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Brown spaces and the Golomb topology

Abstract

A *Brown space* is a topological space X such that for all non-empty open subsets U and V of X, we have $\mathrm{cl}_X(U) \cap \mathrm{cl}_X(V) \neq \varnothing$. It is clear that Brown spaces are connected and not completely Hausdorff. Given $a, b \in \mathbb{N}$, whose greatest common divisor is 1, we consider the arithmetic progression $P_G(a,b) = \{b + an : n \in \mathbb{N} \cup \{0\}\}$. The family $\mathcal{B}_G$ of all such arithmetic progressions is a base for a topology τ_G on $\mathbb{N}$. In this paper we show that for every $d \in \mathbb{N}$, the set $P_G(1,d)$ is a Brown space which is dense in $(\mathbb{N}, \tau_G)$. In particular, $(\mathbb{N}, \tau_G)$ is a Brown space. We also show that for each prime number p and every natural number c, such that the greatest common divisor between p and c is 1, the set $P_G(p,c)$ is totally separated. We write some consequences of such result. For example that the space $(\mathbb{N}, \tau_G)$ is not connected im kleinen at each of its points. This generalizes a result of Kirch AM.[1] We also present a simpler proof of a result presented by Szczuka P.[2] Some general properties of Brown spaces are also presented in this paper.

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Introduction

We denote by $\mathbb{Z}$ and by $\mathbb{N}$ the sets of integers and of natural numbers, respectively, and we let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We also denote by $\mathbb{P}$ the set of prime numbers and consider that $\mathbb{P} \subset \mathbb{N}$. Given $a, b \in \mathbb{N}$, the symbol $\langle a, b\rangle$ denotes the greatest common divisor of a and b and we consider the infinite arithmetic progressions

$$P(a,b) = \{b + an : n \in \mathbb{N}_0\} = b + a\mathbb{N}_0 \text{ and}$$

$$P_G(a,b) = \{b + an : n \in \mathbb{N}_0\} = b + a\mathbb{N}_0, \quad \textit{provided that } \langle a,b\rangle = 1. \tag{1}$$

For $a \in \mathbb{N}$ and $b \in \mathbb{Z}$ we also consider the infinite arithmetic progressions

$$P_F(a,b) = \{b + az : z \in \mathbb{Z}\} = b + a\mathbb{Z} \quad \textit{and} \quad M(a) = \{an : n \in \mathbb{N}\}.$$

Clearly $P(a,b) = P_F(a,b) \cap \mathbb{N}_0$ and $P(a,b) = P_G(a,b)$ if and only if $\langle a,b\rangle = 1$. Note that $M(a) = P(a,a)$. In 1955 Furstenberg H[3] showed in that the family $\mathcal{B}_F = \{P_F(a,b) : (a,b) \in \mathbb{N} \times \mathbb{Z}\}$ is a base for a topology τ_F on $\mathbb{Z}$. The topological space $(\mathbb{Z}, \tau_F)$ is second countable and T_4, and hence metrizable. Moreover each basic set $P_F(a,b)$ is open and closed in $(\mathbb{Z}, \tau_F)$, so this space is zero-dimensional and not connected.

In 1959 and 1962 Golomb SW[4,5] showed inthat the family

$$\mathcal{B}_G = \{P_G(a,b) : (a,b) \in \mathbb{N} \times \mathbb{N} \textit{ and } \langle a,b\rangle = 1\} \tag{2}$$

is a base for a topology τ_G on $\mathbb{N}$. Indeed

$$\tau_G = \{\varnothing\} \cup \{U \subset \mathbb{N} : \textit{for each } b \in U \textit{ there is } a \in \mathbb{N} \textit{ such that } \langle a,b\rangle = 1 \textit{ and } P_G(a,b) \subset U\}.$$

In,[5] whose first edition was published in 1970, τ_G is called the "relatively prime integer topology", though in this paper, as well as in all the papers by Szczuka P,[2,6–10] we call τ_G the *Golomb topology* and the topological space $(\mathbb{N}, \tau_G)$ is called *the Golomb space*. It is known that $(\mathbb{N}, \tau_G)$ is second countable, T_2 and connected Theorems 2 and 3,[4] and Theorems 2 and 3[5]. Using the fact that for every $p \in \mathbb{P}$, the set $M(p)$ is closed in $(\mathbb{N}, \tau_G)$, Golomb SW[4] proved in both Theorem 1[4] & Theorem 1[5] that the set $\mathbb{P}$ is infinite. The proof of the connectedness of $(\mathbb{N}, \tau_G)$, as presented by Golomb SW,[4] uses Number Theory. As it is indicated in[4,5] "a proof of the connectedness of $(\mathbb{N}, \tau_G)$, without reference to Number Theory, was presented by Brown M[11] in the April 1953 meeting of the American Mathematical Society", held in New York. Brown M[11] studied the space $(\mathbb{N}, \tau_G)$, though he did not publish his work. The abstract of his talk, published in[11] is the following one:

A countable connected Hausdorff space. The points are the positive integers. Neighborhoods are sets of integers $\{a + bx\}$, where *a* and *b* are relatively prime to each other ($x = 1, 2, 3, \ldots$). Let $\{a + bx\}$ and $\{c + dx\}$ be two neighborhoods. It is shown that bd is a limit point of both neighborhoods. Thus, the closures of any two neighborhoods have a nonvoid intersection. This is a sufficient condition that a space be connected.

This abstract served Clark PL,[12] in 2017, the authors of,[12] to coin the following term:

Definition 1.1 *A Brown space is a topological space X such that for all non-empty open subsets U and V of X, we have $\mathrm{cl}_X(U) \cap \mathrm{cl}_X(V) \neq \varnothing$. If X is a topological space and $Y \subset X$, we say that Y is a* Brown space in *X if Y, as a subspace of X, is a Brown space.*

The following result appears in [2, Proposition 6].

Theorem 1.2 *Each Brown space X is connected.*

Proof. If X is not connected, then there exist non-empty open and closed subsets U and V of X such that $X = U \cup V$ and $U \cap V = \varnothing$. Then $\mathrm{cl}_X(U) \cap \mathrm{cl}_X(V) = U \cap V = \varnothing$, a contradiction to the fact that X is a Brown space.

In this paper we will present some general properties of Brown spaces. We will give an explicit proof of the fact that $(\mathbb{N}, \tau_G)$ is a Brown space (Theorem 3.4). We will also show that, for every $d \in \mathbb{N}$, the subset $P_G(1,d)$ is a Brown space in $(\mathbb{N}, \tau_G)$ (Theorem 3.3). Since $(\mathbb{N}, \tau_G)$ is second countable, it is also Lindelöf. By the space $(\mathbb{N}, \tau_G)$ is not $T_{3\frac{1}{2}}$.[4] Since every second coutable space is Lindelöf,[14] and every

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Lindelöff and T_3 space is T_4 [14] and hence $T_{2\frac{1}{2}}$, the space $(\mathbb{N},\tau_G)$ is not T_3. Without using all these results from General Topology (that lead to the fact that every non-empty countable connected T_3 space is a one-point-set), in both Theorem~4[4] and Theorem~4,[5] Golomb SW[5] proved that $(\mathbb{N},\tau_G)$ is not T_3 by showing that for the closed set $M(2)$ in $(\mathbb{N},\tau_G)$ that do not contain the point 1, there are no open subsets U y V in $(\mathbb{N},\tau_G)$ such that $1\in U, M(2)\subset V$ and $U\cap V=\varnothing$.

Since compact T_2 spaces as well as locally compact and T_2 spaces are T_3,[14] the space $(\mathbb{N},\tau_G)$ is neither compact nor locally compact (compare with Theorem 5[4] and Theorem 5[5]). The paper is divided in three sections. After this Introduction, in Section 2 we write the notation as well as some preliminary results that we will use in the paper. In this section we also write some general properties of Brown spaces. In Section 3 we write properties of the Golomb space as well as of some subsets of it.

Notation and preliminary results

In this paper we will use notation and results from both Number Theory and from General Topology. Concerning Number Theory, if $c,d\in\mathbb{Z}$ and $c\neq 0$, then the symbol $c\mid d$ means that there exists $a\in\mathbb{Z}$ such that $d=ca$. If $c,d\in\mathbb{Z}$ and $m\in\mathbb{N}-\{1\}$, then the symbol $c\equiv d(mod m)$ means that $m\mid(c-d)$. The next result is proved in.[15]

Theorem 2.1 *Let $a,b,q,r\in\mathbb{Z}$ be such that $a=bq+r$. Then $\langle a,b\rangle=\langle b,r\rangle$.*

The following result was used in both [13, p. 169] and [1, p. 902] without proof.[13,2] We will use it in Section 3 so, for completeness, we present a proof here.

Theorem 2.2 *Let $b\in\mathbb{Z}$. If $p\in\mathbb{P}$ is such that $\langle b,p\rangle=1$, then for each $n,s\in\mathbb{N}_0$, $\langle pn+b,p^s\rangle=1$.* (3)

Proof. If $s=0$, then

$$\langle pn+b,p^s\rangle=\langle pn+b,p^0\rangle=\langle pn+b,1\rangle=1,\quad \textit{for each } n\in\mathbb{N}_0.$$

If $s=1$ then, since $\langle p,b\rangle=1$, by Theorem 2.1,

$$\langle pn+b,p^s\rangle=\langle pn+b,p\rangle=\langle p,b\rangle=1,\quad \textit{for each } n\in\mathbb{N}_0. \quad (4)$$

Now assume that there exist $s_0\in\mathbb{N}-\{1\}$ and $n_0\in\mathbb{N}_0$ such that $g=\langle pn_0+b,p^{s_0}\rangle>1$. Let $q\in\mathbb{P}$ be such that $q\mid g$. Then $q\mid(pn_0+b)$ and $q\mid p^{s_0}$, so $q=p$. Hence $q\mid(pn_0+b)$ and $q\mid p$, so $q\mid\langle pn_0+b,p\rangle$. This implies, using (4) with $n=n_0$, that $q\mid 1$, a contradiction. Thus $\langle pn+b,p^s\rangle=1$, for every $s,n\in\mathbb{N}_0$.

Concerning General Topology, given a topological space X and $A\subset X$, we denote by $\mathrm{cl}_X(A)$ and by $\mathrm{int}_X(A)$ the closure and the interior of A in X, respectively. In particular, for $A\subset\mathbb{N}$, the symbol $\mathrm{cl}_X(A)$ denotes the closure of A in $(\mathbb{N},\tau_G)$. We consider the closed interval $[0,1]$ with its usual topology. Recall that a topological space X is said to be

T_1 if for each $x\in X$, the set $\{x\}$ is closed in X;

T_2 or *Hausdorff* if for every $x,y\in X$ such that $x\neq y$, there exist open sets U and V in X so that $x\in U$, $y\in V$ and $U\cap V=\varnothing$;

$T_{2\frac{1}{2}}$ or *completely Hausdorff* if for every $x,y\in X$ with $x\neq y$, there exist open sets U and V such that $x\in U$, $y\in V$ and $\mathrm{cl}_X(U)\cap\mathrm{cl}_X(V)=\varnothing$;

regular if for every $x\in X$ and each closed set C in X such that $x\notin C$, there exist open sets U and V in X so that $x\in U$, $C\subset V$ and $U\cap V=\varnothing$;

T_3 if X is regular and T_1;

$T_{3\frac{1}{2}}$ if X is T_1 and for each $x\in X$ and every closed set C in X so that $x\notin C$, there is a continuous function $f:X\to[0,1]$ such that $f(x)=0$ and $f(C)\subset\{1\}$;

T_4 if X is T_1 and for every closed subsets C and D of X with $C\cap D=\varnothing$, there exist open sets U and V in X such that $C\subset U$, $D\subset V$ and $U\cap V=\varnothing$.

It is known that $T_4\Rightarrow T_{3\frac{1}{2}}\Rightarrow T_3\Rightarrow T_{2\frac{1}{2}}\Rightarrow T_2\Rightarrow T_1$. It is also known that none of the previous implications is reversible. The following result is easy to prove.

Theorem 2.3 *No Brown space X, with at least two points, is $T_{2\frac{1}{2}}$.*

By Theorem 2.3 no connected $T_{2\frac{1}{2}}$, with at least two points, is a Brown space.

Let X be a topological space and x in X. We say that X is *indiscrete in* x or that x is an *indiscrete point of* X if the only open subset of X that contains x is X itself. We say that X is *indiscrete* if its topology is the indiscrete topology. Note that X is indiscrete if and only if every point of X is indiscrete. The following result is proved in [2, Proposition 6].

Theorem 2.4 *Let X be a topological space. Then*

if X contains an indiscrete point, then X is a Brown space;

if X is a Brown space, then X is regular if and only if X is indiscrete.

Note that a T_1 space contains no indiscrete points. By Theorem 2.4, no connected T_3 space, with at least two points, is a Brown space. We also have that if X is a connected regular space without indiscrete points, then X is not a Brown space. Hence the converse of Theorem 1.2 is not true.

We say that a topological property P is

hereditary if for any space X that has the property P, every subspace of X also has the property P;

multiplicative if for any family $\{X_s:s\in S\}$ of topological spaces with the property P, the Cartesian product $\prod_{s\in S}X_s$, with the product topology, also has the property P;

factorizable if for any family $\{X_s:s\in S\}$ of topological spaces if the Cartesian product $\prod_{s\in S}X_s$, with the product topology, has the

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property P then each factor X_s also has the property P.

Theorem 2.5 *Let X and Y be topological spaces and $f : X \to Y$ be a continuous and surjective function. If X is a Brown space, then Y is a Brown space.*

Proof. Let U and V be non-empty open subsets of Y. Since f is continuous and surjective, $f^{-1}(U)$ and $f^{-1}(V)$ are non-empty open subsets of X. Hence, since X is a Brown space, $\mathrm{cl}_X\left(f^{-1}(U)\right)\cap \mathrm{cl}_X\left(f^{-1}(U)\right)\neq\varnothing$. By continuity of X we have $\varnothing\neq \mathrm{cl}_X\left(f^{-1}(U)\right)\cap \mathrm{cl}_X\left(f^{-1}(U)\right)\subset f^{-1}(\mathrm{cl}_Y(U))\cap f^{-1}(\mathrm{cl}_Y(V)) = f^{-1}(\mathrm{cl}_Y(U)\cap \mathrm{cl}_Y(V))$.

Hence $\mathrm{cl}_Y\left(U\right)\cap \mathrm{cl}_Y\left(V\right)\neq\varnothing$, so Y is a Brown space.

By Theorem 2.5 being a Brown space is a topological property. Moreover.

Theorem 2.6 *Being a Brown space is both a multiplicative and a factorizable property.*

Proof. Let $\{X_s : s\in S\}$ be a family of non-empty topological spaces. Let $X=\prod_{s\in S}X_S$ and assume that X has the product topology. For each $t\in S$ the projection $p_t : X\to X_t$ defined for any $x=(x_s)_{s\in S}\in X$ by $p_t(x)=x_t$ is continuous and surjective. Hence if X is a Brown space then, by Theorem 2.5, X_t is a Brown space too. This shows that being a Brown space is a factorizable property.

Now assume that each X_s is a Brown space. Let U and V be two non-empty open subsets of X. Fix $x\in U$ and $y\in V$ and assume that $B=\prod_{s\in S}B_s$ and $C=\prod_{s\in S}C_s$ are basic subsets of X such that $x\in B\subset U$ and $y\in C\subset V$. For each $s\in S$, the sets B_s and C_s are non-empty and open in the Brown space X_s, so $\mathrm{cl}_{X_s}\left(B_s\right)\cap \mathrm{cl}_{X_s}\left(C_s\right)\neq\varnothing$. Then

$$\mathrm{cl}_X\left(B\right)\cap \mathrm{cl}_X\left(C\right)=\left(\prod_{s\in S}\mathrm{cl}_{X_s}\left(B_s\right)\right)\cap\left(\prod_{s\in S}\mathrm{cl}_{X_s}\left(C_s\right)\right)=\prod_{s\in S}\left(\mathrm{cl}_{X_s}\left(B_s\right)\cap \mathrm{cl}_{X_s}\left(C_s\right)\right)\neq\varnothing.$$

Hence $\varnothing\neq \mathrm{cl}_X\left(B\right)\cap \mathrm{cl}_X\left(C\right)\subset \mathrm{cl}_X\left(U\right)\cap \mathrm{cl}_X\left(V\right)$, so X is a Brown space. This shows that being a Brown space is a multiplicative property.

In Theorem 3.5 we will show that being a Brown space is not a hereditary property. If X and Y are topological spaces, $f : X\to Y$ is a quotient mapping and X is a Brown space then, by Theorem 2.5, Y is a Brown space too.

A topological space X is said to be

hereditarily disconnected if no non-empty connected subset of X contains more that one point;

totally separated if for every $x, y\in X$ with $x\neq y$, there exist open sets U and V in X such that $x\in U$, $y\in V$, $X=U\cup V$ and $U\cap V=\varnothing$;

zero-dimensional is X is T_1 and has a base consisting of open and closed sets.

Hereditarily disconnected space are also called *totally disconnected*. In[14] it is shown that zero-dimensional spaces are hereditarily disconnected. In[14] it is proved that if X is hereditarilly disconnected and locally compact, then X is zero-dimensional. Note that a space X is hereditarily disconnected if and only if, for each $x\in X$, the component C_x of X that contains x is a one-point-set, namely $\{x\}$. Note also that X is totally separated if and only if, for every $x\in X$, the quasicomponet Q_x of X that contains x is a one-point-set, namely $\{x\}$. Since $C_x\subset Q_x$ for each $x\in X$ [14], totally separated spaces are hereditarily disconnected. If X is compact and T_2 then $C_x=Q_x$ for every $x\in X$ [14] so, in compact and T_2 spaces the notions of being hereditarily disconnected and of being totally separated coincide.

Let X be a topological space and $a, b\in X$. We say that X is *connected between a and b* if for every open sets U and V in X such that $a\in U$, $b\in V$ and $U\cap V=\varnothing$, we have $X\neq U\cup V$. It is straight forward to see that if X if T_2 then X is totally separated if and only if for each $x, y\in X$ with $x\neq y$, the space X is not connected between x and y.

Let X be a topological space and $x\in X$. We say that

X is *locally connected at x*, if for any open subset V of X such that $x\in V$, there exists an open and connected subset U of X such that $x\in U\subset V$;

X is *connected im kleinen at x*, if for any open subset V of X such that $x\in V$, there exists a connected subset U of X such that $x\in \mathrm{int}_X\left(U\right)\subset U\subset V$;

X is *almost connected im kleinen at x*, if for any open subset V of X such that $x\in V$, there exists a closed and connected subset U of X such that $\mathrm{int}_X\left(U\right)\neq\varnothing$ and $U\subset V$;

X is *almost connected im kleinen* if X is almost connected im kleinen at each of its points;

X is locally connected if X is locally connected at each of its points.

Clearly, if X is locally connected at x, then X is connected im kleinen at x. The converse of this implication is not true. We construct an example in $\mathbb{R}^2$, with the usual topology. For each $i\in\mathbb{N}$, let $q_i=\left(\frac{1}{i},0\right)\in\mathbb{R}^2$ and let $L_{i,0}$ be the straight line segment from q_i to q_{i+1}. For each $(i,n)\in\mathbb{N}\times\mathbb{N}$, we consider

$$p_{i,n}=\frac{1}{i+1}(1,\frac{1}{n})\in\mathbb{R}^2,$$

as well as the straight line segment $L_{i,n}$ from $p_{i,n}$ to q_i. Formally, for each $(i,n)\in\mathbb{N}\times\mathbb{N}$, let

$$L_{i,n}=\{(x,y)\in\mathbb{R}^2 : \frac{1}{i+1}\le x\le\frac{1}{i} \quad\text{and}\quad y=\frac{1}{n}(1-ix)\},$$

where $y=\frac{1}{n}(1-ix)$ is the equation of the straight line in $\mathbb{R}^2$ that contains the points $p_{i,n}$ y q_i. For each $i\in\mathbb{N}$, we define $X_i=\bigcup_{n\in\mathbb{N}\cup\{0\}}L_{i,n}$.

Then $X_b=\mathrm{cl}_2\left(\bigcup_{i\in\mathbb{N}}X_i\right)$ is a topological space which is compact, connected, connnected im kleinen at $(0,0)$ and not locally connected at $(0,0)$. The space X_b, called the *infinite broom* in [5,

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p. 139], appears in[16] Example ~27.15, p. 201], though there is no any detailed proof of the fact that X_b, is connected im kleinen at $(0,0)$. and not locally connected at $(0,0)$. In[16] there is a detailed proof of this, though it is written in Spanish. In[16] it is shown that if a topological space X is connected im kleinen at each of its points, then X is locally connected. Let

$$Y = \left(\bigcup_{n\in\mathbb{N}} L_{i,0}\right) - \{q_i : i \in \mathbb{N}\} \subset X_b.$$

Note that X_b is almost connected im kleinen at any point $p \in Y$ and not connected im kleinen at such point p. Note also that if a T_3 space is connected im kleinen at $x \in X$, then it is almost connected im kleinen at x.

Properties of the golomb space

In the space $(\mathbb{N}, \tau_G)$ a non-empty subset U of $\mathbb{N}$ is open if and only if, for every $b \in U$, there exists $a \in \mathbb{N}$ such that $\langle a,b\rangle = 1$ and $P_G(a,b) \subset U$ (compare with).[7] Hence U is infinite. In particular any subset of $\mathbb{N}$ with non-empty interior in $(\mathbb{N}, \tau_G)$ is infinite. Let $a, x \in \mathbb{N}$. In Szczuka P[8] presented some results that involve the set $\mathrm{cl}\left(P(a,x)\right)$. In,[7] she showed that if $x_1 \equiv x(mod a)$, $x_1 \le a$ and $\langle a,x\rangle = 1$, then $P(a,x_1) \subset \mathrm{cl}\left(P_G(a,x)\right)$. In[8] she showed that if a and x are odd and $\langle a,x\rangle = 1$, then $\mathrm{cl}\left(P_G(a,x)\right) = \mathrm{cl}\left(P_G(2a,x)\right)$. We show the following result.

Theorem 3.1 *Let $a, x \in \mathbb{N}$ be so that $\langle a,x\rangle = 1$. Then $M(a) \subset \mathrm{cl}\left(P_G(a,x)\right)$.*

Proof. Let $ab \in M(a)$ and let W be an open subset of $(\mathbb{N}, \tau_G)$ such that $ab \in W$. There exists $d \in \mathbb{N}$ such that $\langle d, ab\rangle = 1$ and $P_G(d,ab) \subset W$. Assume first that $a = 1$. Then $P_G(a,x) = \{x + n : n \in \mathbb{N}_0\}$ so if $x \le ab$, then $ab \in P_G(a,x) \subset \mathrm{cl}\left(P_G(a,x)\right)$. If $ab < x$ then, since $P_G(d,ab)$ is infinite, there is $n \in \mathbb{N}$ such that $dn + ab \ge x$, so $dn + ab \in P_G(d,ab) \cap P_G(a,x) \subset W \cap P_G(a,x)$. This shows that $ab \in \mathrm{cl}\left(P_G(a,x)\right)$.

Now assume that $a \ge 2$ By showing that $P_G(d,ab) \cap P_G(a,x) \ne \varnothing$ we will obtain that $W \cap P_G(a,x) \ne \varnothing$. To prove that $P_G(d,ab) \cap P_G(a,x) \ne \varnothing$, assume first that $d = 1$. Then $ab + x \in P_G(a,x)$ and $ab + x = 1 \cdot x + ab \in P_G(d,ab)$. Now assume that $d \ge 2$. Since $d \ge 2$. and $a \ge 2$, an element $z \in P_G(d,ab) \cap P_G(a,x)$ satisfies the system of congruencies

$$z \equiv ab(mod d) \quad and \quad z \equiv x(mod a). \quad (5)$$

Hence, an element in $P_G(d,ab) \cap P_G(a,x)$ is a solution of the system (5). Conversely, every solution of the system (5), is an element of $P_G(d,ab) \cap P_G(a,x)$. Let us show, then, that the system (5) has a solution. If $\langle d,a\rangle \ne 1$, then there exists $p \in \mathbb{P}$ such that $p \mid d$ and $p \mid a$. Then $p \mid d$ and $p \mid ab$, so $\langle d, ab\rangle \ne 1$, a contradiction. Hence $\langle d,a\rangle = 1$ and by the Chinese Reminder Theorem, the system (5) has a solution. This shows that $P_G(d,ab) \cap P_G(a,x) \ne \varnothing$. Hence $W \cap P_G(a,x) \ne \varnothing$ and then $ab \in \mathrm{cl}\left(P_G(a,x)\right)$.

Corollary 3.2 *For every $d \in \mathbb{N}$, the set $P_G(1,d)$ is dense in $(\mathbb{N}, \tau_G)$.*

Proof. Let $d \in \mathbb{N}$. Put $a = 1$. Then $\langle a,d\rangle = 1$, $M(a) = \mathbb{N}$ and, by Theorem 3.1, $\mathbb{N} = M(a) \subset \mathrm{cl}\left(P_G(a,d)\right)$, so $P_G(a,d) = P_G(1,d)$ is dense in $(\mathbb{N}, \tau_G)$.

Theorem 3.3 *For every $d \in \mathbb{N}$, the set $P_G(1,d)$ is a Brown space in $(\mathbb{N}, \tau_G)$. In particular, $P_G(1,d)$ is connected.*

Proof. Let U and V be two non-empty open subsets of $P_G(1,d)$. Since $P_G(1,d)$ is open in $(\mathbb{N}, \tau_G)$, the sets U and V are open in $(\mathbb{N}, \tau_G)$, Let $x \in U$ and $y \in V$. Then there exist $a, b \in \mathbb{N}$ such that $\langle a,x\rangle = \langle b,y\rangle = 1$ and $P_G(a,x) \subset U$ and $P_G(b,y) \subset V$. If $a = 1$, then $P_G(a,x) = \{x + n : n \in \mathbb{N}_0\}$. Since $P_G(b,y)$ is infinite, there is $n_0 \in \mathbb{N}_0$ such that $y + bn_0 \ge x$ and then

$$y + bn_0 \in P_G(a,x) \cap P_G(b,y) \subset U \cap V \cap P_G(1,d) \subset \mathrm{cl}(U) \cap \mathrm{cl}(V) \cap P_G(1,d)$$
$$\mathrm{cl}_{P_G(1,d)}\left(U\right) \cap \mathrm{cl}_{P_G(1,d)}\left(V\right).$$

Hence $\mathrm{cl}_{P_G(1,d)}\left(U\right) \cap \mathrm{cl}_{P_G(1,d)}\left(V\right) \ne \varnothing$. Similarly, if $b = 1$ we have $\mathrm{cl}_{P_G(1,d)}\left(U\right) \cap \mathrm{cl}_{P_G(1,d)}\left(V\right) \ne \varnothing$. Now assume that $a \ge 2$ and $b \ge 2$. By Theorem 3.1, $M(a) \subset \mathrm{cl}\left(P_G(a,x)\right)$ and $M(b) \subset \mathrm{cl}\left(P_G(b,y)\right)$ so, in particular, $abd \in \mathrm{cl}\left(P_G(a,x)\right) \cap \mathrm{cl}\left(P_G(b,y)\right)$. Since $a \ge 2$ and $b \ge 2$, we have $d < abd$, so $abd \in P_G(1,d)$. Then

$$abd \in \mathrm{cl}\left(P_G(a,x)\right) \cap \mathrm{cl}\left(P_G(b,y)\right) \cap P_G(1,d) \subset \mathrm{cl}\left(U\right) \cap \mathrm{cl}\left(V\right) \cap P_G(1,d) =$$
$$\mathrm{cl}_{P_G(1,d)}\left(U\right) \cap \mathrm{cl}_{P_G(1,d)}\left(V\right).$$

Hence $\mathrm{cl}_{P_G(1,d)}\left(U\right) \cap \mathrm{cl}_{P_G(1,d)}\left(V\right) \ne \varnothing$, so $P_G(1,d)$ is a Brown space in $(\mathbb{N}, \tau_G)$. Since Brown spaces are connected, $P_G(1,d)$ is connected.

Now we present an explicit proof of what Brown M[11] claimed in assertion (B) of Section 1.

Theorem 3.4 *$(\mathbb{N}, \tau_G)$ is a Brown space. In particular, $(\mathbb{N}, \tau_G)$ is connected and it is not $T_{2\frac{1}{2}}$.*

Proof. Since $P_G(1,1) = \mathbb{N}$ the result follows from Theorem 3.3 and the fact that Brown spaces are not $T_{2\frac{1}{2}}$.

Now we prove the following result.

Theorem 3.5 *Let $c, p \in \mathbb{N}$ be such that $p \in \mathbb{P}$ and $\langle p,c\rangle = 1$. Take $a, b \in P_G(p,c)$ such that $a < b$ and $n, m \in \mathbb{N}_0$ so that $a = pm + c, b = pn + c$ and $0 \le m < n$. Then*

$$U = \bigcup_{i=0}^{m} P_G(p^{n+1}, pi + c) \text{ and } V = \bigcup_{j=m+}^{p^n - 1} P_G(p^{n+1}, pj + c) \quad (6)$$

are open subset of $(\mathbb{N}, \tau_G)$ such that $a \in U$, $b \in V$, $U \cap V = \varnothing$

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and $P_G(p,c) = U \cup V$. In particular, $P_G(p,c)$ is not connected.

Proof. Since $p \in \mathbb{P}$ and $\langle p,c\rangle = 1$, by Theorem 2.2,

$$\langle p^{n+1}, pi + c\rangle = 1, \quad \textit{for each } i \in \mathbb{N}_0.$$

Then we can consider the sets U and V indicated in (6), and they are open in $(\mathbb{N}, \tau_G)$. Since $0 \le m < n < 2^n \le p^n$, we have $m+1 \le n \le p^n - 1$, so

$$a = pm + c = p^{n+1}(0) + pm + c \in P_G(p^{n+1}, pm + c) \subset \bigcup_{i=0}^{m} P_G(p^{n+1}, pi + c) = U$$

and

$$b = pn + c = p^{n+1}(0) + pn + c \in P_G(p^{n+1}, pn + c) \subset \bigcup_{j=m+1}^{p^n - 1} P_G(p^{n+1}, pj + c) = V.$$

Hence $a \in U$ and $b \in V$ so U and V are infinite. Now assume that there is a point $z \in U \cap V$. Let $i \in \{0,1,\ldots,m\}$ and $j \in \{m+1, m+2, \ldots, p^n - 1\}$ be such that $z \in P_G(p^{n+1}, pi + c)$ and $z \in P_G(p^{n+1}, pj + c)$. Since $p^{n+1} > 1$, it follows that

$$z \equiv pi + c(modp^{n+1}) \textit{ and } z \equiv pj + c(modp^{n+1}).$$

Hence $pi + c \equiv pj + c(modp^{n+1})$, so $pi \equiv pj(modp^{n+1})$. This implies that $i \equiv j(modp^n)$. However, since the set $\{0,1,\ldots,m,m+1,\ldots,p^n - 1\}$ is a complete system of reminders modulus p^n and $i \le m < m+1 \le j$, we have $i \not\equiv j(modp^n)$. From this contradiction we infer that $U \cap V = \varnothing$.

Now we show that $U \cup V = P_G(p,c)$. If $z \in U$, then there exists $i \in \{0,1,\ldots,m\}$ such that $z \in P_G(p^{n+1}, pi + c)$. Let $z_0 \in \mathbb{N}_0$ be such that $z = p^{n+1} z_0 + pi + c$. Then $z = p(p^n z_0 + i) + c \in P_G(p,c)$. Similarly, if $z \in V$, there is $j \in \{m+1, m+2, \ldots, p^n - 1\}$ such that $z \in P_G(p^{n+1}, pj + c)$. Let $z_1 \in \mathbb{N}_0$ be such that $z = p^{n+1} z_1 + pj + c$. Then $z = p(p^n z_1 + j) + c \in P_G(p,c)$. This shows that $U \cup V \subset P_G(p,c)$. To prove the other inclusion, let $z \in P_G(p,c)$. Then there exists $k \in \mathbb{N}_0$ such that $z = pk + c$. Using the Division Algorithm we obtain $s,t \in \mathbb{N}_0$ such that $k = p^n s + t$ and $0 \le t < p^n$. Hence

$$z = pk + c = p(p^n s + t) + c = p^{n+1} s + pt + c. \qquad (7)$$

Since $0 \le t < p^n$ and $0 \le m < p^n$, we obtain

$$t \in \{0,1,\ldots,p^n - 1\} = \{0,1,\ldots,m\} \cup \{m+1, m+2, \ldots, p^n - 1\}.$$

Hence, by (7),

$$z \in \left(\bigcup_{i=0}^{m} P_G(p^{n+1}, pi + c) \right) \cup \left(\bigcup_{j=m+1}^{p^n - 1} P_G(p^{n+1}, pj + c) \right) = U \cup V.$$

This shows that $P_G(p,c) \subset U \cup V$. Hence, $U \cup V = P_G(p,c)$. This completes the first part of the proof. Since $P_G(p,c)$ has been written as the union of two non-empty open subsets of $(\mathbb{N}, \tau_G)$, which are disjoint, the set $P_G(p,c)$ is not connected.

Corollary 3.6 *Let $c, p \in \mathbb{N}$ be such that $p \in \mathbb{P}$ and $\langle p,c\rangle = 1$. Take $a,b \in P_G(p,c)$ such that $a \ne b$. Then there exist open sets U and V in $(\mathbb{N}, \tau_G)$ such that $a \in U$, $b \in V$, $U \cap V = \varnothing$ and $P_G(p,c) = U \cup V$.*

Note that $(\mathbb{N}, \tau_G)$ is a Brown space and, for each $c, p \in \mathbb{N}$ such that $p \in \mathbb{P}$ and $\langle p,c\rangle = 1$, $P_G(p,c)$ is an open subset of $(\mathbb{N}, \tau_G)$ which is not a Brown space, by Theorem 3.5. Hence being a Brown space is not a hereditary property.

Using Corollary 3.6 we obtain the following result.

Theorem 3.7 *Let $c, p \in \mathbb{N}$ be such that $p \in \mathbb{P}$ and $\langle p,c\rangle = 1$. Then $P_G(p,c)$ is totally separated.*

Since totally separated spaces are hereditarily disconnected, for every $c, p \in \mathbb{N}$ such that $p \in \mathbb{P}$ and $\langle p,c\rangle = 1$, the set $P_G(p,c)$ is hereditarily disconnected.

Now we present several consequences of Theorem 3.7. In 1969 Kirch AM[1] showed in[1] Theorem 1, that the space $(\mathbb{N}, \tau_G)$ is not locally connected. Formally he proved that $(\mathbb{N}, \tau_G)$ is not locally conntected at 1. In the following theorem we generalize this result.

Theorem 3.8 *For every $d \in \mathbb{N}$ the space $P_G(1,d)$ is neither connected im kleinen nor almost connected im kleinen at each of its points.*

Proof. Assume, on the contrary, that there exists $d \in \mathbb{N}$ such that $P_G(1,d)$ is either connected im kleinen or almost connected im kleinen at some point $c \in P_G(1,d)$. We define a set $P_G(p,c)$ as follows: if c is odd, we take $p = 2$. If c is even, we take $p \in \mathbb{P}$ such that $c < p$. In any situation, it follows that $p \in \mathbb{P}$, $\langle p,c\rangle = 1$ and $P_G(p,c)$ is an open subset of $P_G(1,d)$ such that $c \in P_G(p,c)$. Since $P_G(1,d)$ is either connected im kleinen or almost connected im kleinen at c, there exists a connected subset C of $P_G(1,d)$ such that $\text{int}_{P_G(1,d)}(C) \ne \varnothing$ and $C \subset P_G(p,c)$. Since $P_G(1,d)$ is open in $(\mathbb{N}, \tau_G)$, we have $\text{int}(C) \ne \varnothing$. Since all the non-empty open subsets of $(\mathbb{N}, \tau_G)$ are infinite, the set C is infinite. This contradicts the fact that, by Theorem 3.7, $P_G(p,c)$ is hereditarilly disconnected. Hence $P_G(1,d)$ is neither connected im kleinen nor almost connected im kleinen at c.

Corollary 3.9 *The space $(\mathbb{N}, \tau_G)$ is neither connected im kleinen nor almost connected im kleinen at each of its points.*

As a consequence of Theorem 3.8 for any $a \in \mathbb{N}$, the space $(\mathbb{N}, \tau_G)$ is not locally connected at a.

For $a \in \mathbb{N}$ we define $\Theta(a) = \{p \in \mathbb{P} : p \mid a\}$. Note that $\Theta(1) = \varnothing$.

Theorem 3.10 *Let $a, c \in \mathbb{N}$. If $P(a,c)$ is connected in $(\mathbb{N}, \tau_G)$ then $\Theta(a) \subset \Theta(c)$. In particular, if $\langle a,c\rangle = 1$ then $P_G(a,c)$ is connected if and only if $a = 1$.*

Proof. Assume first that $P(a,c)$ is connected. If $\Theta(a) \not\subseteq \Theta(c)$, then $a > 1$ and there exists $p \in \Theta(a) - \Theta(c)$. Hence $p \in \mathbb{P}$, $p \mid a$ and $p \nmid c$, so $\langle p,c\rangle = 1$. Since $p \mid a$, we have $P(a,c) \subset P_G(p,c)$. By Theorem 3.7, $P_G(p,c)$ is hereditarily disconnected, so $P(a,c)$ is a one-point-set, a contradiction. This shows that $\Theta(a) \subset \Theta(c)$.

Now assume that $\langle a,c\rangle = 1$ and that $P_G(a,c) = P(a,c)$ is connected. Then, by the first part of the theorem, $\Theta(a) \subset \Theta(c)$. If $a \geq 2$, then there is $p \in \mathbb{P}$ such that $p \mid a$. Then $p \mid c$, contradicting that $\langle a,c\rangle = 1$. Hence $a = 1$. In Theorem 3.3 we proved that $P_G(1,c)$ is connected.

In Theorem 3.3 Szczuka P[2] showed the following result:

Theorem 3.11 *Let* $a, c \in \mathbb{N}$. *The arithmetic progression* $P(a,c)$ *is connected in* $(\mathbb{N}, \tau_G)$ *if and only if* $\Theta(a) \subset \Theta(c)$. *In particular,*

1. the progression $\{an : n \in \mathbb{N}\}$ is connected in $(\mathbb{N}, \tau_G)$
2. if $\langle a,c\rangle = 1$, then $P_G(a,c)$ is connected in $(\mathbb{N}, \tau_G)$ if and only if a=1.

The proof of the "only if" part of Theorem 3.11 is much simpler is we know that the progressions $P_G(p,c)$ are hereditarily disconnected, as we presented in the proof of Theorem 3.10. In[2] it is claimed that the fact that each set $P_G(1,c)$ is connected is obvious.

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Conflicts of interest

Authors declare that there is no conflict of interest.

References

1. Kirch AM. A countable, connected, locally connected Hausdorff space. *Amer Math Monthly.* 1969;76(2):169–171.
2. Szczuka P. The connectedness of arithmetic progressions in Furstenberg's, Golomb's and Kirch's topologies. *Demonstration Math*. 2010;43(4):899–909.
3. Furstenberg H. On the infinitude of primes. *Amer Math Monthly*. 1955;62(58):353.
4. Golomb SW. A connected topology for the integers. *Amer Math Monthly*. 1959;66(8).
5. Golomb SW. *General Topology and its Relations to Modern Analysis and Algebra*, Proceedings of the Symposium held in Prague in Sep 1961. Academia Publishing House of the Czechoslovak Academy of Sciences;1962:179–186.
6. Szczuka P. Connections between connected topological spaces on the set of positive integers. *Cent Eur J Math*. 2013;11(5):876–881.
7. Szczuka P. Regular open arithmetic progressions in connected topological spaces on the set of positive integers. Glasnik *Matematicki.* 2014;49(69):13–23.
8. Szczuka P. The closures of arithmetic progressions in the common division topology on the set of positive integers. *Central European Journal of Mathematics*. 2014;12(7):1008–1014.
9. Szczuka P. The closures of arithmetic progressions in Kirch's topology on the set of positive integers. *Int J Number Theory.* 2015;11(3):673–682.
10. Szczuka P. Properties of the division topology on the set of positive integers. *Int J Number Theory.* 2016;12(3):775–785.
11. Brown M. The April Meeting in New York. *Bull Amer Math Soc.* 1953;59(4):330–371.
12. Clark PL, Lebowitz Lockard N, Pollack P. *A note on Golomb topologies.*
13. Engelking R. General Topology, Revised and Completed Edition. *Sigma Series in Pure Mathematics*, 6. Heldermann Verlag, Berlin; 1989.
14. Fine B, Rosenberger G. *Number Theory: An Introduction via the Density of Primes*. Second Edition, Birkhäuser; 2016.
15. Willard S. *General Topology*, Reprint of the 1970 Original Addison-Wesley, Rending, MA., Inc., Mineola, New York: Dover Publications; 2004.
16. Pachecho-Juárez. Propiedades del Producto de Espacios Topológicos, Facultad de Ciencias Físico Matemáticas, Benemérita Universidad Autónoma de Puebla; 2009.
17. Steen LA, Seebach JA. Counterexamples in Topology, Reprint of the Second (1978) edition Original Springer-Verlag, New York: Dover Publications; 1995.

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