

Evolution of mathematics: a brief sketch

Ancient beginnings: numbers and figures

It is no exaggeration that Mathematics is ubiquitously present in our everyday life, be it in our school, play ground, mobile number and so on. But was that the case in prehistoric times, before the dawn of civilization? Necessity is the mother of invention or discovery and evolution in this case. So mathematical concepts must have been seen through or created by those who could, and use that to derive benefit from the discovery. The creative process of discovery and later putting that to use must have been arduous and slow. I try to look back from my limited Indian perspective, and reflect up on what might have been the course taken up by mathematics during the long journey, since ancient times.

A very early method necessitated for counting of objects (enumeration) was the Tally Marks used in the late Stone Age. In some parts of Europe, Africa and Australia the system consisted of simple vertical bars: I, II, III, IIII, V, for five objects like those of five fingers of a palm. Larger number of objects were marked as IIII I, IIII II, IIII III, IIII IIII, IIII IIII, IIII IIII I, IIII IIII II, etc. In India, the tally markings in Brahmi script attested from the third century BCE, but not in one form. The inscriptions on Askoke's pillars are in Brahmi script. The tally markings of that era use the marks I, II, III, + and a host of other markings, the mark + possibly standing for the four directions. During roughly the same era in Ancient Roman empire, the tally markings became more systematic as: I, II, III, IV, V, VI, VII, VIII, IX, X, XI, XII etc. This usage continued for pretty long time – till the thirteenth century, after learning about the Decimal System originating in India in a much earlier era.

The tally markings and the Roman system suffered from the difficulty in the counting process of aggregation (addition), removal (subtraction), division and subdivision of objects. The decimal system invented in prehistoric times consisted of representing any number in a base of ten consisting of ten symbols 1, 2, 3, 4, 5, 6, 7, 8, 9, and 0, as we all are familiar with. It is emphasised that these symbols are abstractions regarding collection of some objects (I can not refrain from recalling that a number theorist young mathematician burst at me when I asked him in a chat that "What is 1"). For instance, the number 3 may represent three fingers, three books, three chairs or any other triplet. The first nine symbols in the base of ten, were possibly introduced by finding nine geocentric Grahas (Houses, Planets) in ancient Astronomy viz. Surya (Sun), Chandra (Moon), Budh (Mercury), Sukra (Venus), Mangal (Mars), Guru (Jupiter), Shani (Saturn), Rahu and Ketu, the nodal points of intersection of the lunar and solar orbits at which eclipses of the Sun and the Moon occurred. The invention of the tenth digit 0 or Shunya (or Kha) by Pingala (c.300 BCE), is a landmark invention in the emergence of mathematics. Shunya representing emptiness containing the five elements of the world of creation, viz. Agni (fire), Vayu (air), Akash (space), Prithvi (earth) and Jal (water). In the base of ten, the succeeding whole numbers were denoted by

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10, 11, 12, 13, etc., giving place value to each digit. The barrier of writing very large numbers was thus broken. For instance, the numbers mentioned in Yajur Veda could easily be represented as Dasha (10^1), Shata (10^2), Sahsra (10^3), Ayuta (10^4), Niuta (10^5), Prayuta (10^6), Arambud (10^7), Nyarbud (10^8), Samudra (10^9), Madhya (10^{10}), Anta (10^{11}) and Parartha (10^{12}). In the Buddhist literature, up to Talakshana (10^{53}) and in Jain literature, Shirshaprahelika (10^{193}) are mentioned. Later generation Indian mathematicians, especially Brahmagupta (598–670 CE), thereafter devised systematic ways for the basic operations of addition, subtraction, multiplication and division. Fractions, proper and improper, like and unlike were handled as a matter of course by these early mathematicians.

The Rule of Three was developed for the solution of many practical problems. Whole Numbers and Fractions are better known as Rational Numbers in modern mathematical terminology. Irrational Numbers had their origin in geometry. The representation of fractional part of a rational number by full stop or period mark is only a recent standardisation made in 1960, as several types of markings were in vogue. The use of whole numbers and rationals as labels is also of recent origin. For instance, Mobile number, Citizenship number, platform number etc. are examples of the former, and Version 2.1 etc. are of the latter kind of labels. Rationals and irrationals together comprise Real Numbers, and have been used for a long time also as Measure of some quantity, especially in geometry, such as length of a line on a certain scale.

Geo in Greek language means the earth, but usually in the sense of ground and land, and geometry means measurements regarding that. Evidence of geometrical measurements still exist in the forms of pyramids in Egypt, but formulations based on geometric ideas can be traced back to the post Vedic Brahman literature in the Shatpatha Brahman, where preparation of vedis for Yajya (oblation to fire) are described in great detail. These procedures were further extended in the Shulbasutras by Baudhayan, Apastamba, Manava and Katyayan, before the Buddhist era. The sutras not only described a host of different rules for constructing vedis of different forms, but also gave abstractions of the underlying geometrical properties. In the Shulbasutras by Baudhayan, Apastamba and Katyayan, sutras are given verbal statements of the well known Pythagoras theorem, later during the Buddhist era. In

the sutras can be found the discovery of irrationals, $\sqrt{2}, \sqrt{3}$ etc. and approximations of the numbers by rationals. The irrational π , which is the constant ratio of the circumference of circles by their diameters is treated and approximated in the Shatapatha Brahman. However, much later π was more accurately approximated by Aryabhata (c. 476–550), and much later Nilkantha (1444–1544) clearly stated that the value of π so obtained was not the exact value of π . That π is irrational was proved by Lambert JH (1728–1777) as late as in 1761. In the meantime, the subject of geometry was put on logical foundation by the Greek mathematician Euclid (c. 300 BCE), that has become a milestone of observing logical development in mathematics since the advent of nineteenth century during which further developments took place including extension to three dimensions.

Algebraic formalism and meeting with geometry

Algebra or Bijaganit in Sanskrit, is extension of arithmetic in which instead of numbers symbols are used in like manner. Although this kind of mathematics is named after al-Khwarizmi (c.780–850), it originated in Babylon in about (c.2200 BCE) to be followed later in Egypt in (c.1650 BCE). However in India, its use in geometric lengths can be traced to Shatapath Brahman and later in the Shulbasutras for construction of mahavedis. The invention was again necessitated for determining an unknown bija (root) from certain known numbers satisfying given conditions for equations. In this formalism, the solution (root finding) of linear equations was studied providing an easier method than the rule of three that required mental calculations. The solution of quadratic equations was treated by the Babylonians and Greeks by geometrical methods. The problem arose when given a specific area, the length of the sides of certain shapes were required for construction. The formula that we know is due to Brahmagupta (597–698 CE) stated in verse form as was the style in India. During the same era, Sridharacharya (c.750 CE), independently obtained the same formula. Thereafter, the subject made great strides to the present times. AL-Khwarizmi

Trigonometry came in to being from the necessity of measurements regarding triangular forms and somewhat later on to study positions and motion of celestial bodies in Astronomy. Although the beginnings can be traced to Babylon (c. 1900 BCE), Egypt (c. 650 BCE) and Greece (c. 310 BCE), the development of trigonometrical ratios took place in the writings of Aryabhat (476–550 CE), his contemporary Varahmihir (550 CE) and Brahmagupta (628 CE), who introduced sine cosine and versine. Several formulas regarding these ratios were discovered by them and other Indian mathematicians in later centuries that exhibit great mathematical elegance. During this period, great advances in positional and dynamical astronomy were also made by their use of the new mathematical advancement, such as forecasting eclipses and their duration. The trigonometrical ratios of tangent, cotangent, secant and cosecant were however introduced later by Arab and Persian mathematicians, notably by al-Buzjani (c. 940–998 CE). Analytical Geometry was invented independently by the French mathematicians Rene Descartes (1596–1650) and Pierre de Fermat (1607–1665),

to essentially study geometry of curves, especially the conic sections in two dimensions, by using algebraic methods. This development was however preceded much earlier by the study of conic sections by Apollonius (c. 262–190 BCE) of Greece that also contained the idea of coordinates. A definitive book on Analytical Geometry in Three Dimensions (or Solid Geometry) was published by George Salmon (1819–1904). The development of analytic geometry, led to the invention of Infinitesimal Calculus, which up to the modern day makes up another foundational storey of mathematics.

Infinitesimal, infinite, the calculus and real analysis

The study of curves by analytical methods of geometry using coordinates, required tangency condition by straight or curved lines and calculation of areas enclosed by curves. As is well known, these mathematical problems were treated by Isaac Newton (1642–1727) and Gottfried Wilhelm Leibniz (1646–1716) to found Calculus (differential and integral). Newton's approach was naturally geometrical, in terms of what were called fluxions of fluent (a time varying function), or the derivative in modern terminology, where as that of Leibniz was Integration, which he saw as a generalisation of the summation of series. In order to relate the two, Leibniz introduced differentials in infinitesimals. The relation between the two approaches is called the Fundamental Theorem of Calculus, which was systematically stated and proved by James Gregory (1638–1675).

The subjects of Calculus (differential and integral) as also of Analytical Geometry are based on Real Numbers hypothesised as forming a Continuum. The representation of irrationals in digits, requires the concept of Infinity as the occurrence of digits do not terminate. Even though philosophically infinity was envisaged by the Greek Zeno (c. 490 BCE), its symbolism was described by the word Khahar or (Purna in Vedic terminology) by Bhaskeracharya (1114–1185) in his book BijaB ganit, stating its property that a (finite) number added or subtracted from it leaves the symbol unaltered. The symbol ∞ which has no ends, was formally introduced in mathematics by John Wallis (1616–1703) in 1655. The opposite of the concept of infinite is the infinitesimal, and Bhaskeracharya who was an astronomer at the Ujjain Observatory, had a clear idea of infinitesimal as smaller than small, propounded earlier by Aryabhat and Brahmagupta. While studying angular motion of planets by "Mean Anomalies", he studied the change in mean anomaly with change in infinitesimal time, and in the process derived a formula which is equivalent to a statement in regard to the differential of the sine function in modern notation. The way Bhaskeracharya also dealt with the calculation of surface area and volume of a sphere by the method of summation indicates using integration from first principles. These developments regarding infinitesimal calculus predates those of Newton and Leibniz by nearly 500 years. The limiting processes of Calculus led to new advance, from finite procedures to treat their limiting process to infinity led to the discovery of a new component of Calculus—the Infinite Series. This advance was achieved by Madhav (c. 1340–1425), Nilkantha (1444–1545) and their successors. A number of extremely elegant infinite power series were summed including those of the famous sine, cosine and arctan power series. Taylor series expansion of these

functions were also given. These topics now comprise Real Analysis, in which emphasis is laid on rigour as propounded by Augustin-Louis Cauchy (1789–1857) and others during the nineteenth century.

Revelation of mechanics by calculus

Newton also laid the foundation of (Classical) Mechanics, aided by his development of Infinitesimal Calculus. His use of the term fluxion for derivative is a pointer to this simultaneous development of the two subjects. The new development led the Bernoulli brothers Johann (1667–1748) and Jacob (or James) (1655–1705) to make further developments in Mechanics that went hand in hand with that of Calculus. Johann's son Daniel (1700–1782) went on to develop hydrodynamics from the single principle of Conservation of Energy, while his uncle Jacob laid the foundation of the Theory of Elasticity in one dimension, which was generalized to three dimensions much later in 1820 by Claude-Louis Navier (1785–1836). Jacob also discovered the constant e while investigating Compound Interest and solved the dynamical problem of finding the Isochronous Curve as a Cycloid. His brother Johann solved the Brachistochrone Problem to lay the foundation of the Calculus of Variations and proved it also to be a Cycloid. This special problem deals with the question of finding the form of the curve when a particle sliding smoothly down its length takes the least time. The Variational Principle of Virtual Work was also enunciated by Johann, which was much later used to great effect by Joseph-Louis Lagrange (1736–1813) and William Rowan Hamilton (1805–1865).

Leonhard Euler (1707–1783)—one of the greatest mathematicians ever—was a friend and colleague of Daniel Bernoulli. He greatly contributed to the Infinitesimal Calculus and formally developed the calculus of Variations the latter leading to the celebrated Euler-Lagrange Differential Equation in Mechanics. He greatly extended formal operational tools of Calculus and treating Infinite Series, though sometimes lacking in mathematical rigour of modern standards. The proof of the well known exponential series is due to him. The extension of the exponential series to Complex Numbers was also done by him, such numbers having been discovered earlier, in connection with the solution of the Cubic Equation and systematised by Rafael Bombelli (1526–1572). The symbol i for $\sqrt{-1}$ is due to Euler. In this way Euler laid the foundation of Complex Analysis, a subject further developed in the nineteenth century by Augustin Louis-Cauchy (1789–1857) that became a subject of great practical utility in practical applications. Euler applied the powerful tools that he had invented to derive Euler's Dynamical Equations for the motion of a rigid body about a fixed point, The Euler's Equations of inviscid flow in Hydrodynamics, and the Euler-Bernoulli Beam Equation in Elasticity Theory. Applying his methods, he determined with great accuracy, the orbits of Comets and other Celestial Bodies. Euler also laid the foundation of Graph Theory and Topology by solving the Königsberg Seven Bridges Problem. The era of Newton-Leibniz-Bernoullis and Euler truly represents advancement in mathematics, in particular to Analysis, hand in hand with that of mechanics during the eighteenth century, culminating with the work of Joseph-Louis Lagrange (1736–1813) who reduced Mechanics to Analysis.

Calculating objects of analysis: numerical analysis

Euler also contributed to Numerical Analysis, a twentieth century subject classification, that deals with the methods of numerical approximation for the solution of symbolically posed problems of Analysis. The subject is very important from practical point of view in present times, but it was practiced by early astronomers like Aryabhat, Varahmihir (505–587 CE) and Bhaskaracharya as well in presenting data in the form of charts. Starting with the name of Newton who dealt with the problem of Interpolation, prominent names of the eighteenth, nineteenth and the twentieth century mathematicians is associated. Euler's name is associated with the well known Euler-Maclaurin Summation Formula [Colin Maclaurin (1698–1746)] that expresses a definite integral of a function as the sum of the function at some interpolating points, and Euler's Method for numerically integrating a first order differential equation. For facilitating calculations, mechanical devices were invented at first, but later on since the 1940's, the advent of electronic computers greatly facilitated numerical computations.

Uncertainty, chance and probability theory

Gambling was not uncommon for the affluent during the middle ages. Even in such games of uncertainty (or randomness), it was noticed that certain events showed regularity. In games of dice, problem was put to Blaise Pascal (1623–1662) who corresponded with his friend Pierre de Fermat, that paved the way for development of the Theory of Probability. Pascal and Fermat introduced calculation of Probability by two methods: dividing the probability space in to equally likely events, called the Classical method and the Frequency Method depending on the relative frequency of occurrence of an event under study. It was Jacob Bernoulli who proved the equivalence of the two methods in 1713. The well known Binomial Distribution, a generalisation of the Bernoulli Distribution, was later proved to tend to the Normal Distribution by Abraham de Moivre (1667–1754), known as the first Central Limit Theorem. A more systematic theory was given by Pierre-Simon Laplace (1749–1827), who proved the theorem under more general conditions. He also introduced the well known Laplace Distribution, Bayesian Probability, Probability Generating Function, and the Characteristic Function. The Normal Distribution which is dominant in probability theory was also discovered through the development of the Theory of Errors, by Carl Friedrich Gauss (1777–1855), based on ideas similar to that lay behind the central limit theorem. Attempts to put the theory on rigorous foundation and applying it systematically were successful after Andrey Kolmogorov (1903–1987) laid it on axiomatic theory like that of Euclid for Geometry. Subsequently, Kiyosi Itô (1915–2008) formulated a calculus for Stochastic Processes known as the Stochastic Calculus. Now a days Stochastic Modelling using Probability Theory is indispensable in Mathematical Theory of Statistics, Life Insurance, Biology, Finance, and even in Space Technology. The relationship between Probability Theory and practical requirements has been the basic reason for development of Probability Theory during the last three centuries.

Rapid progress in theoretical branches of physics

Ever since the time of Newton and Leibniz, developments in other branches of physics also took place. In studying musical string instruments, Jean le Rond d'Alembert (1717–1783) obtained the well known Wave Equation in one dimension and solved it in 1746. The equation was soon generalised to three dimensions by Euler. After these developments, and before treating the Theory of Probability, Laplace made some outstanding discoveries in Celestial Mechanics. In his time, gravitational attraction was already known and Lagrange had discovered the Gravitational Potential a name due to George Green (1793–1841) Laplace showed that the potential satisfies the celebrated Laplace's Equation. The equation occurs frequently in other branches of physics as well. The mathematical study of this equation and its solution later on formed the subject of the Theory of Potential. Theories for heat exchange and propagation were also being propounded during the turn of the eighteenth century. The one propounded by Jean-Baptiste Joseph Fourier (1768–1830) had a profound impact on Analysis, when he derived the Heat Equation for conduction of heat and solved it in the form of a Fourier Series. Later on Franz Ludwig Fick (1829–1901) found that the equation was equally applicable for diffusion of one fluid medium in to another.

Fourier's series solution was somewhat informal but confound mathematicians for over one hundred years. Greater precision and formality was later given by PeP ter Gustav Lejune Dirichlet (1805–1859) and Georg Friedrich Bernhard Riemann (1826–1866). In this direction Riemann developed the Theory of Riemann Integration for Bounded Functions and Dirichlet introduced the Functions of Bounded Variation. The theory of integration was further extended to unbounded functions by Henri Lebesgue (1875–1941) for what are known as Measurable Functions. Since then Trigonometric Series and in general Harmonic Analysis became a topic of intense study over a long period of time. Questions on differentiability of such series arose when Karl Weierstrass (1815–1897) obtained a trigonometric series, which is nowhere differentiable. The Fourier Series solution of the Wave Equation was not twice differentiable for it to be a solution—a question settled much later by Laurent L Moise Schwartz (1915–2002), who introduced the concept of Distributions instead of Functions, and the definition of Weak Solutions instead of pointwise or Strong Solutions.

The thermal processes in gases, in which molecules can move about, led to Statistical Mechanics, as the motion of molecules becomes random. For Ideal Gases in which particles do not interact at all, the Probability Density Function of the molecular speeds was obtained by James Clerk Maxwell (1831–1879) and Ludwig Boltzmann (1844–1906), the former obtaining it based on molecular collision and symmetry arguments, and the latter by dwelling more in to physical origins of the mechanical motion. The probability distribution function is known as the Maxwell-Boltzmann Distribution, and holds when the thermodynamic system is in a state of equilibrium. Boltzmann proceeded further to consider a system not in equilibrium state and the particles interactive. The resulting Boltzmann Equation is an integro-

differential equation for the Probability Density Function in six dimensional space of molecular position and momentum. The equation contains a complex Collision Term. The best known simplified model of the term is due to Prabhu Lal Bhatnagar (1912–1976), Eugene P. Gross (1926–1991) and Max Krook (1913–1985) given as late as in 1954, which makes the equation far more amenable to solution. Recently in 2010, it has been proved by Philip T. Gressman and Robert M. Strain that the solution of the exact equation is always well-behaved; that is to say, if a system obeying Boltzmann equation is perturbed, then it will return to equilibrium, rather than diverge or behave otherwise.

Magnetism and Electricity were discovered long ago, particularly the former, as the property was used by sailors in their device Mariner's Compass. Around 1600 CE, William Gilbert (1544–1603) who is regarded as the father of Electricity and Magnetism, experimentally demonstrated that the earth itself was a huge magnet.

In the year Benjamin Franklin (1705–1790), one of the founding fathers of U.S.A., demonstrated the flow of electricity in his famous kite experiment during a thunderstorm. The aggregation of numerous experiments in the nineteenth century, led to numerous mathematical equations by Charles-Augustin de Coulomb (1736–1806), Hans Christian Orsted (1777–1851), Gauss, Jean-Baptiste Biot (1774–1862), Felix Savart (1791–1841), Andre-Marie Ampere (1775–1836), and Michael Faraday (1791–1867). The apparently disparate laws of Electricity and Magnetism were integrated by Maxwell that required, on mathematical ground, the induction of Displacement Current in Ampere's Law. He showed that these equations imply that light propagates as electromagnetic waves satisfying the wave equation. His equations were reformulated by Oliver Heaviside (1850–1925) in the vector formalism, which is followed up to our times. Vector Analysis was introduced earlier by Josiah Willard Gibbs (1839–1903) for its usefulness in mechanics, and Heaviside Vector Calculus for use in Maxwell's Equations. Joseph Larmor (1857–1942) and later Hendrik Antoon Lorentz (1853–1928) showed that the Maxwell's Equations remained invariant under a Space-Time Transformation, which Henri Poincarre (1854–1912) later called the Lorentz Transformation. These developments led Albert Einstein (1879–1955) to conclude that light has a fixed speed, and propound the Special Theory of Relativity by 1907, but struggled to put it up in final form encompassing Gravitational effect. During subsequent years, David Hilbert (1862–1943) who was a Pure Mathematician having mastery in Geometry, got interested in physics, in particular to Einstein's work, invited Einstein during the summer of 1915 to Göttingen his place of work, for giving a week of lectures. During this very time Amalie Emmy Noether (1882–1935) was also invited to join the group. After a few months, both Einstein and Hilbert put up the theory in two separate papers: "The Field Equations of Gravitation" by Einstein and "The Foundations of Physics" by Hilbert. Recent historical research shows that Noether made the critical breakthrough in the formal completion of the theory. It is interesting to note that no Nobel Prize in Physics was awarded for the Special or the General Theory of Relativity. Ironically Hilbert commented that "Physics is too hard for Physicists," as he began to understand physics and how physicists were using mathematics.

Stepping in to the Modern Age: Twentieth Century

The mathematical framework of General Relativity is Riemannian Geometry developed earlier by Riemann. Another development in the early twentieth century physics is the development of Quantum Mechanics for sub-atomic scale physics. The subject was developed in essentially two different mathematical frameworks. Matrix Mechanics by Werner Heisenberg (1901–1976) and Max Born (1882–1970), and in wave formalism by Erwin Schrodinger (1887–1995). Matrix Algebra developed earlier by Arthur Cayley (1821–1895) and the Wave Equation were both formally well known at that time. Quantum Mechanics uses the concept of uncertainty, which is mathematically dealt with by Probability Theory. The mathematical formulation of General Relativity and Quantum Mechanics are very different, and up to the present times no single mathematical framework has been found that unifies the two. The former theory is applicable on cosmic scale, while the latter holds at subatomic scale. On terrestrial scale of length, Newtonian Mechanics succeeds.

The progress of sciences during the eighteenth and nineteenth century took place hand in hand with that in mathematics, and the scientists of that era used to describe themselves as mathematicians. However, especially during the middle of nineteenth century, some of them began to lay emphasis on logic in the many procedures that were developed by their predecessors. Breaking away from Euclid Lobachevsky (1792–1856) and Janos Bolyai (1802–1860) created Hyperbolic Geometry and Riemann, the elliptic or Riemannian Geometry that proved vital for the development of the General theory of Relativity. The Boolean Algebra was created by George Boole (1815–1864) consisting of only two elements true denoted by 1 and false denoted by 0. The Relation Algebra was created earlier by India born Augustus De Morgan (1806–1871). While studying switching circuits, Claude Shannon (1916–2001) found that the Boolean Algebra was eminently suitable for designing logic gates and development of the Computer later on by Howard Aiken (1900–1973). Modern VLSI circuit design depends on efficient representation of what are known as Boolean functions. In Analysis, Richard Dedekind (1831–1916) pursued a logical basis for the real number system, so did Weierstrass. Georg Cantor (1845–1918) created the Theory of Sets, in which a set of uncountable points on a line is nowhere dense. Set theoretic notations are now standard in many branches of mathematics, such as in Switching Algebra of Shannon. Cantor's effort to prove the Continuum Hypothesis of real numbers failed and can not be proved by logic. Meanwhile, moving from the pointwise Euclidean space to finite dimensional Vector Spaces, employing the set of n -dimensional vectors and matrices possessing a certain norm, has been found to be very important in Numerical Analysis, and in Image Compression routines through Fourier expansion. Further extension to infinite dimensional Function Spaces defined over set of functions with certain defined norm lead to the subject of Functional Analysis a subject that has proved vital for proving the existence and uniqueness of partial differential equations. In particular starting with the work of Jean Leray (1906 – 1998), on Euler's hydrodynamical equations in the year 1934, and its generalization to Navier – Stokes equations for

viscous fluids, a vast amount of mathematical literature has been created around the two equations. The creation of Turbulence at higher velocities of flow, due to viscosity of real fluids, is vastly important in applications, that brings in stochasticity treated by Probability Theory; eventually leaving the equations indeterminate posing another kind of challenge. Integral Transforms such as the Fourier Transform and the Laplace Transform are also topics of Functional Analysis that have proved to be immensely useful, in the solution of linear ordinary and partial differential equations occurring in different sciences and engineering. The Calculus of Variations and the Theory of Distributions alluded earlier, also fall under the purview of Functional Analysis. The twentieth century saw the application of methods of Real, Complex, Functional and Numerical Analyses to solve great many problems of different sciences to usher in the modern technological advancement that has taken place. However, the current twenty-first century faces the challenges of atmospheric and climate, energy harnessing, natural disasters like oceanic tsunamis and tectonic earthquakes, global financial markets, social unrest, biosphere and many more that need critical study. These are all large, multiple scales phenomena that may require further developments in mathematics for definitive solutions. In this connection it is necessary for all of us to note the following words of the famous Probabilist Mathematician Pafnuty Lvovich Chebyshev (1821–1894): “The link-up between theory and practice yields the most salutary results, and the practice yields the most salutary results, and the practical side is not only one that benefits; the sciences themselves advance under its influence, for it opens up to them new objects of investigation or fresh aspects of familiar objects. If the theory gains much from new applications of an old method or from its new developments, then it benefits still more from the discovery of new methods, and in this case too, science finds itself a true guide in practical affairs.”

It is imperative upon us to follow the path laid down by Chebyshev, as was envisaged by ancient sages. The question is whether the journey is going to be arduous? Maybe, but if one elaborately goes through the developments that has taken place up to now, most parts are charmingly elegant in form, if one looks at the beautiful formulae, equations, patterns, arrangements etc. with an eye of sensitivity of discovery after overcoming challenges of arduous study. The whole or Natural Numbers themselves possess such elegance and beauty that mathematicians from Fermat to Srinivasa Ramanujan (1887–1920) to Andrew Wiles (1953) engaged themselves to uncover those.

In closing the above account of Evolution of Mathematics, many branches and sub-branches of the vast subject have been missed. Names of numerous brilliant minds have inevitably been missed; the intent being to draw the vastly big picture of the evolution by a very wide brush. In the short narration with my limited resources the information scattered in the Internet proved useful. The other useful references are given in the References given below. Concise information on the contribution of the Indian mathematicians of yore, was available from the book by Professor Satyabachi Sar, and is thankfully acknowledged here. Informal exchange of views with him were also very useful. The other useful references are given in the References given below.¹⁻¹⁰

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Conflicts of interest

Authors declare that there is no conflict of interest.

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