Research Note

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Non–existence of potential vorticity in a stationary axisymmetric adiabatic fluid configuration

Abstract

It is found that the value of relativistic potential vorticity is zero in the case of a stationary axisymmetric adiabatic fluid flows. The fluid's vorticity flux vector lies in the level surfaces of constant entropy per baryon and thereby leads to conservation of fluid helicity.

Keywords: potential vorticity, fluid's vorticity, vector, entropy, kelvin's circulation theorem, fluid helicity, euler's equation

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Prasad G
Department of Physics, Kamla Nehru Institute of Physical and Social Sciences, India

Correspondence: Gunraj Prasad, Department of Physics, Kamla Nehru Institute of Physical and Social Sciences, Sultanpur, Uttar Pradesh, India, Tel 9452328706, Email gunrajpr@gmail.com

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Fluid vorticity flux and potential vorticity

The conservation of relativistic potential vorticity along adiabatic fluid flow lines in connection with Kelvin's circulation conservation has been demonstrated by Katz.[1] In particular, the stream line invariance of potential vorticity is intimately related to a weak version of Kelvin's circulation theorem. Bekenstein[2] has pointed out that there is an intrinsic relation between potential vorticity and fluid helicity. The purpose of present note is to obtain an explicit expression of fluid vorticity flux[3] in a stationary axisymmetric adiabatic fluid flows and use this expression to show that the value of potential vorticity is zero and the fluid helicity is conserved.

We begin with Euler's equation of motion given by[4]

$$W_{ab}u^{b} = Ts_{,a}\,, \qquad (1)$$

Where T and s denote, respectively, the local temperature and the entropy per baryon measured in the fluid's rest frame. u^{a} is the 4-velocity of the fluid and obeys the normalization condition $u^{a}u_{a} = -1$. The particle vorticity 2-form W_{ab} is expressible as

$$W_{ab} = \left(\mu u_{b}\right)_{;a} - \left(\mu u_{a}\right)_{;b}\,, \qquad (2)$$

Where $\mu = \dfrac{\rho + p}{n}$ denotes the relativistic enthalpy per baryon. Adiabaticity condition $u^{a}s_{,a} = 0$ follows from (1). The proper energy density and pressure are respectively, designated by ρ and p . The baryon conservation law is given by

$$\left(nu^{a}\right)_{;a} = 0\,, \qquad (3)$$

Where n is baryon number density. It follows from (2) that

$$W_{[ab;c]} = 0\,, \qquad (4)$$

Where the square bracket around indices denotes skew–symmetrization. Covariant derivative is denoted by semicolon while partial derivative is indicated by comma throughout the present text. It is evident from (1) and (4) that Euler's equation appears as Maxwell's like equation. We now assume that the fluid flows are stationary and axisymmetric. We take t and φ to be the time and axial coordinates, respectively, in spherical coordinate system and r , θ are poloidal coordinates. All physical quantities including the metric tensor g_{ab} are independent of toroidal coordinates t and φ . In this coordinate system using symmetry assumption we obtain from (1)–(4) the first integrals as follows:

$$W_{\varphi r} = An\sqrt{-g}\,u^{\theta}\,, \qquad (5)$$

$$W_{\varphi\theta} = -An\sqrt{-g}\,u^{r}\,, \qquad (6)$$

$$\frac{W_{tr}}{W_{\varphi r}} = \frac{W_{t\theta}}{W_{\varphi\theta}} = -(say) \qquad (7)$$

$$W_{tr} = -An\sqrt{-g}\,u^{\theta}\,, \qquad (8)$$

$$W_{t\theta} = An\sqrt{-g}\,u^{r}\,, \qquad (9)$$

$$W_{r\theta} = An\sqrt{-g}\left(-u^{t} + u^{\varphi}\right) + \left(\frac{T}{u^{\theta}}\right)s_{,r} \qquad (10)$$

Where A is a constant of integration along the fluid flow lines and is called streamline invariant. $\bar{\Omega}$ represents the mechanical rotation of fluid's vortex lines. The magnetic part of W_{ab} can be defined as

$$V^{a} = -{}^{*}W^{ab}u_{b}\,, \qquad (11)$$

Where $V^{a} = 2\mu\omega^{a}$ and ${}^{*}W^{ab} = \frac{1}{2}\eta^{abcd}W_{cd}$. The vector field ω^{a} is the fluid's vorticity and is defined according as $\omega^{a} = \frac{1}{2}\eta^{abcd}u_{b;c}u_{d}$.[5] η^{abcd} is the Levi-Civita alternating tensor. The quantity V^{a} has been referred to as the fluid's vorticity flux.[3] Making use of (5), (6), (8), (9) and (10) in (11), a straightforward calculation yields that

$$V^{a} = An\left[\left(\delta^{a}_{t} + \delta^{a}_{\varphi}\right) + \left(u_{t} + u_{\varphi}\right)u^{a}\right] + \left(\frac{T}{u^{\theta}}\right)s_{,r}\left(\delta^{a}_{\varphi}u_{t} - \delta^{a}_{t}u_{\varphi}\right) \quad , (12)$$

Which exhibits that the spatial variation of the entropy per baryon contributes to the generation of fluid's vorticity flux in addition to the mechanical rotation of vortex lines. On account of (12) and adibaticity condition, we find that

$$V^a s_{,a} = 0\,, \qquad (13)$$

Which shows that the vorticity flux vector lies in the level surfaces of constant entropy per baryon. The expression given by (12) is an explicit expression for the fluid's vorticity flux. The potential vorticity defined by Katz[1] is $e = \frac{\mu\omega^a s_{,a}}{n}$. Because of (13), it is seen that $e = 0$. Thus the potential vorticity is zero in a stationary axisymmetric adiabatic fluid flows. The fluid helicity introduced by Bekenstein[2] is expressible as

$$H^a = {}^*W^{ab}\mu u_b\,, \qquad (14)$$

Which in our case takes the form $H^a = \mu V^a$. The fluid helicity is conserved, i.e. $H^a_a = 0$ because of (13). Further results will be published elsewhere.

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Conflict of interest

Author declares there is no conflict of interest.

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