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Some concepts of boundedness in modular metric spaces

Abstract

We study the concept of boundness in the settings of modular metric spaces. This concept has been studied in metric spaces by different authors with different motivations. In this article, we extend the results of boundedness in metric spaces to the frame- work of modular metrics. For example, we prove that if a modular metric is a continuous from the right on the set of positive real numbers then the total boundedness in (X_w, d_w) is equivalent to w -total boundedness. We also show that modular set is separable if and only if the family of its bounded subsets agrees with the family of totally bounded subsets. In short, we study those collections of subsets on a modular set that are bornologies of bounded sets and totally bounded sets.

Keywords: metric, bounded set, totally bounded set, bornology, modular metric and modular set

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Introduction

The concept of modular metric spaces was introduced by V Chistyakov in 2010. The author presented a complete description of generators of Lipschitz continous functions and classes of superposition operators. Chistyakov defined a modular metric in the following way: Let X be a nonempty set, then the function $w:(0,\infty)\times X\times X\to[0,\infty]$ is said to be a modular metric on X if the following conditions are satisfied: (a) $w(\lambda,x,x)=0$ for all $x\in X$ and $\lambda\in 0,\infty)$, (b) $w(\lambda,x,y)=w(\lambda,y,x)$ whenever $x,y\in X$ and $\lambda>0$. (c) $w(\lambda+\mu,x,y)\le w(\lambda,x,z)+w(\mu,z,y)$ whenever $x,y,z\in X$ and $\lambda,\mu\in(0,\infty)$.

Chistyakov provided an application of modular metrics which consists of an extended kind of Helly's theorem on the pointwise selection principle. This was obtained by building a special modular space, the set of all bounded and regulated mappings on an interval. He also considered the description of superposition operators acting in modular spaces, the existence of regular selections of set-valued maps, the new interpretation of Lipschitzian and absolutely continuous maps, and the existence of solutions to the Carathéodory-type ordinary differential equations in Banach spaces with the right-hand side from the Orlicz. Several extensions to his findings then followed. One such study is by Abdou where he investigated 1-local retracts in modular metric spaces with focus on the existence of common fixed points of modular nonexpansive mappings. In addition, the well-known and important concept of hyperconvexity in a metric space has been successfully investigated and applied in many areas of mathematics and other fields. For instance, Otafudu and Sebogodi introduced the theory of hyper- onvexity in the setting of modular pseudometric that is herein called w-Isbell-convexity. In their paper, they showed that on a modular set, w-Isbell-convexity is equivalent to hyperconvexity whenever the modular pseudometric is continuous from the right on the set of positive numbers. They also discussed the boundedness of a set endowed with a modular pseudometric. The aim of this article is to study the theory of boundedness in modular metrics and introduce connections between boundedness and totally boundedness on a modular set. We characterize those bornologies on a modular set that are bornologies of bounded sets and totally bounded sets.[1–12]

Preliminaries

For the comfort of the reader and in preparation of the terminology that we are going to use throughout this article, we recall the following concepts that can be found in.

Definition 1. *Let X be a set. A function $w:(0,\infty)\times X\times X\to[0,\infty]$ is said to be a modular metric on X if the following conditions are satisfied:*

I. $w(\lambda,x,x)=0$ *for all* $x\in X$ *and* $\lambda\in 0,\infty)$.

II. $w(\lambda,x,y)=w(\lambda,y,x)$ *whenever* $x,y\in X$ *and* $\lambda>0$.

III. $w(\lambda+\mu,x,y)\le w(\lambda,x,z)+w(\mu,z,y)$ *if* $x,y,z\in X$ *and* $\lambda,\mu\in(0,\infty)$.

IV. The mapping w is said to be a modular metric provided that it also satisfies the following condition:

V. $w(\lambda,x,y)=0$ *for all* $\lambda>0$ *imply* $x=y$.

For $x_0\in X$, the modular set $X_w(x_0)$ is defined by

$$X_w(x_0)=\left\{x\in X:\lim_{\lambda\to\infty}w(\lambda,x,x_0)=0\right\}.$$

In what follows, we are not going to reference the point $x_0\in X$ any more in order to simplify the notation $X_w(x_0)$, thus $X_w=X_w(x_0)$ in the sequel.

The set $X_w(x_0)$ is also referred to as w-modular set (around x_0) and x_0 is called the center of X_w. Moreover, the function q_w defined by

$$d_w(x,y)=\inf\{\lambda>0:w(\lambda,x,y)\le\lambda\}$$

for all $x,y\in X_w$, is a metric on X_w, whenever w is modular metric on X.

Remark 1. *For any $x\in X_w$ and $\lambda,\mu>0$, we define the sets $B^w_{\lambda,\mu}(x)$ and $C^w_{\lambda,\mu}(x)$ by $B^w_{\lambda,\mu}(x)=\{z\in X_w:w(\lambda,x,z)<\mu\}$ and $C^w_{\lambda,\mu}(x)=\{z\in X_w:w(\lambda,x,z)\le\mu\}$. The set $B^w_{\lambda,\mu}(x)$ is called an open ball about x relative to λ and μ, and the set $C^w_{\lambda,\mu}(x)$ is called a closed ball about x relative to λ and μ.*

Example 1. *If we equip* X_w *with the pseudometric* $d(x,y)$ *then* $w(\lambda,x,y)=d(x,y)$ *for* $\lambda>0$ *and* $x,y\in X_w$ *is modular pseudometric.*

Example 2. *Let* $X=\mathbb{R}$. *Define* $w:(0,\infty)\times\mathbb{R}\times\mathbb{R}\to[0,\infty]$ *by* $w(\lambda,x,y)=\begin{cases}\infty, & \text{if } x>y\\ 0, & \text{otherwise}\end{cases}$ *whenever* $\lambda>0$. *Then* w *is a modular metric on* $\mathbb{R}$.

We notice that $X_w=X_w(x_0)=\{x_0\}$, where $x_0\in\mathbb{R}$ and $d_w(x,y)=0$ for all $x,y\in X_w$. Moreover, for any $r>0$, we have

$$B_{d_W}(x_0,r)=\{y\in X_w:0=d_w(x_0,y)<r\}=\{x_0\}$$

and

$$B^w_{r,r}(x_0)=\{y\in X_w:w(r,x_0,y)<r\}=\{x_0\}=C^w_{r,r}(x_0).$$

Example 3. *Consider a pseudometric space* (X,d). *If we equip* X *with the modular pseudometric* $w(\lambda,x,y)=\dfrac{d(x,y)}{\lambda^p}$ $x,y\in X$ *and* λ, *where* p *is a strictly positive constant.*

Then it follows that $X_w=X$ *and* $d_w(x,y)=[d(x,y)]^{p+1}$. *Furthermore, for any* $\lambda>0$, *we have*

$$\begin{aligned}B_{d_W}(x,\lambda) &= \{y\in X_w:[d(x,y)]^{p+1}<\lambda\}\\ &= \{y\in X_w:\frac{[d(x,y)]}{\lambda^p}<\lambda\}\\ &= \{y\in X_w:w(\lambda,x,y)<\lambda\}=B^w_{r,r}(x).\end{aligned}$$

If w is a modular pseudometric on a nonempty set X, the a subset A of X_w is said to be $\tau(w)$-open (or w-open) if for any $x\in A$ and $\lambda>0$, there exists $\mu:=\mu(x,\lambda)>0$ such that $B^w_{\lambda,\mu}(x)\subset A$. Note that $B^w_{\lambda,\mu}(x)$ is not $\tau(w)$-open, in general.

Lemma 1. *Let* w *be a modular pseudometric on* X *and* $x\in X_w$. *Then, whenever* $\lambda>0$ *we have*

1. $B_{d_W}(x,\lambda)\subseteq B^w_{\lambda,\lambda}(x)$
2. $C_{d_W}(x,\lambda)\subseteq C^w_{\lambda,\lambda}(x)$ *where* $B_{d_W}(x,\lambda)$ *and* $C_{d_W}(x,\lambda)$ *are respectively open and closed balls with centre* x *and radius* $\lambda>0$ *with respect to the pseudometric* d_w.

The following important concept of continuity of a modular pseudometric space was introduced in.

Definition 2. *Let* w *be a modular metric on a set* X. *Given* $x,y\in X$,

thelimitfromtherightof w *ateachpoint* $\lambda>0$ *denotedby* $w_{+0}(\lambda,x,y)$ *is defined by* $w_{+0}(\lambda,x,y)=\lim\limits_{\mu\to\lambda^+}w(\mu,x,y)=\sup\{w(\mu,x,y):\mu>\lambda\}$.

1. *thelimitfromtheleftof* w *ateachpoint* $\lambda>0$ *denotedby* $w_{-0}(\lambda,x,y)$ *is defined by* $w_{-0}(\lambda,x,y)=\lim\limits_{\mu\to\lambda^-}w(\mu,x,y)=\inf\{w(\mu,x,y):0<\mu<\lambda\}$.
2. w *is said to be continuous from the right on* $(0,\infty)$ *if for any* $\lambda>0$ *we have* $w(\lambda,x,y)=w_{+0}(\lambda,x,y)$.
3. w *is said to be continuous from the left on* $(0,\infty)$ *if for any* $\lambda>0$ *we have* $w(\lambda,x,y)=w_{-0}(\lambda,x,y)$.
4. w *is said to be continuous on* $(0,\infty)$ *if* w *is continuous from the right and continuous from the left on* $(0,\infty)$.

Definition 3. *A sequence* $\{x_n\}$ *in a metric modular set* X_w *is said to be a* w*-Cauchy sequence if and only if for each* ò >0, $\lambda>0$, *there is* $n_0>0$ *such tha* $w_\lambda(x_{n+m},x_n)<$ ò *t for all* $n>n_0$, $m>0$.

If every w-Cauchy sequence is convergent in $\tau(w)$-topology, then X_w is called w-complete metric modular set.

Definition 4 (). *Let* w *be a modular metric on a set* X. *Then we say that a map* $T:X_w\to X_w$ *is a* w*-Lipschitz if there exists* $k>0$ *such that* $w(k\lambda,T(x),T(y))\le w(\lambda,x,y)$ *for all* $\lambda>0$ *and* $x,y\in X_w$. *If* $k=1$, *then the map* T *is called a* w*-nonexpansive map.*

Definition 5. *A bornology on a set* X *is a collection* \$\mathcalligra{B}\$ *of subsets of* X *which satisfies the following conditions:*

(a) \$\mathcalligra{B}\$ *forms a cover of* X, *i.e.* \$X=\bigcup \ mathcalligra{B}\,\,\$;

(b) for any \$B \in \mathcalligra{B}\$ *and* $A\subseteq B$, *then* \$A \in \ mathcalligra{B}\,\,\$;

(c) \$\mathcalligra{B}\$ *is stable under finite unions, i.e. if* \$X_1,X_2,\cdots,X_n \in \mathcalligra{B}\,\,\$, *then* \$\$\bigcup_ {i=1}^n X_i \in \mathcalligra{B}\,\,.\$\$

If we take a nonempty set X and a bornology \$\mathcalligra{B}\ hspace{0.1cm}\$ on X, then the pair \$(X,\mathcalligra{B} \ hspace{0.15cm})\$ is called a bornological universe. For every nonempty set X the family \$\mathcalligra{B}=\{B\subset X: B \ hspace{0.1cm} \text{is finite}\}\$ is the smallest bornology on X

Boundedness in modular metric spaces

In this section we discuss the concepts of boundedness of a modular set of a modular metric. Some of the results can be found in.

Definition 6. *Let* W *be a modular metric on* X. *A nonempty subset* A *of* X_w *is said to be* w*-bounded if there exists* $x\in X_w$ *such that* $A\subseteq B^w_{\lambda_1\lambda}(x)$ *for some* $\lambda>0$.

Definition 7. *Let* W *be a modular metric on* X. *A nonempty subset* A *of* X_w *is said to be* d_W- *bounded if there exists* $x\in X_w$ *such that* $A\subseteq B_{d_w}(x,\lambda)$ *for some* $\lambda>0$.

Definition 8. *Let* A *be a* $w-$*bounded subset of* X_w. *The diameter of* A, *denoted by* $\mathrm{diam}_w(A)$, *is defined by* $diam_w(A)=\sup\{w(\lambda,x,y):x,y\in A\}$ *for some* $\lambda>0$.

Lemma 2. *Let* w *be a modular pseudometric on X. If* A *is* $w-$ *bounded then* $diam_w(A)<\infty$.

Proof. Suppose that A is $w-$ bounded. Then for some $x\in X_w$, we have $A\subseteq B^w_{\lambda_1\lambda}(x)$ for some $\lambda>0$. If $z,y\in A$ then $w(\lambda,x,y)<\lambda$ and $w(\lambda,y,z)<\lambda$. It follows that

$$w(\lambda+\lambda,x,z)\le w(\lambda,x,y)+w(\lambda,y,z)<\lambda+\lambda.$$

For $\lambda'=2\lambda>0$, we have

$$sup\{w(\lambda',x,y):x,y\in A.\}<\lambda'.$$

Thus, $diam_w(A)<\infty$.

\$\hfill{\Box}\$

Lemma 3. *Let* w *be a modular metric on* X *and A be a subset of* X_w. *Then* $diam_w(A)$ $diam_w(A)\le diam_{qw}(A)$.

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Remark 2. *If w is continuous from the right on* $(0,\infty)$, *then for any* $x, y \in X_w$ *and* $\lambda > 0$ *we have that* $q_w(x, y) \leq \lambda$ *if and only if* $w(\lambda, x, y) \leq \lambda$.

Remark 3. *Let w be a modular metric on* X. *In the sequel, we denote by $\mathcalligra {B}_{w}(X_{w})$ the collection of all $w-$ bounded subsets in* X_w. *We observe that $\mathcalligra {B}_{w} (X_{w})$ satisfies the following conditins:*

whenever $x \in X_w$, *then* $\{x\}$ *$\mathcalligra {B}_{w}(X_{w})$;*

whenever $F \in \mathcalligra {B}_{w}(X_{w})$ and $G \subseteq F$, *then $G\in \mathcalligra {B}_{w}(X_{w})$;*

whenever $F, G\in \mathcalligra {B}_{w}(X_{w})$, then $F\cup G\in \mathcalligra {B}_{w}(X_{w})$.

Therefore, the collection $\mathcalligra {B}_{w}(X_{w})$ forms a bornology on X that we call the bornology of w*-bounded sets in* X_w.
We also denote the collection of all $d_W -$ bounded subsets of X_w by $\mathcalligra {B}_{d_{w}}(X_{w}).$ One would wonder indeed what the relationships between these collections.

Remark 4. *Let w be a modular metric on a set* X, *then from Lemma* 1, *we have the following: $$\mathcalligra {B}_{d_{w}}(X_{w}) \ subseteq \mathcalligra {B}_{w}(X_{w})$$*

Proposition 1. *Let w be a modular metric on* X. *If w is continuous from the right on* $(0,\infty)$, *then $$\mathcalligra {B}_{w}(X_{w})=\ mathcalligra {B}_{d_{w}}(X_{w}).$$*

Proof. Since we know that $\mathcalligra {B}_{d_{w}}(X_{w}) \ subseteq \mathcalligra {B}_{w}(X_{w})$ from Remark 4, we only prove the reverse inclusion, i.e, showing that $B^w_{\lambda,\lambda}(x) \subseteq B_{dw}(x, y)$

If $A \in \mathcalligra {B}_{w}(X_{w})$ then $A \subseteq B^w_{\lambda,\lambda}(x)$ for some $x \in X_w$ and $\lambda > 0$. If $y \in A$, then $w(\lambda, x, y) < \lambda$. By the right continuity of w on $(0,\infty)$, we have $d_w(x, y) < \lambda$ for $\lambda > 0$ and some $x \in X_w$. Thus, $A \subseteq B_{dw}(x,\lambda)$ and therefore $A \in \ mathcalligra {B}_{d_{w}}(X_{w})$. $\hfill{\Box}$

Definition 9 *Let w be a modular metric on* X. *We say that:*

I. *The collection $\mathcalligra{B}_{w}(X_{w})$ is* w *-compact if every descending chain of nonempty subsets of $\ mathcalligra{B}_{w}(X_{w})$ has a nonempty intersection.*
II. The collection $\mathcalligra{B}_{w}(X_{w})$ is normal if for any $A \in \mathcalligra {B}_{w}(X_{w})$ with A *having more than one point, there exists* $\lambda > 0, \mu > 0$ *such that* $\lambda < diam_w(A)$ *and* $\mu < diam_w(A)$ *and for* $x_0 \in A$ *with* $A \subseteq C^w_{\lambda,\lambda}(x_0)$.

Lemma 4. *Let w be a modular metric on* X. *Then*

I. If $\mathcalligra{B}_{w}(X_{w})$ is *w-compact, then $\ mathcalligra{B}_{d_{w}}(X_{w})$ is* $d_w -$ *compact.*
II. *If $\mathcalligra{B}_{w}(X_{w})$ is normal, then $\ mathcalligra{B}_{d_{w}}(X_{w})$ is normal.*

Proof.

I. This is a consequence of the definition of compactness above and the fact that $\mathcalligra {B}_{d_{w}}(X_{w}) \subseteq \ mathcalligra {B}_{w}(X_{w})$ from Remark 4.
II. Let $A \in \mathcalligra{B}_{w}(X_{w})$ and since $\ mathcalligra{B}_{w}(X_{w})$ is normal, $\lambda < diam_w(A)$ and $A \subseteq B^w_{\lambda,\lambda}(a)$ for some $\lambda > 0$ and for $a \in A$.

We only need to show that $A \subseteq C_{dw}(x, y)$. Let $x \in A$ and since $A \subseteq B^w_{\lambda,\lambda}(a)$, $w(\lambda, a, x) < \lambda$ for some $\lambda > 0$. Thus $d_w(a, x) < \lambda$ and $A \subseteq B_{dw}(x,\lambda)$, implying that $\mathcalligra{B}_{d_{w}}(X_ {w})$ is normal. $\hfill{\Box}$

Theorem 1. *Let w be a modular pseudometric on X. If X_w is q_w -bounded and $T : X_w \to X_w$ is a w-nonexpansive map, then T has atleast one fixed point whenever $\mathcalligra {B}_{w}(X_{w})$ is w-compact and normal.*

Proof. Suppose that X_w is d_w-bounded and since $\mathcalligra {B}_{w}(X_{w})$ is w-compact and normal by Lemma 4 indeed $\mathcalligra {B}_{d_{w}}(Σ {w})$ is d_w-compact and normal too. Let $T : X_w \to X_w$ be a w-nonexpansive map then $w(\lambda, T(x), T(y)) \leq w(\lambda, x, y)$ for all $x, y \in X_w$ and $\lambda > 0$. It follows from the corollary of with $k = 1$ that $d_w(T(x), T(y)) \leq d_w(x, y)$.

Thus $T : (X_w, d_w) \to (X_w, d_w)$ is a nonexpansive map too and since $\mathcalligra {B}_{q_{w}}(X_{w})$ is d_w-compact and normal by the map T has at least one fixed point. $\hfill{\Box}$

Totally Boundedness in modular metric spaces

The concept of total boundedness in metric spaces has been studied by main authors. In this section, we introduce and study this notion on modular sets of modular metrics.

Definition 10. *Let w be a modular metric on X. A subset B of X_w is said to be w-totally bounded if for each $\mu > 0$ and $\lambda > 0$ there exists a finite subset $\{x_1, x_2, x_3, \ldots x_k\}$ of B such that $B \subseteq \bigcup_{j=1}^{k} B^w_{\lambda,\mu}(x_j)$.*

Definition 11. *Let w be a modular metric on X. A subset B of X_w is said to be d_w-totally bounded if for each $\lambda > 0$ there exists a finite subset $\{x_1, x_2, x_3, \ldots x_k\}$ of B such that $B \subseteq \bigcup_{j=1}^{k} B_{d_W}(x_j, \lambda)$.*

Remark 5. *Let w be a modular metric on X. In the sequel, we denote by $\mathcalligra {TB}_{w}(X_{w})$ the collection of all w-totally bounded subsets in X_w. We observe that $\mathcalligra {TB}_{w} (X_{w})$ satisfies the following conditins:*

I. whenever $x \in X_w$, *then* $\{x\}$ *$\mathcalligra {TB}_{w}(X_{w})$;*
II. *whenever $F \in \mathcalligra {TB}_{w}(X_{w})$ and $G \subseteq F$, then $G\in \mathcalligra {TB}_{w}(X_{w})$;*
III. *whenever $F, G\in \mathcalligra {TB}_{w}(X_{w})$, then $F\ cup G\in \mathcalligra {TB}_{w}(X_{w})$.*

Therefore, the collection $\mathcalligra {TB}_{w}(X_{w})$ forms a bornology on X_w that we call the bornology of w*-totally bounded sets in X_w*

Given the modular metric d_w, we denote the collection of all d_w -totally bounded subsets of X_w by $\mathcalligra {TB}_{d_{w}} (X_{w}).$

The next example shows that for finite dimension spaces, total boundedness concide with boundedness.

Example 4. *If we equip X $\mathbb{R}$ with the moodular metric defined in Example* 2. *Then any subset B of $X_w = \{x_0\}$ is both in $\mathcalligra {B}_{w}(X_{w})$ and in $\mathcalligra {TB}_{w}(X_{w}).$*

Lemma 5. *Let w be a modular pseudometric on X. Given the modular set X_w we have $$\mathcalligra {TB}_{w}(X_{w}) \ subseteq \mathcalligra {B}_{w}(X_{w}).$$*

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Proof. Let $A \in \mathcalligra {TB}_{w}(X_{w})$ then for $\lambda>0$ and $\mu>0$ there exists a finite set $\{x_0, x_2, x_3, \ldots, x_n\} \subset A$ such that

$$A \subseteq \bigcup_{j=1}^{k} B^w_{\lambda,\mu}(x_j)$$

Now fix x_1 and put $\delta = \max\{w(\lambda, x_1, x_j) : j = 1,2,3,4,\ldots\} + \lambda$ and we have $A \subset B^w_{\mu,\delta}(x)$. Thus, $A \in \mathcalligra {B}_{w}(X_{w})$.

$\hfill{\Box}$

The following example shows that for infinite dimension spaces, total boundedness and boundedness are two different notions.

Example 5. *Let us equip the set of real numbers $X=\mathbb{R}$ with the modular metric $w(\lambda, x, y) = \dfrac{d(x,y)}{\lambda}$ for $\lambda>0$ and $x,y\in X$ for the discrete metric $d(x,y)=\begin{cases}0 & \text{if } x=y\\ 1 & \text{if } x\neq y\end{cases}$ It is shown in that $X_w=\mathbb{R}$. We now show that the set $\mathbb{N}\subset X_w$ is in $\mathcalligra {B}_{w}(X_{w})$ but not in $\mathcalligra {TB}_{w}(X_{w})$.*

Proof. ›r all $x,y\in\mathbb{N}$ we can find $\lambda>0$ such that $w(\lambda,x,y)<\lambda$. Thus $\mathbb{N}$ is w-bounded. But any finite set $\{x_1,x_2,x_3,\ldots x_n\}$ in $\mathbb{N}$ with the modular metric w for $\lambda>0$ and $\mu>0$, the set $\mathbb{N}\subset\mathbb{R}$ cannot be covered by $B^w_{\mu,\lambda}(x_i)$ for $1\le i\le n$. Hence, $\mathbb{N}$ is not w-totally bounded.

$\hfill{\Box}$

Let (X,w) be a modular metric space. Then, for each set A in X, we write $\mathrm{cl}_{\tau(w)}(A)$ for its closure with respect to $\tau(w)$ and $\mathrm{int}_{\tau(w)}(A)$ for its interior with respect to $\tau(w)$.

The next proposition is a consequence of Proposition 1, Remark 2.

Proposition 2. *Let w be a modular pseudometric on X which is continuous from the right on $(0,\infty)$. Then $$\mathcalligra {TB}_{w}(X_{w})=\mathcalligra {TB}_{d_{w}}(X_{w}).$$*

Definition 12. *Let w be modular metric on X. A subset B of X_w is called w-compact if each of its w-open covers has a finite subcover. Indeed, B is w-compact if for every collection M of w-open subsets of B with $B=\bigcup_{U\in M}U$ there is a finite subset F of M such that $B=\bigcup_{U\in F}U$*

A modular metric space is said to be boundedly compact if its every $\tau(w)$-closed bounded subset is compact.

Theorem 2. *Let w be modular metric on X. Then, every w-compact subset A of X_w is both w-totally bounded and w-bounded.*

Proof. Suppose that $\lambda>0$, $\mu>0$. Consider an w-open cover $\{B^w_{\lambda,\mu}(x) : x\in A\}$ of A. Since A is w-compact, there exists $x_1,x_2,\cdots,x_n \in A$ such that $A\subseteq B^w_{\lambda,\mu}(x_j)$. By Definition 10 and Lemma 5, A is both w-totally bounded and w-bounded.$\hfill{\Box}$

Theorem 3. *Let w be a modular metric on X_w. A set $B\subseteq X_w$ is w-compact if and only if B is both w-complete and w-totally bounded.*

Proof. Let B be w-compact and $\{x_n\}_{n\in\mathbb{N}}$ be a w-Cauchy sequence in B by the compactness of B, the w-Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ has a subsequence that converges to x. Thus B is w-complete. Since $\{x_n\}_{n\in\mathbb{N}}$ has a w-subsequence which converges that subsequence is actually w-Cauchy. Thus, B is indeed a totally bounded set.

On the other hand, if B is both w-complete and w-totally bounded. Then a sequence $\{x_n\}$ in B has a w-Cauchy subsequence that converges in B. Thus, w-compact. $\hfill{\Box}$

Corollary 1. *Let w be a modular metric on X. Then a subset A of X_w is w-totally bounded if and only it is w-compact.*

Proposition 3. *Let W be a modular metric on X. A w-totally bounded subset B of X_w is W-separable.*

Proof. Suppose B is W-totally bounded subset of B, for any positive interge n, we can find a finite set $A_n\subseteq B$ such thatfor all $x\in B$, $\lambda>0$ and $\mu>0$ we have $B^w_{\lambda,\mu}(A_n)<\dfrac{1}{n}$. Now let $B=\cup_{n\in\mathbb{N}}A_n$. The set B is either finite or infinitely countable thus countable. To show the density of B, let us pick $x\in B$, then we have $w(\lambda,x,B)<\dfrac{1}{n}$ implying that $w(\lambda,x,B)=0$ and $x\in\mathrm{cl}(B)$. This proves that x is a w-limit point of B and hence B is a dense subset of B. Thus, B w-separable. $\hfill{\Box}$

Theorem 4. *Every w-separable modular metric space is embeddable as subspace of the Hilbert cube $H=[0,1]^{\mathbb{N}}$.*

Theorem 5 (Tychonoff's Theorem). *The topological product of a family of compact spaces is compact*

Theorem 6. *(Compare.) Let w be a modular metric on X and let $x_0\in X_w$. The following conditions are equivalent:*

1. *There exists an equivalent modular metric m such that $\mathcalligra{B}_w(X_w)=\mathcalligra{T}\tiny{\mathcalligra{B}}_{m}(X_m)$*
2. *The modular set X_w is w-separable.*
3. *There is an embedding ϕ of X_w into some modular set Y_w such that the family $\{\mathrm{cl}_{Y_W}(C^w_{\lambda,n}(x_0) : n\in\mathbb{N}\}$ is w-compact and nonempty.*
4. *There exists an equivalent modular metric m with $\mathcalligra{B}_w(X_w)=\mathcalligra{T}\tiny{\mathcalligra{B}}_{m}(X_m)=\tiny{\mathcalligra{B}}_{m}(X_m).$*

Proof.: $1\Rightarrow2$ If there exists an equivalent modular metic m such that $\mathcalligra{B}_w(X_w)=\mathcalligra{T}\tiny{\mathcalligra{B}}_{{m}}(X_m)$, then $X_w=\bigcup_{i=1}^n B_i$ where B_i are m-totally bounded subsets. This means that X_w is a countable union of m-totally bounded sets, thus its m-totally bounded and by Proposition 3, the modular set X_w is w-separable.

$2\Rightarrow3$: Let w be a bounded modular metric, then by Theorem 4, we can find an embedding ϕ of X_w into $[0,1]^{\mathbb{N}}$. Let Y be the closure of $\phi(X_w)$ in the product and choose $n\in\mathbb{N}$ with $X_w=C^w_{\lambda,n}(x_0)$. The set $Y_w=\mathrm{cl}_{Y_W}(C^w_{\lambda,n}(x_0))$ is w-compact and nonempty, this is so since $[0,1]^{\mathbb{N}}$ is w-compact with respect to product topology so its subset Y_w is w-compact and non-empty.

$3\Rightarrow4$: If if m is a modular metric equivalent to w then $\mathcalligra{B}_w(X_w)=\tiny{\mathcalligra{B}}_{m}(X_m)$

by . To prove that $\mathcalligra{B}_w(X_w)=\mathcalligra{T}\tiny{\mathcalligra{B}}_{m}(X_w),$ let $B \in \mathcalligra{T}\tiny{\mathcalligra{B}}_m(X_m)$ and C_m be a completion of Y_m. Since C_m, is a complete modular set, the closure of B with respect to completion is w-compact. Let us choose $n \in \mathbb{N}$ with $\mathrm{cl}_{(Y_m)}(B) \subseteq \mathrm{cl}_{Y_W}\left(C^w_{\lambda,n}(x_0)\right)$. But this means that

$$B \subseteq \mathrm{cl}_{(Y_m)}(B) \subseteq \mathrm{cl}_{Y_W}\left(C^w_{\lambda,n}(x_0)\right) = C^w_{\lambda,n}(x_0).$$

Thus $B\in \mathcalligra{B}_w (X_w)$ and it follows that $\mathcalligra{T}\tiny{\mathcalligra{B}}_{m}(X_m)\subseteq \mathcalligra{B}_w(X_w).$ For the reverse inclusion. If $B \in \mathcalligra{B}_w(X_w)$, we can choose $n \in \mathbb{N}$ with $B \subseteq C^w_{\lambda,n}(x_0)$. Thus, $B \subseteq \mathrm{cl}_{(Cm)}\left(C^w_{\lambda,n}(x_0)\right)$ is m-compact and C_m-totally bounded. Therefore, $B \in \mathcalligra{T}\tiny{\mathcalligra{B}}_{m}(X_m).$

The equivalence $4 \Rightarrow 1$ follows from. $\hfill{\Box}$

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Conflicts of interest

The author declares there is no conflict of interest.

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