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Compilation of a system equations dynamics of a six-link mechanism with branching using complex functions

Abstract

The article proposes a new method for constructing a mathematical model of a six-link mechanism with branching in the form of a system of differential equations of motion in the form of Lagrange equations of the second kind based on complex variable functions. The use of complex variable functions simplifies and speeds up the process of determining the coordinates of the links, which are defined by a single complex function in exponential form, when constructing the system of differential equations of motion. This simplifies the calculation of the squares of the velocities of the links' centers of mass and speeds up the process of calculating the kinetic energy of the system, which is necessary when using Lagrange equations of the second kind. The method is illustrated by a mechanism model with six movable links, two of which branch off from the branching point. In this case, the free end of one of the links is assumed to be fixed on a stationary cylindrical hinge that is rigidly connected to the support surface. The angles are measured from the horizontal coordinate axis of the fixed reference system, in accordance with the definition and method of measuring the argument of a complex number. The model of a mechanism with six movable links connected by hinges has drives in the hinges that create control moments and uniquely determine the positions of the links. In practice, the proposed model can be used to create robotic manipulators for industrial production, medical robots, and anthropomorphic mechanisms such as robots, exoskeletons, spacesuits, prosthetics, and simulators. It can be used to model and study the biomechanics of the human musculoskeletal system, solve direct and inverse problems of anthropoid structure control, select drives and materials, and optimize the design.

Keywords: complex variable function, mechanism with six movable links, lagrange differential equations of the second kind, cylindrical hinge, control torques, rotation angles, angular velocities, angular accelerations, robot, exoskeleton

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Introduction

The creation of adequate control systems for robotic manipulators, anthropoid mechanisms such as exoskeletons, simulators, spacesuits, anthropomorphic robots, and other mechanical systems with a significant number of degrees of freedom requires fundamental research on the analytical modeling of dynamic processes and the development of effective methods for compiling systems of differential equations of motion for such mechatronic robotic systems. The challenge lies in the significant time required for analytical transformations when compiling Lagrange equations of the second kind, as the number of operations increases exponentially with the number of links. Thus, adding the next link to the model leads to a significantly non-linear increase in time costs. To solve this problem, matrix and recursive methods for writing systems of differential equations of motion have been proposed.[1–3] However, these methods have the disadvantage of requiring time-consuming preparation for analyzing models with one, two, or three links, and then using the resulting generalizing formulas to construct a multi-link model, such as a model with six movable links. In order to find an alternative way to simplify operations in order to speed up the writing of systems of differential equations of motion, this paper proposes the use of the complex variable function apparatus. In this approach, each link is described by a single complex function in the exponential form,[4] rather than by two trigonometric expressions as in the traditional approach.[5] Accordingly, the use of complex functions reduces the number of operations performed by approximately two times, and the use of the exponential form of the record simplifies and speeds up the calculation of derivatives. Thus, a significant optimization of the process of compiling a system of differential equations of motion for the considered multi-link model is achieved. The results of the study can be applied to anthropomorphic and other robotic mechatronic systems.[6–12]

Material and methods

Consider a mechanism model with six movable inertial links that do not change their length (Figure 1). A feature of the proposed design solution is the symmetry of the model with respect to the attachment point of the links H_2. Let's call the point H_2 the point of branching. From this point, two links emerge in three different directions. The position of the emerging links is determined by the angles of inclination to the horizontal. Due to such a symmetrical design, the proposed mechanism can find wide application in the creation of manipulators, industrial, medical robots. For definiteness, we give the mechanism an anthropoid appearance. At the same time, we fix the end of one of the links. Without violating the generality of the consideration, we assume that the point H_0 is rigidly connected to the support. We place the origin of the right-handed Cartesian coordinate system at this point, with the H0x axis pointing horizontally to the right and the H_0y axis pointing vertically upwards. We assume that the mechanism moves in the vertical plane defined by these coordinate axes.

The links are coupled with cylindrical hinges at the points H_0, H_1, … H_6. The cylindrical hinge at the point H_0 is firmly fixed to the supporting surface. The lengths, mass, moments of inertia, multiplier that sets the position of the center of mass of the links shown in Figure1. The links position depends on just one parameter and is determined by the angles $\psi_1(t)$, $\psi_2(t)$, …, $\psi_6(t)$ – the generalized coordinates. M_1, M_2, …, M_6 – the controlling torque developed in the hinges.

The position of the centers of mass C_1, C_2, ..., C_6 of each moving link in Figure 1 is described by a complex variable function in exponential form:

$$w_1 = l_1 n_1 e^{j\psi_1}, \quad (1)$$

$$w_2 = l_1 e^{j\psi_1} + l_2 n_2 e^{j\psi_2}, \quad (2)$$

$$w_3 = l_1 e^{j\psi_1} + l_2 e^{j\psi_2} + l_3 n_3 e^{j\psi_3}, \quad (3)$$

$$w_4 = l_1 e^{j\psi_1} + l_2 e^{j\psi_2} + l_3 e^{j\psi_3} + l_4 n_4 e^{j\psi_4}, \quad (4)$$

$$w_5 = l_1 e^{j\psi_1} + l_2 e^{j\psi_2} + l_5 n_5 e^{j\psi_5}, \quad (5)$$

$$w_6 = l_1 e^{j\psi_1} + l_2 e^{j\psi_2} + l_5 e^{j\psi_5} + l_6 n_6 e^{j\psi_6}, \quad (6)$$

where j – imaginary unit.

In the proposed model, there is a branching point where three or more links meet. In this model, the branching point is H_2. It meets three links with numbers two, three, and five. When passing through the branching point, when considering link number 3, for example, links numbers five and six should not be considered, and the summation should be stopped at link number four, formula (4). When considering link number five, links numbers three and four should be skipped, formula (6). This is also used in the subsequent results.

The positions of the cylindrical hinges H_1, H_2, ..., H_6 connecting the movable links in Figure 1 are described by complex variable functions in exponential form, which are obtained from formulas (1)-(6) by replacing each of the multipliers n_1, n_2, ..., n_6 with one. The position of the hinge H_0, which is rigidly connected to the support surface, is obtained from formula (1) by replacing the multiplier n1 with zero. The proposed method of recording the coordinates of the centers of mass of links (1)-(6) using an exponential form of recording and multipliers, n_1, n_2, ..., n_6, which determine the positions of the centers of mass, is universal. By varying the values of the multipliers n_1, n_2, ..., n_6, it is possible to obtain information about the motion of any point on each link of the mechanism. For example, if we consider each rod to be absolutely solid and homogeneous, then the position of the center of mass in such a rod model is located at the geometric center of the rod. In this case, to set the positions of the centers of mass of the links in formulas (1)-(6), it is sufficient to set each value of the multiplier n_1, n_2, ..., n_6 equal to one-half, i.e.: $n_1 = ½$, $n_2 = ½$, …, $n_6 = ½$.

By differentiating the formulas (1)-(6) with respect to time, we determine the complex velocities of the centers of mass of the anthropoid links:

$$\frac{dw_1}{dt} = l_1 n_1 e^{j\psi_1} j \frac{d\psi_1}{dt}, \quad (7)$$

$$\frac{dw_2}{dt} = l_1 e^{j\psi_1} j \frac{d\psi_1}{dt} + l_2 n_2 e^{j\psi_2} j \frac{d\psi_2}{dt}, \quad (8)$$

$$\frac{dw_3}{dt} = l_1 e^{j\psi_1} j \frac{d\psi_1}{dt} + l_2 e^{j\psi_2} j \frac{d\psi_2}{dt} + l_3 n_3 e^{j\psi_3} j \frac{d\psi_3}{dt}, \quad (9)$$

$$\frac{dw_4}{dt} = l_1 e^{j\psi_1} j \frac{d\psi_1}{dt} + l_2 e^{j\psi_2} j \frac{d\psi_2}{dt} + l_3 e^{j\psi_3} j \frac{d\psi_3}{dt} + l_4 n_4 e^{j\psi_4} j \frac{d\psi_4}{dt}, \quad (10)$$

$$\frac{dw_5}{dt} = l_1 e^{j\psi_1} j \frac{d\psi_1}{dt} + l_2 e^{j\psi_2} j \frac{d\psi_2}{dt} + l_5 n_5 e^{j\psi_5} j \frac{d\psi_5}{dt}, \quad (11)$$

$$\frac{dw_6}{dt} = l_1 e^{j\psi_1} j \frac{d\psi_1}{dt} + l_2 e^{j\psi_2} j \frac{d\psi_2}{dt} + l_5 e^{j\psi_5} j \frac{d\psi_5}{dt} + l_6 n_6 e^{j\psi_6} j \frac{d\psi_6}{dt}, \quad (12)$$

The kinetic energy of the mechanism:

$$T = T_{H_0H_1} + T_{H_1H_2} + T_{H_2H_3} + T_{H_3H_4} + T_{H_2H_5} + T_{H_5H_6}. \quad (13)$$

$$\begin{gathered}
2T = \left[I_1 + (m_2 + m_3 + m_4 + m_5 + m_6)l_1^2\right]\dot\psi_1^2 + \left[I_2 + (m_3 + m_4 + m_5 + m_6)l_2^2\right]\dot\psi_2^2 + \left[I_3 + m_4 l_3^2\right]\dot\psi_3^2 + \\
+ I_4\dot\psi_4^2 + \left[I_5 + m_6 l_5^2\right]\dot\psi_5^2 + I_6\dot\psi_6^2 + (m_2n_2 + m_3 + m_4 + m_5 + m_6)l_1l_2\cos(\psi_1 - \psi_2)\dot\psi_1\dot\psi_2 + \\
+(m_3n_3 + m_4)l_1l_3\cos(\psi_1 - \psi_3)\dot\psi_1\dot\psi_3 + (m_3n_3 + m_4)l_2l_3\cos(\psi_2 - \psi_3)\dot\psi_2\dot\psi_3 + \\
+m_4n_4l_1l_4\cos(\psi_1 - \psi_4)\dot\psi_1\dot\psi_4 + m_4n_4l_2l_4\cos(\psi_2 - \psi_4)\dot\psi_2\dot\psi_4 + \\
+m_4n_4l_3l_4\cos(\psi_3 - \psi_4)\dot\psi_3\dot\psi_4 + (m_5n_5 + m_6)l_1l_5\cos(\psi_1 - \psi_5)\dot\psi_1\dot\psi_5 + \\
+(m_5n_5 + m_6)l_2l_5\cos(\psi_2 - \psi_5)\dot\psi_2\dot\psi_5 + m_6n_6l_1l_6\cos(\psi_1 - \psi_6)\dot\psi_1\dot\psi_6 + \\
+m_6n_6l_2l_6\cos(\psi_2 - \psi_6)\dot\psi_2\dot\psi_6 + m_6n_6l_5l_6\cos(\psi_5 - \psi_6)\dot\psi_5\dot\psi_6.
\end{gathered} \quad (14)$$

The potential energy of the mechanism:

$$\begin{gathered}
\Pi = \left(m_1n_1 + m_2 + m_3 + m_4 + m_5 + m_6\right)gl_1\sin\psi_1 + \left(m_2n_2 + m_3 + m_4 + m_5 + m_6\right)gl_2\sin\psi_2 + \\
+\left(m_3n_3 + m_4\right)gl_3\sin\psi_3 + m_4n_4gl_4\sin\psi_4 + \left(m_5n_5 + m_6\right)gl_5\sin\psi_5 + m_6n_6gl_6\sin\psi_6.
\end{gathered} \quad (15)$$

A system of differential equations is constructed using the second-order Lagrange equations.

Results

The system of differential equations of motion for the three-link exoskeleton model has been composed using the software for building the system of differential equations of motion

$$\begin{gathered}
\left[I_1 + (m_2 + m_3 + m_4 + m_5 + m_6)l_1^2\right]\ddot\psi_1 + \\
+(m_2n_2 + m_3 + m_4 + m_5 + m_6)\cos(\psi_1 - \psi_2)l_1l_2\ddot\psi_2 + \\
+(m_3n_3 + m_4)\cos(\psi_1 - \psi_3)l_1l_3\ddot\psi_3 + m_4n_4\cos(\psi_1 - \psi_4)l_1l_4\ddot\psi_4 + \\
+(m_5n_5 + m_6)\cos(\psi_1 - \psi_5)l_1l_5\ddot\psi_5 + m_6n_6\cos(\psi_1 - \psi_6)l_1l_6\ddot\psi_6 + \\
+(m_2n_2 + m_3 + m_4 + m_5 + m_6)\sin(\psi_1 - \psi_2)l_1l_2\dot\psi_2^2 + \\
+(m_3n_3 + m_4)\sin(\psi_1 - \psi_3)l_1l_3\dot\psi_3^2 + m_4n_4\sin(\psi_1 - \psi_4)l_1l_4\dot\psi_4^2 + \\
+(m_5n_5 + m_6)\sin(\psi_1 - \psi_5)l_1l_5\dot\psi_5^2 + m_6n_6\sin(\psi_1 - \psi_6)l_1l_6\dot\psi_6^2 + \\
+\left(m_1n_1 + m_2 + m_3 + m_4 + m_5 + m_6\right)gl_1\cos\psi_1 = M_{\psi1} - M_{\psi2},
\end{gathered} \quad (16)$$

$$\begin{gathered}
(m_2n_2 + m_3 + m_4 + m_5 + m_6)\cos(\psi_1 - \psi_2)l_1l_2\ddot\psi_1 + \\
+\left[I_2 + (m_3 + m_4 + m_5 + m_6)l_2^2\right]\ddot\psi_2 + \\
+(m_3n_3 + m_4)\cos(\psi_2 - \psi_3)l_2l_3\ddot\psi_3 + m_4n_4\cos(\psi_2 - \psi_4)l_2l_4\ddot\psi_4 + \\
+(m_5n_5 + m_6)\cos(\psi_2 - \psi_5)l_2l_5\ddot\psi_5 + m_6n_6\cos(\psi_2 - \psi_6)l_2l_6\ddot\psi_6 - \\
-(m_2n_2 + m_3 + m_4 + m_5 + m_6)\sin(\psi_1 - \psi_2)l_1l_2\dot\psi_2^2 + \\
+(m_3n_3 + m_4)\sin(\psi_2 - \psi_3)l_2l_3\dot\psi_3^2 + m_4n_4\sin(\psi_2 - \psi_4)l_2l_4\dot\psi_4^2 + \\
+(m_5n_5 + m_6)\sin(\psi_2 - \psi_5)l_2l_5\dot\psi_5^2 + m_6n_6\sin(\psi_2 - \psi_6)l_2l_6\dot\psi_6^2 + \\
+\left(m_2n_2 + m_3 + m_4 + m_5 + m_6\right)gl_2\cos\psi_2 = M_{\psi2} - M_{\psi3} - M_{\psi5},
\end{gathered} \quad (17)$$

$$
\begin{aligned}
&(m_3n_3+m_4)\cos(\psi_1-\psi_3)l_1l_3\ddot{\psi}_1+\\
&+(m_3n_3+m_4)\cos(\psi_2-\psi_3)l_2l_3\ddot{\psi}_2+\\
&+(I_3+m_4l_3^2)\ddot{\psi}_3+m_4n_4\cos(\psi_3-\psi_4)l_3l_4\ddot{\psi}_4-\\
&-(m_3n_3+m_4)\sin(\psi_1-\psi_3)l_1l_3\dot{\psi}_1^2-\\
&-(m_3n_3+m_4)\sin(\psi_2-\psi_3)l_2l_3\dot{\psi}_2^2+\\
&+m_4n_4\sin(\psi_3-\psi_4)l_3l_4\dot{\psi}_4^2+\\
&+\left(m_3n_3+m_4\right)gl_3\cos\psi_3=M_{\psi 3}-M_{\psi 4},
\end{aligned}
\tag{18}
$$

$$
\begin{aligned}
&m_4n_4\cos(\psi_1-\psi_4)l_1l_4\ddot{\psi}_1+\\
&+m_4n_4\cos(\psi_2-\psi_4)l_2l_4\ddot{\psi}_2+\\
&+m_4n_4\cos(\psi_3-\psi_4)l_3l_4\ddot{\psi}_3+I_4\ddot{\psi}_4-\\
&-m_4n_4\sin(\psi_1-\psi_4)l_1l_4\dot{\psi}_1^2-\\
&-m_4n_4\sin(\psi_2-\psi_4)l_2l_4\dot{\psi}_2^2-\\
&-m_4n_4\sin(\psi_3-\psi_4)l_3l_4\dot{\psi}_3^2+\\
&+m_4n_4gl_4\cos\psi_4=M_{\psi 4},
\end{aligned}
\tag{19}
$$

$$
\begin{aligned}
&(m_5n_5+m_6)\cos(\psi_1-\psi_5)l_1l_5\ddot{\psi}_1+\\
&+(m_5n_5+m_6)\cos(\psi_2-\psi_5)l_2l_5\ddot{\psi}_2+\\
&+(I_5+m_6l_5^2)\ddot{\psi}_5+m_6n_6\cos(\psi_5-\psi_6)l_5l_6\ddot{\psi}_6-\\
&-(m_5n_5+m_6)\sin(\psi_1-\psi_5)l_1l_5\dot{\psi}_1^2-\\
&-(m_5n_5+m_6)\sin(\psi_2-\psi_5)l_2l_5\dot{\psi}_2^2+\\
&+m_6n_6\sin(\psi_5-\psi_6)l_5l_6\dot{\psi}_6^2+\\
&+\left(m_5n_5+m_6\right)gl_5\cos\psi_5=M_{\psi 5}-M_{\psi 6},
\end{aligned}
\tag{20}
$$

$$
\begin{aligned}
&m_6n_6\cos(\psi_1-\psi_6)l_1l_6\ddot{\psi}_1+\\
&+m_6n_6\cos(\psi_2-\psi_6)l_2l_6\ddot{\psi}_2+\\
&+m_6n_6\cos(\psi_5-\psi_6)l_5l_6\ddot{\psi}_5+I_6\ddot{\psi}_6-\\
&-m_6n_6\sin(\psi_1-\psi_6)l_1l_6\dot{\psi}_1^2-\\
&-m_6n_6\sin(\psi_2-\psi_6)l_2l_6\dot{\psi}_2^2-\\
&-m_6n_6\sin(\psi_5-\psi_6)l_5l_6\dot{\psi}_5^2+\\
&+m_6n_6gl_6\cos\psi_6=M_{\psi 6},
\end{aligned}
\tag{21}
$$

Thus, the research goal has been achieved: a new method for writing dynamic equations based on the use of complex variable functions has been developed. This method has been used to create a system of differential equations of motion for a symmetric mechanism with six movable links. This system serves as a mathematical model for the mechanical system under consideration (Figure. 1).

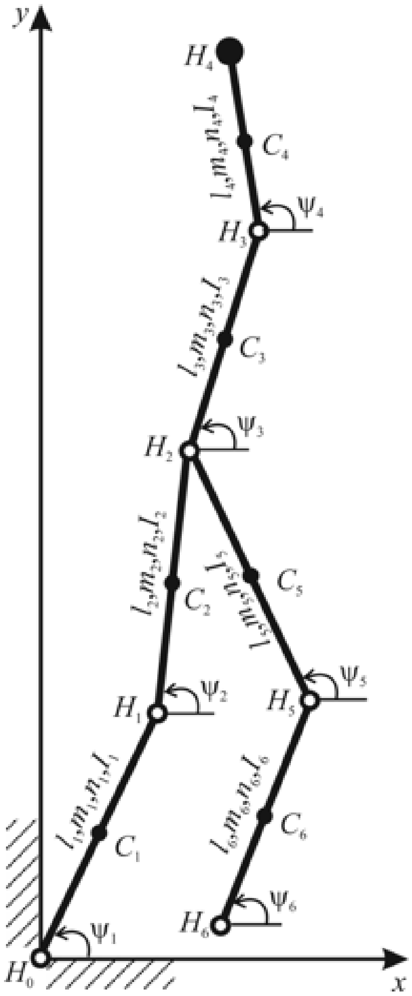

Figure 1 The model of the mechanism with six mobile links and local systems of coordinates, moving in the vertical plane.

Discussion

In the future, it is possible to use the compiled system of differential equations of motion to solve the first and second problems of dynamics for the considered model of a mechanism with six movable rigid links. For example, using the method of program control of motion, by setting the kinematic functions of changing angles over time, which synthesize anthropomorphic walking in the form of:

$$
\psi_j\left(t\right)=\pi/2+\varsigma_j\sin\left[\varpi_j-\left(1-\cos\left[2\pi t/T\right]\right)\pi/2\right],\qquad j=1,2,...,6,
\tag{22}
$$

where T – walking period, ς_j, ϖ_j – walking parameters, measured in seconds and radians, respectively.

It is possible, by substituting the functions (22) into the system (16)-(21) and solving the system algebraically with respect to the unknown control moments standing on the right-hand side of the equations, to

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find the control that should be synthesized by the mechanisms drives. The first problem of dynamics is solved in this way. To solve the second problem of dynamics, it is necessary to set the control forces that determine the rotations of the mechanism links on the plane, the initial or boundary conditions of motion, and to determine the law of motion – the change of the angles of inclination of the links over time. To do this, it is necessary to solve a system of nonlinear differential equations of the twelfth order (16)-(21). Analytical methods do not solve such a system of differential equations even on a small time interval in a linearized form. Therefore, the use of numerical methods is necessary to solve the system of differential equations (16)-(21).

Conclusion

As a result of the conducted research, a new, more rational method of compiling systems of differential equations of motion has been created, based on the application of the apparatus of functions of a complex variable. It allows to reduce the number of necessary actions, determining the position of each link not by two projections using trigonometric functions, but by one function of a complex variable, presented in an exponential form of recording. Compared to trigonometric functions, the use of complex variable functions reduced the number of operations from 88 to 49. The advantage of this approach becomes more evident when using models with a large number of links, with the number of operations increasing almost exponentially with the increase in the number of links. As an example, a mechanical system with six movable links of an anthropoid structure is considered, for which a system of differential equations of motion is composed.

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None.

Conflicts of interest

The author declares there is no conflict of interest.

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