannot be non–Archimedean. They are Archimedean even for a stronger condition: because of the Archimedean metric in $\mathbb{R}$ and the fact that $\lambda \in \mathbb{R}_+$.

Metrics in $\mathbb{R}^n$ are defined by function–norms defined in a vector space for which the following conditions are satisfied:

a) The unit ball is convex (in the case of an elliptic metric);

b) The set $\{x \mid |x| \geq 1\}$ convex (for a hyperbolic metric).

In a linear space, the concepts of convexity and concavity of metrics can also be expressed in another way, and they are equivalent to the properties of the triangle inequality and the opposite triangle inequality, respectively. The notion of strict convexity (or concavity) is also naturally defined here–when it follows from the equality in the indicated inequalities that the points x, y, z lie on the same line.

Note that the hyperbolic metric is defined only for the interior of some cone and has the character of "directivity to the future", which specifies the ordering: if the distance $y - x$ is measurable, then the event y occurs later. If also the distance $z - y$ is measurable, then $z - x$ is also measurable, and $|z - x| \geq |y - x| + |z - y|$. This corresponds to the (strengthened) transitivity property of the order "later".

Usually, the cone of measurable vectors, where a hyperbolic (concave) norm is defined, is a set of vectors for which all coordinates are non–negative, or it contains such a set. The metric on $\mathbb{R}^n$ also defines metric in the ad joint space of linear functional. The ad joint space is also a vector space $\mathbb{R}^n_+$. For the hyperbolic metric defined in the cone $\mathbb{R}^n_+$, the adjoint metric is also defined in $\mathbb{R}^n_+$. Namely:

$$|p| = sup_{|x|=1} px,$$

In the elliptic case, and

$$|p| = inf_{|x|=1} px$$

In the hyperbolic case. As a general example for both types of the metric we can consider the metric defined by the norm in L_p in an n –dimensional space:

$$|x| = (\frac{1}{n}\sum_{i=1}^{n} |x_i|^p)^{1/p}. \qquad (2)$$

Here, we took the positive coefficients for convenience, in the case of a hyperbolic metric (taking into account the orientation) that the metric is defined only in the case when all the coordinates $\{x_i\}$ are positive.

Theorem 3.1: For $p \geq 1$, the metric (2) defines an elliptic norm, for $p < 1$–a hyperbolic. The adjoint space has the same metric L_q, where $p + q = pq, (q = \frac{p}{p-1})$.

Proof: For $p > 1$, the triangle inequality (and its strictness) follows from the Minkowski inequality, and the statement for the adjoint space is a consequence of Holder's inequality.[1] In this case, a strict inequality holds. For $p = 1$, the triangle inequality is not strictly satisfied, and there exist many paths from the point 0 (the origin) to the point x with length $|x|$. The adjoint space in this case is the space L_∞.

Now consider the case $p < 1$ and $x_i > 0$ for all i. The opposite triangle inequality for $0 < p < 1$ is also a direct consequence of Minkowski's theorem for this case.[14] For $p < 0$, this can be proved using the inequalities for the case $p > 0$ and the Holder's inequality. Fortunately, in Edvin et al.,[14] there are already corresponding generalizations, the direct consequences of which are our assertions for vectors with positive coordinates. This completes the proof.

In the case $p = 1$, the metric will be both ordinary and hyperbolic. The latter is defined only in the case when each coordinate x_i is nonnegative.

By continuity, we can define the metric (2) and in the case $p = 0$. Moreover, the adjoint metric corresponds to the case $q = 0$, i.e. coincides with the original one. This self–adjoint hyperbolic metric is called the Berwald–Moore metric:

$$|x|^n = x_1 * x_2 ... * x_n.$$

Here the norm value is the geometric mean of the family of positive numbers. Note that the norm on the set of functional corresponds to the norm with another index $L_q : \frac{1}{p} + \frac{1}{q} = 1 \Rightarrow pq = p + q$. Self–adjoint (Hilbert) spaces are spaces with $p = 2$ (for an elliptic metric) and $p = 0$ (the Berwald–Moore metric for the hyperbolic metric). In part, so, some physicists consider the Berwald–Moore metric to be the proper metric of space–time, and not the Minkowski metric. In fact, these metrics are close in the following sense. Newton's geometry, based on Galileo transformations, coincides with these geometries in the first approximation, differing only by orders of $O((\frac{v}{c})^2)$, where v is a speed, and c –the speed of light. The geometries of Minkowski and Berwald–Moore themselves coincide in the second order of accuracy, differing only in the third order. And in dimension 2, these geometries are completely identical. Indeed, by changing the variables $x_1 = t - x, x_2 = t + x$, the Minkowski metric $ds^2 = dt^2 - dx^2$ becomes the Berwald–Moore metric $ds^2 = dx_1 dx_2$.

Consider now quadratic pseudometrics:

$$|x|^2 = x_1^2 + ... + x_k^2 - x_{k+1}^2 - ... - x_n^2, n \geq k \geq 1. \qquad (3)$$

Prove the following theorem:

Theorem 3.2: The pseudo metric (2) defines an elliptic metric for $k = n$, hyperbolic–for $k = 1 < n$, and none in other cases. In the latter case, for $1 < k < n$, for any two points of the space there exist smooth paths of any positive length connecting them.

Proof: The case $k = n$ is well known. The case $k = 1 < n$ is considered in Edvin et al.,[14] where a more general inequality is proved: let $\varphi(x) = (x_1^p - x_2^p - ... - x_n^p)^{1/p}, p > 1$, then

$$\varphi(x + y) \geq \varphi(x) + \varphi(y), x \in \mathbb{R}^n_+, y \in \mathbb{R}^n_+.$$

This inequality in the case $p = 2$ is equivalent to the opposite triangle inequality for the Minkowski metric. In the case $k = 0$, the pseudometric is negative definite and does not represent a hyperbolic metric.

For the remaining case $1 < k < n$, we find smooth measurable paths issuing from the origin and ending at another point $r = (x_1, .., x_k, x_{k+1}, ..., x_n)$. Denote $r_1 = (x_1, x_2, .., x_k, 0, ..., 0)$ byand $r_2 = (0, ..., 0, x_{k+1}, ..., x_n)$ the decomposition of the position vector into two parts. Let $a = \sqrt{x_1^2 + x_2^2 + ... + x_k^2}, b = \sqrt{x_{k+1}^2 + ... + x_n^2}$. Choosing a path of the form $(x_1(t), x_2(t), ..., x_n(t)) = (\frac{x_1(a+b)}{2a}t, \frac{x_2(a+b)}{2a}t, ..., \frac{x_k(a+b)}{2a}t, \frac{x_{k+1}(a+b)}{2b}t, ..., \frac{x_n(a+b)}{2b}t), 0 \leq t \leq 1$, we get to the point $(y_1, ..., y_k, y_{k+1}, ..., y_n)$ for a distance of zero.

Further, connecting this point with r by a segment of zero length

$$(y_1+\frac{x_1(ab)}{2a}t, y_2+\frac{x_2(ab)}{2a}t,\dots,y_k+\frac{x_k(ab)}{2a}t, y_{k+1}+\frac{x_{k+1}(ba)}{2b},\dots,y_n+\frac{x_n(ba)}{2b}),0\le t\le 1$$

, we obtain a path of zero length. Since the end point does not coincide with the origin, then $a+b>0$. When $a=b$, the second segment is not needed. If $ab=0$, then by a small perturbation we make $ab\neq 0$. We smooth the path at the junction point with a slight perturbation and obtain a measurable path with any small length.

The path with any great length is obtained by making circles in the plane $r_2=0$. Note that for $k=1$ we cannot go through closed paths, and for an achievable point there exists a maximal length (a segment) among the paths connecting this point to the origin. This completes the proof.

Note that the usual symmetric polynomials σ_k on positive numbers x_i also give the hyperbolic metric $|x|^k=\sigma_k(x_1,x_2,\dots,x_n)$. In this case σ_2 corresponds to the Minkowski metric, σ_n –to Berwald–Moore metric. The fulfilment of the opposite triangle inequality for these cases is proved in Edvin et al.,[14].

For a hyperbolic metric, expressed as a homogeneous symmetric function of coordinates, the maximum of the norm for a fixed value $\sigma_1=x_1+x_2+\dots+x_n$ is attained when all variables are equal. So, it is more convenient to write the norm in terms of the variables:

$$y_0=\frac{a}{n}\sigma_1,\ y_i=\alpha z_i+\beta\sum_{j\neq i}z_j,\ z_i=(x_{i+1}-\frac{\sigma_1}{n}),\ i=1,\dots,n-1. \quad (4)$$

Then the norm is written in the form:

$$|x|=y_0 f(v_1,v_2,\dots,v_{n-1}),\ v_i=\frac{y_i}{y_0}. \quad (5)$$

In accordance with the above, $f(0,0,\dots,0)=1=max f(v_1,v_2,\dots,v_{n-1})$. In this case y_0 expresses the time coordinates, v_i –speed coordinates.

Consider the adduction of the metricto $|x|^k=\sigma_k(x_1,\dots,x_n), n\ge k$ the form?? Using the coordinate transformation (4). Let's start with the case $k=2$. We get:

$$\sigma_2=\frac12(\sigma_1^2-s_2)=\frac12\sigma_1^2-\frac12\sum_i(x_i-\frac{\sigma_1}{n})^2-\sigma_1^2(\frac1n-\frac{1}{2n^2})=$$

$$=\frac12((1-\frac1n)\sigma_1)^2-\frac12(z_1+\dots+z_{n-1})^2-\frac12\sum_{i=1}^{n-1}z_i^2=\frac12((1-\frac1n)\sigma_1)^2-\sum_{i=1}^{n-1}(\alpha z_i+\beta\sum_{j\neq i}z_j)^2.$$

Here, the values of the parameters α and β are computed so that the equality to be satisfied. Thus, in the variables

$$y_0=\frac{n-1}{n\sqrt{2}}\sum_{i=1}^{n}x_i,\ y_i=(\alpha-\beta)z_i+\beta\sum_{j=1}^{n-1}z_j,\ z_i=x_{i+1}-\frac{\sigma_1}{n}$$

The metric is adducted to the form of the Minkowski metric if the numbers α,β satisfy the system of equations:

$$\alpha^2+\beta^2(n-2)=1,\ 4\alpha\beta+(n-3)\beta^2=1.$$

Here we assume that $n\ge 3$.

The case $n=2$ is solved immediately: $\sigma_2=x_1x_2=(\frac{x_1+x_2}{2})^2-(\frac{x_1-x_2}{2})^2$. Such α,β are the following:

$$\alpha=\sqrt{\frac{7n-13\pm 4\sqrt{3}(n-2)}{n^2+10n-23}},\ \beta=\sqrt{\frac{n+5\mp 4\sqrt{3}}{n^2+10n-23}}.$$

For $k=3$ we obtain:

$$\sigma_3=\frac16\sigma_1^3-\frac12\sigma_1 s_2+\frac13 s_3.$$

The metric reduces to

$$\sigma_3=y_0^3-y_0\sum_{i=1}^{n-1}y_i^2+a\sum_i y_i^3.$$

For $k=4=n$ enter the coordinates:

$$2y_0=x_1+x_2+x_3+x_4, 2y_1=x_1+x_2-x_3-x_4, 2y_2=x_1-x_2-x_3+x_4, 2y_3=x_1-x_2+x_3-x_4.$$

The reverse transition has an analogous form:

$$2x_1=y_0+y_1+y_2+y_3, 2x_2=y_0+y_1-y_2-y_3, 2x_3=y_0-y_1-y_2+y_3, 2x_4=y_0-y_1+y_2-y_3.$$

This transformation is similar to the transformation in the case $k=2=n$ at the transition from the metric $|x|^2=x_1x_2$ to $\frac12(y_0^2-y_1^2)$ by the transformation $y_0=\frac{1}{\sqrt2}(x_1+x_2),\ y_1=\frac{1}{\sqrt2}(x_1-x_2)$ preserving the sums of squares $y_0^2+y_1^2=x_1^2+x_2^2$, $x_1^2+x_2^2+x_3^2+x_4^2=y_0^2+y_1^2+y_2^2+y_3^2$. Here y_0 plays the role of the time coordinate, and the rest–the role of spatial coordinates, and take arbitrary values. A positive value of the norm have vectors for which y_0 is greater than the maximal root of the corresponding polynomial. In our case:

$$x_1x_2x_3x_4=\frac{1}{16}(y_0^4-2y_0^2r^2+8y_0y_1y_2y_3+2(y_1^4+y_2^4+y_3^4)-r^4), r^2=y_1^2+y_2^2+y_3^2.$$

At first glance this metric is anisotropic. As will be seen later, this does not mean anisotropy of space, but reduces only to the possible different compression of space in the direction of motion for different directions of velocity. And the speed of light will remain the same, independent of direction. This compression is apparent from the side of another observer.

An analogue of these transformations for the case $k=n$ is the transformation using the Hadamard matrix with the normalization $\frac{1}{\sqrt{n}}$. In this case, the role of the time coordinates plays $y_0=\frac{1}{\sqrt{n}}(x_1+x_2+\dots+x_n)$. However, not for all n there is a Hadamard matrix. Correspondingly, the orthonormal system of coordinates y_i must be sought as in the case $k=2$.

Let $E_k(x)=\frac{1}{C_n^k}\sigma_k(x_1,x_2,...,x_n)$. An analogue of the Holder inequality:

$$\left(\frac{1}{n}\sum_{i=1}^{n}x_i y_i\right)^k \geq E_k(x)E_k(y)$$

Is not satisfied without the additional requirement of the same ordering:

$$(x_i-x_j)(y_i-y_j)\geq 0 \ \forall i,j.$$

Only for $k=n$ this inequality holds for of non–negative numbers, which means that the Berwald–Moore metric is self–adjoint.

In general, any metric (norm) in R^n can be adducted to the form:

$$|x|=E_1\varphi(e_2,e_3,...,e_n),$$

where $e_i=\frac{E_i(x)}{E_1^i(x)}$. This corresponds to dimension theory in mechanics. Removing the value of $E_1(x)$ we get a dimensionless formula, and the multiplier is expressed as a function of the dimensionless variables $e_i(x)$. For a hyperbolic metric, the opposite triangle inequality reduces to the negative definiteness of the quadratic form $\frac{\partial^2 s}{\partial x_i \partial x_j}dx_i dx_j$ on the indicatrix. The latter condition is often easier to verify.

Physicists often seem strange to have hyperbolic metrics defined in the region $x_i\geq 0$. But really, there are no grounds for this. We can always take the coordinate system $y_0,...,y_{n-1}$, obtained by a linear transformation from the initial coordinates, where $y_1,y_2,...,y_{n-1}$ are arbitrary, and the restriction $y_0>0$ is superimposed on measurable vectors directivity to the future). And the strict opposite triangle inequality imposes on the indicatrix given by the homogeneous equation $f(y_0,...,y_{n-1})=1$ of the first degree, or arbitrary $y_0=\varphi(y_1,...,y_{n-1})$, which is the solution of the first, one condition:

The quadratic form is positive–definite.

$$\sum_{ij}\frac{\partial^2\varphi}{\partial y_i \partial y_j}x_i x_j \qquad (6)$$

In a hyperbolic metric, there is always a $(n-1)$ –dimensional light cone, the set of vectors with the norm 0. Any vector with a positive norm can be represented as the sum of two vectors with zero norms. The latter follows from the fact that two light cones with vertices at the point O (origin) and at the point X at the end of the measurable vector, intersect.

Conclusion

One of the problems considered in the paper is a minimization of the set of axioms for geometries describing the physical space–time continuum. We were convinced that hyperbolicity determines the cause–effect and does not require a special definition of the signature of space. The continuity property follows from the Archimedean property of the standard topology on ℝ and from homogeneity axiom. For a quantum description of space–time, where the cause–and–effect can be not valid at short distances, probably it would be better to replace the standard topology on ℝ with a quasi–topology: when all the sequences (filters) $\{x_i\}$ converging to the point x are defined by the fact that, starting from some i, their elements fall into an ε –neighborhood of the point x, with the fixed value ε as a quantum number.

In Guts et al.,[8] is presented a justification for STR, expressed in the principles of the space uniformity and the identity of the finite light speed in all inertial reference systems, like in the Minkowski geometry. In the future, we plan to show that STR represents itself a hyperbolic homogeneous and isotropic geometry. The property of isotropy corresponds more closely to the Berwald–Moore metric, rather than Minkowski one. The isotropy of hyperbolic geometries corresponds more closely to the invariance property with respect to the choice of scale for all possible variants, different in various directions. The role of generalized conformal transformations, as shown at the end of the second section of this article, is played by transformations of the metric with the hyperbolic functions. The Berwald–Moore geometry fully corresponds to the homogeneity property when replacing scales in all possible directions.

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Conflict of interest

Author declares there is no conflict of interest.

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