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Modified exponential convex functions and inequalities

Abstract

In this article, we introduce the concept of modified exponentially convex function deduce several new integral inequalities for the class of modified exponentially convex function functions. The idea and techniques used in the paper may stimulate further research in this fascinating field.

Keywords: convex function, exponentially convex function, Hermite-Hadamard inequality, holder's inequality and power-mean inequality

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Introduction

Several branches of mathematical and engineering science have been developed by using the crucial and significant concepts of convex analysis. Inequalities presently very active and is a fascinating field of research. In recent years, a wide class of integral inequalities is being derived via different concepts of convexity.[1–19] The extensive development of research on big data analysis and deep learning has recently increased the interest in information theory involving exponentially convex functions. The significance of exponentially convex function is used to manipulate for statistical learning, sequential prediction and stochastic optimization,[1, 2, 14] and the references therein.

The class of exponentially convex functions was introduces by Antczak,[2] Dragomir et al.[6] and Noor et al.[12] It's natural to unify these different concepts. Motivated by these facts, Awan et al.[3] introduced and investigated another class of convex functions, which is called exponential convex function and can be viewed as significantly different from the class introduced by.[2, 6, 11, 17] We would like to emphasize that a class of convex functions which unifies and naturally distinct from other classes of convex functions is the class of *h*-convex functions, introduced by Varosanec.[19] In the present paper, we introduce a new class of exponential convex functions with respect to an arbitrary nonnegative function *h*, which is called modified exponentially convex function. We have obtained several new Hermite-Hadamard inequality and related inequalities for modified exponentially convex function. The techniques and the ideas of this paper may stimulate further research in this dynamic field.

Preliminaries

We now discuss the new classes of convex functions involving an arbitrary function *h*. Let $\Omega\subset\mathbb{R}$ be a convex set in the finite dimensional Euclidean space $\mathbb{R}^n$. From now onwards we take $\Omega=[a,b]$, unless otherwise specified.

First of all, we recall the following well known concepts and results.

Definition 2.1: *A set $\Omega\subset\mathbb{R}^n$ is said to be convex, if* $ta+(1-t)b\in\Omega,\qquad \forall a,b\in\Omega, t\in[0,1]$

Definition 2.2: *A function f on the convex set Ω is said to be a convex function, if and only if,*

$$f(ta+(1-t)b)\le tf(a)+(1-t)f(b),\qquad \forall a,b\in\Omega, t\in[0,1]$$

We now recall the concept of h-convex function.

Definition 2.3: *Let $h:(0,1)\subseteq J\to\mathbb{R}$ be a nonnegative function. We say that $f:\Omega\to\mathbb{R}$ is said to be h-convex function, if f is nonnegative, then*

$$f(ta+(1-t)b)\le h(t)f(a)+(1-h(t))f(b),\qquad \forall a,b\in\Omega, t\in[0,1].$$

We now consider class of exponentially convex function, which are mainly due to Antczak,[2] Dragomir et al[6] and Noor et al.[12]

Definition 2.4: *Let $f:\Omega=[a,b]\subset\mathbb{R}\to\mathbb{R}$ is exponentially convex function, if 'f' is positive, then*[2, 6, 12]

$$e^{f(ta+(1-t)b)}\le te^{f(a)}+(1-t)e^{f(b)},\qquad a,b\in\Omega, t\in[0,1]. \tag{2.1}$$

We now consider a new class of exponentially convex function with respect to arbitrary nonnegative functions h is called modified exponentially convex function function.

Definition 2.5: *Let $h:(0,1)\subseteq J\to\mathbb{R}$ be a nonnegative function. A function $f:\Omega\to\mathbb{R}$ is said to be a modified exponentially convex function, if and only*

$$e^{f(ta+(1-t)b)}\le h(t)e^{f(a)}+(1-h(t))e^{f(b)},\qquad \forall a,b\in\Omega, t\in[0,1]. \tag{2.2}$$

For $t=\frac{1}{2}$, we have Jensen type modified exponentially convex function function.

$$e^{f\left(\frac{a+b}{2}\right)}\le e^{f(b)}+h\left(\frac{1}{2}\right)\left[e^{f(a)}-e^{f(b)}\right],\qquad \forall a,b\in\Omega. \tag{2.3}$$

We now discuss some new special cases of Definition (2.5).

1. If h(t)=t in (2.2), then Definition 2.5 reduces to the Definition 2.4.

If $h(t)=t^{s}$ in Definition 2.2, then we have a class of exponentially *s*-convex functions.

Definition 2.6: *Let* $s\in[0,1]$ *be a real number. We say that* $f:\Omega\to\mathbb{R}$ R *is an exponentially s-convex function, if*

$$e^{f(ta+(1-t)b)}\le t^{s}e^{f(a)}+(1-t^{s})e^{f(b)},\qquad \forall a,b\in\Omega, t\in[0,1].$$

If $h(t)=t^{-s}$ in Definition 2.2, then we have a class of exponentially *s*-convex functions.

Definition 2.7. *Let* $s\in[0,1]$ *be a real number. We say that* $f:\Omega\to\mathbb{R}$ *is an exponentially Godunova-Levin type convex function, if*

$$e^{f(ta+(1-t)b)}\le t^{-s}e^{f(a)}+(1-t^{-s})e^{f(b)},\qquad \forall a,b\in\Omega, t\in[0,1].$$

For appropriate and suitable choice of functions h, one can obtain several new and known classes of convex functions as special cases. This shows that the concept of modified exponentially convex function is quite a general and unifying one.

Proposition 2.8: *Let* $f_n:\Omega\to\mathbb{R}$ *is sequence of functions which point wise converge to* $f:\Omega\to\mathbb{R}$ *and* $h_n:(0,1)\subseteq J\to R$ *be nonnegative sequence of functions which point wise converge to* $h:(0,\,1)\subseteq J\to\mathbb{R}$, *so there is* $a\,\phi>0$ *such that fn is an modified exponentially convex function function then f is an modified exponentially convex function for* $n\ge\phi$.

Proof: Given that *fn* is an modified exponentially convex function function, then

$$e^{f(ta+(1-t)b)}=\lim_{n\to+\infty}e^{f_n(ta+(1-t)b)}$$

$$\le\lim_{n\to+\infty}\left(h_n(t)e^{f(a)}+(1-h_n(t)e^{f(b)})\right.$$

$$\le h(t)e^{f(a)}+(1-h(t))e^{f(b)},$$

which shows *f* be an modified exponentially convex function function.

Main results

We now derive Hermite-Hadamard type inequalities for modified exponentially

convex function function.

Theorem 3.1: *Let* $h:(0,1)\subseteq J\to R$ *be a nonnegative function and* $f:\Omega=[a,\,b]\subset R\to R$ *be an modified exponentially convex function function. If* $f\in L[a,b]$, *then*

$$e^{f\left(\frac{a+b}{2}\right)}\le\frac{1}{b-a}\int_a^b e^{f(x)}dx\le\frac{e^{f(a)}+e^{f(b)}}{2} \tag{3.1}$$

Proof: Let *f* be an modified exponentially convex function function. Then from (2.3), we have

$$e^{f\left(\frac{x+y}{2}\right)}\le h\left(\frac{1}{2}\right)e^{f(x)}+\left(1-h\left(\frac{1}{2}\right)\right)e^{f(y)},\qquad \forall x,y\in\Omega$$

Integrating the above inequality with respect to *t* on [0, 1], and using the change of variable technique, $x=(1-\ t)a+tb$ and $y=ta+(1-\ t)b$, we have

$$f\left(\frac{a+b}{2}\right)\le h\left(\frac{1}{2}\right)\int_0^1 e^{f((1-t)a+tb)}dt+\left(1-h\left(\frac{1}{2}\right)\right)\int_0^1 e^{f(ta+(1-t)b)}dt$$

$$=\frac{h\left(\frac{1}{2}\right)}{b-a}\int_a^b e^{f(x)}dx+\frac{1-h\left(\frac{1}{2}\right)}{b-a}\int_a^b e^{f(x)}dx$$

$$=\frac{1}{b-a}\int_a^b e^{f(x)}dx$$

$$=\int_0^1 e^{f(ta+(1-t)b)}dt$$

$$\le\int_0^1\left[h(t)e^{f(a)}+(1-h(t))e^{f(b)}\right]dt$$

$$=e^{f(b)}+\left[e^{f(a)}-e^{f(b)}\right]\int_0^1 h(t)dt. \tag{3.2}$$

Again, we have $\int_0^1 e^{f((1-t)a+tb)}dt\le e^{f(a)}+\left[e^{f(b)}-e^{f(a)}\right]\int_0^1 h(t)dt.$ (3.3)

Summing up (3.2) and (3.3), we get

$$\int_0^1 e^{f(ta+(1-t)b)}dt+\int_0^1 e^{f((1-t)a+tb)}dt=\frac{1}{b-a}\int_a^b e^{f(x)}dx$$

$$\le\frac{e^{f(a)}+e^{f(b)}}{2}, \tag{3.4}$$

the required result.

Theorem 3.2: *Let* $h_1,h_2:(0,1)\subseteq J\to\mathbb{R}$ *be two nonnegative functions and* $f:\Omega=[a,b]\subseteq\mathbb{R}\to\mathbb{R}$ *be an modified exponentially convex function and* $g:\Omega=[a,b]\subset\mathbb{R}\to\mathbb{R}$ *If* $f\,g\in L[a,b]$, *then*

$$\frac{1}{b-a}\int_a^b e^{f(x)}e^{g(x)}dx$$

$$\le e^{f(b)}e^{g(b)}+Q(a,b)\int_0^1 h_1(t)h_2(t)dt+P(a,b)\int_0^1 h_1(t)dt+S(a,b)\int_0^1 h_2(t)dt,$$

where

$$P(a,b)=\left[e^{f(a)}e^{g(a)}-e^{f(a)}e^{g(b)}-e^{f(b)}e^{g(a)}+e^{f(b)}e^{g(b)}\right],$$

$$Q(a,b)=\left[e^{f(a)}e^{g(b)}-e^{f(b)}e^{g(b)}\right],$$

$$R(a,b)=\left[e^{f(b)}e^{g(a)}-e^{f(b)}e^{g(b)}\right].$$

Proof: Since f and g are modified exponentially convex function functions, then we have

$$e^{f(at+(1-t)b)}\le h_1(t)e^{f(a)}+(1-h_1(t))e^{f(b)},\qquad \forall a,b\in\Omega, t\in[0,1],$$

$$e^{g(at+(1-t)b)}\le h_2(t)e^{g(a)}+(1-h_2(t))e^{g(b)},\qquad \forall a,b\in\Omega, t\in[0,1].$$

Multiplying above inequalities, we obtain

$$e^{f(at+(1-t)b)}e^{g(at+(1-t)b)}$$

$$\le e^{f(b)}e^{g(b)}+\left[e^{f(a)}e^{g(a)}-e^{f(a)}e^{g(b)}-e^{f(b)}e^{g(a)}+e^{f(b)}e^{g(b)}\right]h_1(t)h_2(t)$$

$$+\left[e^{f(a)}e^{g(b)}-e^{f(b)}e^{g(b)}\right]h_1(t)+\left[e^{f(b)}e^{g(a)}-e^{f(b)}e^{g(b)}\right]h_2(t).$$

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Integrating above inequality over [0, 1] and by using change of variable, we have $\frac{1}{b-a}\int_a^b e^{f(x)}e^{g(x)}dx$

$$\leq e^{f(b)}e^{g(b)} + P(a,b)\int_0^1 h_1(t)h_2(t)dt + Q(a,b)\int_0^1 h_1(t)dt + R(a,b)\int_0^1 h_2(t)dt,$$

the required result.

Corollary 3.1: *If $f = g$ and $h_1(t) = h_2(t) = h(t)$ in Theorem 3.2, then we have a new result* $\frac{1}{b-a}\int_a^b e^{2f(x)}dx$

$$\leq e^{2f(b)} + P(a,b)\int_0^1 h^2(t)dt + [Q(a,b) + R(a,b)]\int_0^1 h(t)dt,$$

Corollary3.2: *If we take $h_1(t) = h_2(t) = t$ in Theorem 3.2, then we have a new result* $\frac{1}{b-a}\int_a^b e^{f(x)}e^{g(x)}dx \leq e^{f(b)}e^{g(b)} + \frac{1}{3}P(a,b) + \frac{Q(a,b)+R(a,b)}{2}$.

Corollary 3.3: *If we take $h_1(t) = h_2(t) = t^8$ in Theorem 3.2, then we have a new result*

$$\frac{1}{b-a}\int_a^b e^{f(x)}e^{g(x)}dx \leq e^{f(b)}e^{g(b)} + \frac{1}{2s+1}P(a,b) + \frac{Q(a,b)+R(a,b)}{s+1}.$$

Theorem 3.3: *Let $h_1, h_2 : (0,1) \subseteq J \to \mathbb{R}$ be two nonnegative functions and $f : \Omega = [a,b] \subset \mathbb{R} \to \mathbb{R}$ be an modified exponentially convex function and $g : \Omega = [a,b] \subset \mathbb{R} \to \mathbb{R}$. If $f\ g \in L[a,b]$, then*

$$e^{f\left(\frac{a+b}{2}\right)}e^{g\left(\frac{a+b}{2}\right)} - \frac{1}{b-a}\int_a^b e^{f(x)}e^{g(x)}dx$$

$$\leq \frac{h_1\left(\frac{1}{2}\right) + h_2\left(\frac{1}{2}\right) - 2h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}{b-a}\int_a^b e^{f(x)[g(a+b-x)-g(x)]}dx,$$

Proof: Take $\frac{a+b}{2} = \frac{ta+(1-t)b}{2} + \frac{(1-t)a+tb}{2}$, we have

$$e^{f\left(\frac{a+b}{2}\right)}e^{g\left(\frac{a+b}{2}\right)}$$

$$= e^{f\left(\frac{ta+(1-t)b}{2}+\frac{(1-t)a+tb}{2}\right)}e^{g\left(\frac{ta+(1-t)b}{2}+\frac{(1-t)a+tb}{2}\right)}$$

$$\leq \left[h_1\left(\frac{1}{2}\right)e^{f(ta+(1-t)b)} + \left(1-h_1\left(\frac{1}{2}\right)\right)e^{f((1-t)a+tb)}\right]$$

$$\times\left[h_2\left(\frac{1}{2}\right)e^{g(ta+(1-t)b)} + \left(1-h_2\left(\frac{1}{2}\right)\right)e^{g((1-t)a+tb)}\right]$$

$$= h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\left[e^{f(ta+(1-t)b)}e^{g(ta+(1-t)b)} - e^{f(ta+(1-t)b)}e^{g((1-t)a+tb)} - e^{g(ta+(1-t)b)}e^{f((1-t)a+tb)}\right]$$

$$+h_1\left(\frac{1}{2}\right)\left[e^{f(ta+(1-t)b)}e^{g((1-t)a+tb)} - e^{f((1-t)a+tb)}e^{g((1-t)a+tb)}\right]$$

$$+h_2\left(\frac{1}{2}\right)\left[e^{g(ta+(1-t)b)}e^{f((1-t)a+tb)} - e^{f((1-t)a+tb)}e^{g((1-t)a+tb)}\right] + e^{f((1-t)a+tb)}e^{g((1-t)a+tb)}$$

Integrating above inequality over [0, 1], we have

$$\leq \frac{h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}{b-a}\int_a^b e^{2f(x)[g(x)-g(a+b-x)]}dx + \frac{h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}{b-a}\int_a^b e^{f(x)[g(a+b-x)-f(x)g(x)]}dx$$

$+\frac{1}{b-a}\int_a^b e^{f(x)g(x)}dx$, which give $e^{f\left(\frac{a+b}{2}\right)}e^{g\left(\frac{a+b}{2}\right)} - \frac{1}{b-a}\int_a^b e^{f(x)}e^{g(x)}dx$

$$\leq \frac{h_1\left(\frac{1}{2}\right) + h_2\left(\frac{1}{2}\right) - 2h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}{b-a}\int_a^b e^{f(x)[g(a+b-x)-g(x)]}dx,$$

which is the required result.

Lemma 3.4: *Suppose that $f : \Omega \to R$ is a differentiable exponential h-convex function on the interior Ω^o of Ω. If $f' \in L[a,b]$ is h-convex function, then*

$$e^{f\left(\frac{a+b}{2}\right)} - \frac{1}{b-a}\int_a^b e^{f(x)}dx =$$

$$\frac{b-a}{4}\times\left[\int_0^1(1-t)e^{f\left(\frac{1+t}{2}a+\frac{1+t}{2}b\right)}f'\left(\frac{1+t}{2}a+\frac{1+t}{2}b\right)dt + \int_0^1(1-t)e^{f\left(\frac{1+t}{2}b+\frac{1+t}{2}a\right)}f'\left(\frac{1+t}{2}b+\frac{1+t}{2}a\right)dt\right]. \tag{3.5}$$

Theorem 3.5: *Let $f : \Omega \to R$ be differentiable exponentially h-convex function on the interior Ω^o of Ω. If $f' \in L[a,b]$ and $|f'|q$ is h-convex function, then we have*

$$\left|e^{f\left(\frac{a+b}{2}\right)} - \frac{1}{b-a}\int_a^b e^{f(x)}dx\right| \leq \frac{b-a}{4}$$

$$\times\left\{\left|e^{f\left(\frac{a+b}{2}\right)}f'\left(\frac{a+b}{2}\right)\right| + \left\{\left[\left|e^{f(a)}\right| + \left|e^{f(b)}\right|\right]\left|f'\left(\frac{a+b}{2}\right)\right|_{-}4\left|e^{f\left(\frac{a+b}{2}\right)}f'\left(\frac{a+b}{2}\right)\right|\right.\right.$$

$$\left.+\left[\left|f'(a)\right| + \left|f'(b)\right|\right]\left|e^{f\left(\frac{a+b}{2}\right)}\right|\right\}\xi_1(h,t) + \left\{\left(\left|f'(b)\right|_{-}\left|f'\left(\frac{a+b}{2}\right)\right|\right)\left(\left|e^{f(b)}\right| - e^{f\left(\frac{a+b}{2}\right)}\right)\left(\left|f'(a)\right| - \left|f'\left(\frac{a+b}{2}\right)\right|\right)\left(\left|e^{f(a)}\right| - e^{f\left(\frac{a+b}{2}\right)}\right)\right\}\xi_2(h,t)\Bigg\},$$

Where

$$\xi_1(h,t) = \int_0^1(1-t)h(t)dt, \tag{3.6}$$

$$\xi_2(h,t) = \int_0^1(1-t)h^2(t)dt. \tag{3.7}$$

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Proof: Using lemma 3.4 and f is an exponentially h-convex function, we have

$$\left|e^{f\left(\frac{1+t}{2}a+\frac{1+t}{2}b\right)}f'\left(\frac{1+t}{2}a+\frac{1+t}{2}b\right)\right|$$

$$\pounds\left[h(t)\left|e^{f(a)}\right|+\left(1-h(t)\left|e^{f\left(\frac{a+b}{2}\right)}\right|\right)\right]\left[h(t)\left|f'(a)\right|+\left(1-h(t)\left|f'\left(\frac{a+b}{2}\right)\right|\right)\right]$$

$$=\left[\left|e^{f\left(\frac{a+b}{2}\right)}\right|+h(t)\left(\left|e^{f(a)}\right|-\left|e^{f\left(\frac{a+b}{2}\right)}\right|\right)\right]\left[\left|f'\left(\frac{a+b}{2}\right)\right|+h(t)\left(\left|f'(a)\right|-\left|f'\left(\frac{a+b}{2}\right)\right|\right)\right]$$

$$\pounds\left\{\left|e^{f\left(\frac{a+b}{2}\right)}f'\left(\frac{a+b}{2}\right)\right|+h(t)\left(\left|e^{f(a)}f'\left(\frac{a+b}{2}\right)\right|-2\left|e^{f\left(\frac{a+b}{2}\right)}f'\left(\frac{a+b}{2}\right)\right|+\left|f'(a)e^{f\left(\frac{a+b}{2}\right)}\right|\right)+h^2(t)\left(\left|f'(a)\right|-\left|f'\left(\frac{a+b}{2}\right)\right|\right)\left(\left|e^{f(a)}\right|-e^{f\left(\frac{a+b}{2}\right)}\right)\right\}$$

Similarly, one can have

$$\left|e^{f\left(\frac{1+t}{2}b+\frac{1+t}{2}a\right)}f'\left(\frac{1+t}{2}b+\frac{1+t}{2}a\right)\right|$$

$$\leq\left\{\left|e^{f\left(\frac{a+b}{2}\right)}f'\left(\frac{a+b}{2}\right)\right|+h(t)\left(\left|e^{f(b)}f'\left(\frac{a+b}{2}\right)\right|-2\left|e^{f\left(\frac{a+b}{2}\right)}f'\left(\frac{a+b}{2}\right)\right|+\left|f'(b)\right|e^{f\left(\frac{a+b}{2}\right)}\right)+h^2(t)\left(\left|f'(b)\right|-\left|f'\left(\frac{a+b}{2}\right)\right|\right)\left(\left|e^{f(b)}\right|-e^{f\left(\frac{a+b}{2}\right)}\right)\right\}$$

Now

$$\left|e^{f\left(\frac{a+b}{2}\right)}-\frac{1}{b-a}\int_a^b e^{f(x)}dx\right|\leq\frac{b-a}{4}\int_0^1(1-t)\left\{\left|e^{f\left(\frac{1+t}{2}a+\frac{1+t}{2}b\right)}f'\left(\frac{1+t}{2}a+\frac{1+t}{2}b\right)\right|+\left|e^{f\left(\frac{1+t}{2}b+\frac{1+t}{2}a\right)}f'\left(\frac{1+t}{2}b+\frac{1+t}{2}a\right)\right|\right\}dt$$

$$+\left[\left|f'(a)\right|+f'(b)\right]\left|e^{f\left(\frac{a+b}{2}\right)}f'\left(\frac{a+b}{2}\right)\right|\Bigg\}\int_0^1(1-t)h(t)dt+\left(\left|f'(b)\right|-\left|f'\left(\frac{a+b}{2}\right)\right|\right)\left(\left|e^{f(b)}\right|-\left|e^{f\left(\frac{a+b}{2}\right)}\right|\right)$$

$$\left(\left|f'(a)\right|-f'\left(\frac{a+b}{2}\right)\right)\left(\left|e^{f(a)}\right|-\left|e^{f\left(\frac{a+b}{2}\right)}\right|\right)\int_0^1(1-t)h^2(t)dt\Bigg\}$$

$$=\frac{b-a}{4}\times\left\{\left|e^{f\left(\frac{a+b}{2}\right)}f'\left(\frac{a+b}{2}\right)\right|+\left\{\left[\left|e^{f(a)}\right|+\left|e^{f(b)}\right|\right]\left|f'\left(\frac{a+b}{2}\right)\right|-4\left|e^{f\left(\frac{a+b}{2}\right)}f'\left(\frac{a+b}{2}\right)\right|\right.\right.$$

$$\left.+\left[\left|f'(a)\right|+f'(b)\right]\left|e^{f\left(\frac{a+b}{2}\right)}f'\left(\frac{a+b}{2}\right)\right|\right\}\xi_1(h,t)+\left(\left|f'(b)\right|-\left|f'\left(\frac{a+b}{2}\right)\right|\right)\left(\left|e^{f(b)}\right|-e^{f\left(\frac{a+b}{2}\right)}\right)$$

$$\left.\left(\left|f'(a)\right|-f'\left(\frac{a+b}{2}\right)\right)\left(\left|e^{f(a)}\right|-\left|e^{f\left(\frac{a+b}{2}\right)}\right|\right)\xi_2(h,t)\right\},$$

Now

$$\left|e^{f\left(\frac{a+b}{2}\right)}-\frac{1}{b-a}\int_a^b e^{f(x)}dx\right|$$

$$\leq\frac{b-a}{4}\times\int_0^1(1-t)\left\{\left|e^{f\left(\frac{1+t}{2}a+\frac{1+t}{2}b\right)}f'\left(\frac{1+t}{2}b+\frac{1+t}{2}a\right)\right|+\left|e^{f\left(\frac{1+t}{2}b+\frac{1+t}{2}a\right)}f'\left(\frac{1+t}{2}b+\frac{1+t}{2}a\right)\right|\right\}dt$$

the required result.

Theorem 3.6: *Let* $f:\Omega\to R$ *be differentiable exponentially h-convex function on the interior* Ω^o *of* Ω. *If* $f'\in L[a,b]$ *and* $|f'|q$ *is h-convex function on* Ω *for* $p,q>1, p^{-1}+q^{-1}=1$, *then we have*

$$\left|e^{f\left(\frac{a+b}{2}\right)}-\frac{1}{b-a}\int_a^b e^{f(x)}dx\right|$$

$$\leq\frac{b-a}{4}\left(\frac{1}{2}\right)^{1-\frac{p}{q}}\left[\left\{\left(\left|e^{f\left(\frac{a+b}{2}\right)}f\left(\frac{a+b}{2}\right)\right|^q+\left[\left|f(a)e^{f\left(\frac{a+b}{2}\right)}\right|^q-2\left|e^{f\left(\frac{a+b}{2}\right)}f\left(\frac{a+b}{2}\right)\right|^q\right.\right.\right.\right.$$

$$+\left\{\left(\left|e^{f\left(\frac{a+b}{2}\right)}f\left(\frac{a+b}{2}\right)\right|^q+\left[\left|f(b)e^{f\left(\frac{a+b}{2}\right)}\right|^q-2\left|e^{f\left(\frac{a+b}{2}\right)}f\left(\frac{a+b}{2}\right)\right|^q\right.\right.\right.$$

$$+\left|f\left(\frac{a+b}{2}e^{f(b)}\right)\right|^q]\xi_1(h,t))+[\left|e^{f(b)}\right|^q-\left|e^{f\left(\frac{a+b}{2}\right)}\right|^q][\left|f(b)\right|^q-\left|f\left(\frac{a+b}{2}\right)\right|^q]\xi_2(h,t)\}^{\frac{1}{q}}],$$

where ξ_1 and ξ_2 are given in (3.6) and (3.7).

Proof: Using lemma 3.4 and the Holder's inequality, we have

$$\left|e^{f\left(\frac{a+b}{2}\right)}-\frac{1}{b-a}\int_a^b e^{f(x)}dx\right|$$

$$\leq\frac{b-a}{4}\times\left[\left(\int_0^1(1-t)dt\right)^{1-\frac{1}{q}}\left\{\int_0^1(1-t)\left|e^{f\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)}f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right.$$

$$+\int_0^1 (1-t)\left|e^{f\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)} f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|^q dt)^{\frac1q}\Big\}\Big]$$

$$\le \frac{b-a}{4}\left(\frac12\right)^{1-\frac1q}\Big\{(\int_0^1(1-t)\Big[\Big|e^{f\left(\frac{a+b}{2}\right)}\Big|^q + h(t)\Big[\Big|e^{f(a)}\Big|^q - \Big|e^{f\left(\frac{a+b}{2}\right)}\Big|^q\Big]\Big]$$

$$\Big[\Big|f(\frac{a+b}{2})\Big|^q + h(t)\Big[|f(a)|^q - \Big|f(\frac{a+b}{2})\Big|^q\Big]\Big]dt)^{\frac1q} + (\int_0^1(1-t)\Big[\Big|e^{f(\frac{a+b}{2})}\Big|^q + h(t)\Big[\Big|e^{f(b)}\Big|^q - \Big|e^{f(\frac{a+b}{2})}\Big|^q\Big]\Big]$$

the required result.

Theorem 3.7: *Let* $f:\Omega\to R$ *be differentiable modified exponentially convex function on the interior* Ω^o *of* Ω. *If* $f'\in L[a,b]$ *and* $|f'|q$ *is h-convex function on Ω for* $p,q>1, p^{-1}+q^{-1}=1$, *then we have*

$$\left|e^{f\left(\frac{a+b}{2}\right)} - \frac{1}{b-a}\int_a^b e^{f(x)}dx\right|$$

$$\le \frac{b-a}{4}\left(\frac{1}{p+1}\right)^{\frac1p}\Big[\Big\{\Big(\Big(\Big|e^{f(\frac{a+b}{2})} f\left(\frac{a+b}{2}\right)\Big|^q + \Big[\Big|f(a)e^{f(\frac{a+b}{2})}\Big|^q - 2\Big|e^{f(\frac{a+b}{2})}f\left(\frac{a+b}{2}\right)\Big|^q$$

$$+\Big|f\left(\frac{a+b}{2}e^{f(a)}\right)\Big|^q\Big]\xi_3(h,t)\Big) + \Big[\Big|e^{f(a)}\Big|^q - \Big|e^{f(\frac{a+b}{2})}\Big|^q\Big]\Big[|f(a)|^q - \Big|f(\frac{a+b}{2})\Big|^q\Big]\xi_4(h,t)\Big)^{\frac1q}$$

$$+\Big\{\Big(\Big(\Big|e^{f(\frac{a+b}{2})} f(\frac{a+b}{2})\Big|^q + \Big[\Big|f(b)e^{f(\frac{a+b}{2})}\Big|^q - 2\Big|e^{f(\frac{a+b}{2})} f(\frac{a+b}{2})\Big|^q$$

$$+\Big|f\left(\frac{a+b}{2}e^{f(b)}\right)\Big|^q\int_0^1 h(t)dt(h,t)\Big) + \Big[\Big|e^{f(b)}\Big|^q - \Big|e^{f(\frac{a+b}{2})}\Big|^q\Big]\Big[|f(b)|^q - \Big|f(\frac{a+b}{2})\Big|^q\int_0^1 h^2(t)dt\Big\}^{\frac1q}\Big],$$

Proof: Using lemma 3.4 and the Holder's inequality, we have the required result.

$$\left|e^{f\left(\frac{a+b}{2}\right)} - \frac{1}{b-a}\int_a^b e^{f(x)}dx\right| \le \frac{b-a}{4}\times\int_0^1(1-t)\Big\{\Big|e^{f\left(\frac{1+t}{2}a+\frac{1+t}{2}b\right)}f'\left(\frac{1+t}{2}a+\frac{1+t}{2}b\right)\Big| + \Big|e^{f\left(\frac{1+t}{2}b+\frac{1+t}{2}a\right)}f'\left(\frac{1+t}{2}b+\frac{1+t}{2}a\right)\Big|\Big\}dt$$

$$\le\frac{b-a}{4}\times\Big[\Big(\int_0^1(1-t)^p dt\Big)^{\frac1p}\Big\{\int_0^1\Big|e^{f\left(\frac{1+t}{2}a+\frac{1+t}{2}b\right)}f'\left(\frac{1+t}{2}a+\frac{1+t}{2}b\right)\Big|^q dt)^{\frac1q} + \int_0^1\Big|e^{f\left(\frac{1+t}{2}b+\frac{1+t}{2}a\right)}f'\left(\frac{1+t}{2}b+\frac{1+t}{2}a\right)\Big|^q dt)^{\frac1q}\Big\}\Big]$$

$$\le\frac{b-a}{4}\left(\frac{1}{p+1}\right)^{\frac1p}\Big\{\Big(\int_0^1\Big[\Big|e^{f\left(\frac{a+b}{2}\right)}\Big|^q + h(t)\Big[\Big|e^{f(a)}\Big|^q - \Big|e^{f\left(\frac{a+b}{2}\right)}\Big|^q\Big]\Big]$$

$$\Big[\Big|f\left(\frac{a+b}{2}\right)\Big|^q + h(t)\Big[|f(a)| - \Big|f\left(\frac{a+b}{2}\right)\Big|^q\Big]\Big]dt)^{\frac1q} + \Big(\int_0^1\Big[\Big|e^{f\left(\frac{a+b}{2}\right)}\Big|^q + h(t)\Big[\Big|e^{f(b)}\Big|^q - \Big|e^{f\left(\frac{a+b}{2}\right)}\Big|^q\Big]\Big]$$

$$\Big[\Big|f\left(\frac{a+b}{2}\right)\Big|^q + h(t)\Big[|f(b)|^q - \Big|f\left(\frac{a+b}{2}\right)\Big|^q\Big]\Big]dt\Big)^{\frac1q}\Big\}$$

$$\le\frac{b-a}{4}\left(\frac{1}{p+1}\right)^{\frac1p}\Big[\Big\{\Big(\Big|e^{f\left(\frac{a+b}{2}\right)}f\left(\frac{a+b}{2}\right)\Big|^q + \Big[\Big|f(a)e^{f\left(\frac{a+b}{2}\right)}\Big|^q - 2\Big|e^{f\left(\frac{a+b}{2}\right)}f\left(\frac{a+b}{2}\right)\Big|^q + \Big|f\left(\frac{a+b}{2}e^{f(a)}\right)\Big|^q\Big]\int_0^1 h(t)dt\Big)$$

$$+\Big[\Big|e^{f(a)}\Big|^q - \Big|e^{f\left(\frac{a+b}{2}\right)}\Big|^q\Big]\Big[|f(a)|^q - \Big|f\left(\frac{a+b}{2}\right)\Big|^q\Big]\int_0^1 h^2(t)dt\Big\}^{\frac1q} + \Big\{\Big(\Big|e^{f\left(\frac{a+b}{2}\right)}f\left(\frac{a+b}{2}\right)\Big|^q$$

$$+\Big[\Big|f(b)e^{f\left(\frac{a+b}{2}\right)}\Big|^q - 2\Big|e^{f\left(\frac{a+b}{2}\right)}f\left(\frac{a+b}{2}\right)\Big|^q + \Big|f\left(\frac{a+b}{2}e^{f(b)}\right)\Big|^q\Big]\int_0^1 h(t)dt\Big)$$

$$+\Big[\Big|e^{f(b)}\Big|^q - \Big|e^{f\left(\frac{a+b}{2}\right)}\Big|^q - |f(b)|^q - \Big|f\left(\frac{a+b}{2}\right)\Big|^q\Big]\int_0^1 h^2 dt\Big\}^{\frac1q}\Big]$$

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$$= \frac{b-a}{4}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\left[\left\{\left(\left|e^{f\left(\frac{a+b}{2}\right)}f\left(\frac{a+b}{2}\right)\right|^{q}+\left[\left|f(a)e^{f\left(\frac{a+b}{2}\right)}\right|^{q}-2\left|e^{f\left(\frac{a+b}{2}\right)}f\left(\frac{a+b}{2}\right)\right|^{q}+\left|f\left(\frac{a+b}{2}e^{f(a)}\right)\right|^{q}\right]\int_0^1 h(t)\,dt\right)\right.\right.\right.$$

$$+\left[\left|e^{f(b)}\right|^{q}-\left|e^{f\left(\frac{a+b}{2}\right)}\right|^{q}\right]\left[\left|f(a)\right|^{q}-\left|f\left(\frac{a+b}{2}\right)\right|^{q}\right]\int_0^1 h^2\,dt\}^{\frac{1}{q}}$$

$$+\left\{\left(\left|e^{f\left(\frac{a+b}{2}\right)}f\left(\frac{a+b}{2}\right)\right|^{q}+\left[\left|f(b)e^{f\left(\frac{a+b}{2}\right)}\right|^{q}-2\left|e^{f\left(\frac{a+b}{2}\right)}f\left(\frac{a+b}{2}\right)\right|^{q}+\left|f\left(\frac{a+b}{2}e^{f(b)}\right)\right|^{q}\right]\int_0^1 h(t)dt\right)\right.$$

$$+\left[\left|e^{f(b)}\right|^{q}-\left|e^{f\left(\frac{a+b}{2}\right)}\right|^{q}\right]\left[\left|f(b)\right|^{q}-\left|f\left(\frac{a+b}{2}\right)\right|^{q}\right]\int_0^1 h^2\,dt\}^{\frac{1}{q}}\Bigg],$$

Now, we derive the following integral inequality for exponential h-convex function.

Theorem 3.8: *Let* $h:(0,1)\subseteq J\to\mathbb{R}$ *be an integrable and non-negative function such that* $\lim_{t\to 0}\left(\frac{h(t)}{t}=k;k\neq 0\right)$. *Suppose that* $f:\Omega=[a,b]\to\mathbb{R}$ *be a differentiable modified exponentially convex function function. Then*

$$\int_a^b e^{f(y)}\,dy \le k\int_a^b\left[e^{f(x)}-e^{f(y)}\right]dy+(x-a)e^{f(a)}+(b-x)e^{f(b)}. \qquad (3.8)$$

Proof: Let f be an modified exponentially convex function function. Then

$$e^{f(tx+(1-t)y)}\le e^{f(y)}+h(t)\left[e^{f(x)}-e^{f(y)}\right], \qquad \forall x,y\in\Omega, t\in[0,1].$$

This implies that

$$\frac{e^{f(tx+(1-t)y)}-e^{f(y)}}{t}\le\frac{h(t)}{t}\left[e^{f(x)}-e^{f(y)}\right]. \qquad (3.9)$$

Taking limit as $t\to 0$ on both sides of the (3.9), we have

$$\left\langle e^{f(y)}f'(y),x-y\right\rangle\le k\left[e^{f(x)}-e^{f(y)}\right]. \qquad (3.10)$$

Integrating (3.10) with respect to x on [a, b] and dividing by (b − a), we have

$$\left\langle e^{f(y)}f'(y),\frac{b+a}{2}-y\right\rangle\le\frac{k}{b-a}\int_a^b\left[e^{f(x)}-e^{f(y)}\right]dx.$$

Similarly, integrating (3.10) with respect to y on [a, b], we have

$$\int_a^b e^{f(y)}f'(y)(x-y)\,dy\le k\int_a^b\left[e^{f(x)}-e^{f(y)}\right]dy. \qquad (3.11)$$

Now consider

$$\int_a^b e^{f(y)}f'(y)(x-y)\,dy=\int_a^b e^{f(y)}dy-(x-a)e^{f(a)}-(b-x)e^{f(b)}. \qquad (3.12)$$

Substituting the value from (3.12) in (3.11), we have

$$\int_a^b e^{f(y)}dy\le k\int_a^b\left[e^{f(x)}-e^{f(y)}\right]dy+(x-a)e^{f(a)}+(b-x)e^{f(b)}.$$

This completes the proof.

Corollary 3.4: *If h(t) = t, then, under the assumptions of Theorem 3.8,[12] we have*

$$\int_a^b e^{f(y)}dy\le\int_a^b\left[e^{f(x)}-e^{f(y)}\right]dy+(x-a)e^{f(a)}+(b-x)e^{f(b)}.$$

Corollary 3.5: *If we take x = a or x = b, and* $k\neq 1$, *then, under the assumptions of Theorem 3.8, we have*

$$\frac{1}{b-a}\int_a^b e^{f(x)}dx\le\frac{1+k}{1-k}\left[\frac{e^{f(a)}+e^{f(b)}}{2}\right].$$

Corollary 3.6: *If we take* $x=\frac{a+b}{2}$ *and* $k\neq -1$, *then, under the assumptions of Theorem 3.8, we have*

$$\frac{1}{b-a}\int_a^b e^{f(x)}dx\le\frac{1}{(1-k)}\left[ke^{f\left(\frac{a+b}{2}\right)}+\left(\frac{e^{f(a)}+e^{f(b)}}{2}\right)\right].$$

Corollary 3.7: *If we take* $x=\frac{2ab}{a+b}$ *and* $k\neq -1$, *then, under the assumptions of Theorem 3.8, we have*

$$\frac{1}{b-a}\int_a^b e^{f(x)}dx\le\frac{1}{(1+k)}\left[ke^{f\left(\frac{2ab}{a+b}\right)}+\left(ae^{f(a)}+be^{f(b)}\right)\right].$$

Remark: From (3.10), we have

$$f(x)-f(y)\ge\log\left(1+\frac{1}{k}\langle f'(y),x-y\rangle\right), \qquad x,y\in\Omega, \quad k\neq 0.$$

One can define, the L-divergence by

$$T(f(x)/f(y)):\log\left(1+\frac{1}{k}\nabla\phi(f(y))[f(x)-f(y)]-[\phi(f(x))-\phi(f(y))]\right), \qquad k\neq 0.$$

This quantity is often considered as the error approximation and has played a significant role in the study of Information theory see, for example.[14]

Conclusion

For appropriate and suitable choice of functions h, we obtain several new and known classes of convex functions as special cases. we derived the Hermite-Hadamard inequalities hold for modified exponentially convex function functions (independent of h) and also calculated some special cases which are new and unifying one.

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Conflicts of interest

Authors declare that there is no conflicts of interest.

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