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Gauge unfixing formalism of the O(N) nonlinear sigma model

Abstract

In a few years back, one of us has proposed a new scheme of embedding constrained systems based mainly on the Gauge Unfixing (GU) formalism and it is known as an extended GU formalism. The proposition was to modify directly the original phase space variables of an arbitrary system in order to turn the system a gauge invariant one. Since the new theory is gauge invariant, we can say that the new system is a first class one in Dirac terminology. In this way, the GU method is a constraint conversion technique. In this work, by using this extended GU formalism we have obtained two different versions of the first class system related on the O(N) nonlinear sigma model

Keywords: constrained systems, embedding systems, gauge invariant hamiltonians

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Introduction

In this work we will discuss a subject that is still of extreme importance in today's theoretical physics. The issue is gauge invariance, one of the main ingredients of Standard Model. One path to explore the subject is through the constraint analysis, since it is well known that first-class constraints system, in Dirac's constraints classification,[1] is a gauge invariant one. Hence, when we have a second-class constrained system, the gauge invariance is lost and it is very convenient to recover this gauge invariance. Thus, we have to convert this second-class into a first−class one. The literature offers several ways to do that. In this work we will accomplish the task through the so−called Gauge Unfixing (GU) conversion technique.

As an example of what we have just said about constraints, let us consider the Abelian pure Chern-Simons (CS) theory, which is a mixed constrained system, where one of its four constraints must be, let us say, "converted," in order to be a first−class one. Then, after this step, we would have well established algebras of two first-class constraints and two second−class constraints.

One of the "conversion" methods, the well known BFT formalism,[2–11] which enlarges the phase space variables through the introduction of the so−called Wess Zumino (WZ) fields, has been used since its first design, to embed the CS theory.[12] As a result, the authors demonstrate many important features.

In another paper,[13] the authors have also employed the BFT formalism to analyze a non−Abelian version of the CS theory. In this paper, the authors suggest two methods that overcome the problem of embedding mixed constrained systems. In an alternative view of the BFT formalism, there is the GU method which embeds second-class constrained systems. It was introduced by Mitra & Rajaraman[14,15] and developed by Vytheeswaran.[16,17]

The GU formalism considers part of the whole group of second−class constraints as being the gauge symmetry generators. The remaining constraints are now the gauge fixing terms. The corresponding second−class Hamiltonian must be adapted, i.e., modified, in such a convenient way in order to satisfy the first−class algebra together with the constraints that were chosen at the beginning as being the gauge symmetry generators. The GU method has a classy property which prevents us from extending the phase space with extra variables. Some time ago, one of us has provided the constraint literature with an alternative procedure concerning the GU formalism and applied it to the CS theory.[18] The objective was to redefine the original phase space variables of a certain constrained system, without introducing any WZ terms, in order to be gauge invariant fields. After that, functions of these gauge invariant fields, which will be gauge invariant quantities, were constructed. This so called "extended" GU formalism begins with a kind of mixed constrained system, which was the CS theory, at that occasion, and, applying the technique, it was obtained a first−class system which was written just in terms of the original phase space variables with many new features. As many important constrained systems have only two second−class constraints, then, in principle, the formalism was introduced only for systems with two second−class constrains without any loss of generality.

In this work we will use this extended GU constraint conversion method to explore the O(N) nonlinear sigma model (NLSM). We have some experience with the NLSM[19] and the motivation to work with it is based on the fact that, although the NLSM were first introduced in high energy physics in the context of chiral symmetry breaking, NLSM also plays an underlying role in condensed matter issues, where it appears naturally as effective field theories depicting the low energy long−wave−length limit of several microscopic models.

Having said that and, in order to clarify the exposition of the subject, this paper is organized as follows: in Section 2, we give a short review of the usual GU formalism. In Section 3, we present our formalism. In Section 4, we apply our procedure to the CS theory. In Section 5, we make our concluding remarks.

A brief review of the gauge unfixing formalism

Let us study a second−class constrained system described by its correspondent Hamiltonian which has, for example, two second-class constraints T_1 and T_2. The main idea of GU formalism is to convert a second−class system into a first−class one by selecting one of the two second−class constraints to be the gauge symmetry generator, i.e., this constraint will be "defined" *ad hoc* as being first−class. The other constraint will be discarded since a new first−class Hamiltonian will be constructed. However, since we have two constraints, the next step is to build another conversion procedure with the second constraint that was discarded. Now this second constraint will be the chosen one, and the first constraint will be discarded. To sum up, we have two cases in this GU formalism, namely, two ways to obtain gauge invariance. This will be clear in a moment.

The idea is to understand the original non invariant gauge theory as being a gauge fixed version of the gauge invariant system. If we choose T_1 as the symmetries generator, then the second−class Hamiltonian have to be modified in order to satisfy a first−class algebra. To accomplish the task, both the new and gauge invariant Hamiltonian can be constructed through a power series of T_2 in order to not generate any new constraints, of course. Hence, with this procedure obligation well established, we can write conveniently that

$$\tilde{H} = H + T_2\{H,T_1\} + \frac{1}{2!}T_2^2\{\{H,T_1\},T_1\} + \frac{1}{3!}T_2^3\{\{\{H,T_1\},T_1\},T_1\} + \ldots, \tag{1}$$

where it can be shown that $\{\tilde{H},T_1\} = 0$ (i.e., there are no secondary or any new constraints) and T_1 must satisfy the first-class algebra $\{T_1,T_1\} = 0$. In this way this final system was shown precisely to be a first−class one, and consequently, gauge invariant.

The O(N) nonlinear sigma model

The O(N) nonlinear sigma model is described by the Lagrangian density

$$L = \frac{1}{2}\partial_\mu \varnothing^a \partial^\mu \varnothing^a + \frac{1}{2}\lambda\left(\varnothing^a\varnothing^a - 1\right) \tag{2}$$

where the $\mu = 0,\ 1$ and a is an index related to the O(N) symmetry group, and the corresponding canonical Hamiltonian density is given by

$$H = \frac{1}{2}\pi_a\pi_a - \frac{1}{2}\partial_i\varnothing_a\partial^i\varnothing_a - \frac{1}{2}\lambda\left(\varnothing_a\varnothing_a - 1\right) \tag{3}$$

The second−class constraints of the system in Eq. (1), in Dirac's constraints classification, are

$$T_1(x) = \varnothing_a(x)\varnothing_a(x) - 1 \text{ and } T_2(x) = \varnothing_a(x)\pi_a(x) \tag{4}$$

The second−class constraint algebra is

$$\left\{T_1(x),T_2(y)\right\} = 2\varnothing_a\varnothing_a\,\delta(x-y) \tag{5}$$

which shows clearly that the Poisson brackets of both constraints is not zero and consequently the system is not gauge invariant.

The infinitesimal gauge transformations generated by symmetry generator T_1 are

$$\delta\varnothing_a = \epsilon\left\{\varnothing_a(x),T_1(y)\right\} = 0 \tag{6}$$

and

$$\delta\pi_a = \varepsilon\left\{\pi_a(x),T_1(y)\right\} = -2\varepsilon\varnothing_a\,\delta(x-y) \tag{7}$$

The gauge invariant field $\tilde{\varnothing}_a$ will now be constructed by using a known Taylor expansion in series of powers of T_2, namely,

$$\tilde{\varnothing}_a = \varnothing_a + b_1^a T_2 + b_2^a T_2^2 + \cdots + b_2^a T_2^n \tag{8}$$

From the invariance condition, $\delta\tilde{\varnothing}_a = 0$, we can calculate all the set of correction terms b_n. For the linear correction term in order of T_2, we have that

$$\delta\tilde{\varnothing}_a = 0 \;\Rightarrow\; \epsilon\left\{\tilde{\varnothing}_a(x),T_1(y)\right\} = 0 \tag{9}$$

Substituting Eq. (8) into the last equation, and using the algebra in Eq. (5) to equate the b_n^a terms, we can easily arrive the result that shows that

$$b_1^a = b_2^a = \cdots = b_n^a = 0 \tag{10}$$

Namely, due to this last result, all the correction terms b_n^a where $n \geq 1$ are zero. Therefore, the gauge invariant field $\tilde{\varnothing}_a$ is

$$\tilde{\varnothing}_a = \varnothing_a \tag{11}$$

And, by using Eq. (6), it is readily to show that $\delta\tilde{\varnothing}_a = 0$. The gauge invariant field $\tilde{\pi}_a$ is also constructed via Taylor series in powers of T_2

$$\tilde{\pi}_a = \pi_a + c_1^a T_2 + c_2^a T_2^2 + \cdots + c_n^a T_2^n \tag{12}$$

From the invariance condition $\delta\tilde{\pi}_a = 0$, we can work out all the correction terms c_n^a. For the linear correction term in order of T_2, we have that

$$\delta\pi_a + c_1^a\varnothing_b\,\delta\pi_b = 0 \tag{13}$$

and consequently,

$$c_1^a = -\frac{\varnothing_a}{\varnothing^2} \tag{14}$$

For the quadratic term, we can write that $c_2^a = 0$, since $\delta c_1^a = \varepsilon\left\{c_1^a,\tilde{T}\right\} = 0$. Thereby, using this result, we can say that all the correction terms c_n^a with $n \geq 2$ is zero. Hence, the gauge invariant field $\tilde{\pi}_a$ is

$$\tilde{\pi}_a = \pi_a - \varnothing_a\frac{\varnothing_b\pi_b}{\varnothing^2} \tag{15}$$

where, by using Eq. (7), It is direct to demonstrate that $\delta\tilde{\pi}_a = 0$. The gauge invariant Hamiltonian, written only in terms of the original phase space variables, is obtained by substituting $\varnothing_a$ by $\tilde{\varnothing}_a$, and π_a by $\tilde{\pi}_a$, Eqs. (11) and (15), respectively, into the canonical Hamiltonian, Eq.(3), as follows

$$\tilde{H} = \frac{1}{2}\pi_a\pi_a - \frac{1}{2}\frac{(\varnothing\pi)^2}{\varnothing^2} - \frac{1}{2}\partial_i\varnothing_a\partial^i\varnothing_a \tag{16}$$

From Eqs. (2) and (3) we have that

$$\frac{\delta\tilde{H}_c}{\delta\pi_b} = \dot{\varnothing}_b \tag{17}$$

And substituting this result into Eq. (15), we can write that

$$\pi_b - \frac{\varnothing_a\pi_a}{\varnothing^2}\varnothing_b = \dot{\varnothing}_b \tag{18}$$

Multiplying this last equation by $\varnothing_b$, we can see that

$$\varnothing_a\dot{\varnothing}_a = 0 \tag{19}$$

which is not a constraint considering the new Hamiltonian. Hence, using this constraint to construct the new Lagrangian we know that

$$\tilde{L} = \tilde{\pi}\dot{\tilde{\varnothing}} - \tilde{H} \tag{20}$$

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Using Esq. (16) and (19) we obtain that

$$\tilde{L} = \frac{1}{2}\partial_\mu \varnothing^a \partial^\mu \varnothing^a \tag{21}$$

And the gauge invariant condition is

$$\delta\varnothing_a = 0 \tag{22}$$

Now, let us follow the extended GU technique in order to consider the infinitesimal gauge transformations generated by symmetry generator T_2 given by Eq. (4), i.e., $T_2 = \varnothing_a(x)\pi_a(x)$. The gauge transformations are

$$\delta\varnothing_a(x) = \varepsilon\{\varnothing_a(x), T_2(y)\} = \varepsilon\varnothing_a(y)\delta(x-y) \tag{23}$$

The gauge transformations are

$$\delta\varnothing_a(x) = \varepsilon\{\varnothing_a(x), T_2(y)\} = \varepsilon\varnothing_a(y)\delta(x-y) \text{ and}$$

$$\delta\pi_a(x) = \varepsilon\{\pi_a(x), T_2(y)\} = -\varepsilon\pi_a(y)\delta(x-y) \tag{24}$$

In order to construct gauge invariant fields, we can write $\tilde{\varnothing}_a(x)$ as

$$\tilde{\varnothing}_a = \varnothing_a\left(1 - \frac{1}{2}\frac{\varnothing^2-1}{\varnothing^2} - \frac{1}{8}\frac{\left(\varnothing^2-1\right)^2}{\varnothing^4} + \cdots\right) \tag{25}$$

Hence, this last result in Eq. (25) suggest that we can write the last equation in the following convenient form,

$$\tilde{\varnothing}_a = f\left(\varnothing^2\right)\varnothing_a \tag{26}$$

From invariance condition $\delta\tilde{\varnothing}_a(x) = 0$ we can obtain $f\left(\varnothing^2\right)$, which is

$$f\left(\varnothing^2\right) = \frac{1}{\sqrt{\varnothing^2}} \tag{27}$$

Substituting it in Eq. (26) we have that

$$\tilde{\varnothing}_a = \frac{1}{\sqrt{\varnothing^2}}\varnothing_a \tag{28}$$

By making the same procedure for $\tilde{\pi}_a$ we have that

$$\tilde{\pi}_a = \pi_a\left(1 + \frac{1}{2}\frac{\varnothing^2-1}{\varnothing^2} + \frac{3}{8}\frac{\left(\varnothing^2-1\right)^2}{\varnothing^4} + \cdots\right) \tag{29}$$

which, analogously, suggests that $\tilde{\pi}_a$ can be written such as

$$\tilde{\pi}_a = g\left(\varnothing^2\right)\pi_a \tag{30}$$

from the invariance condition $\delta\tilde{\pi}_a = 0$, we can obtain $g\left(\varnothing^2\right)$

$$g\left(\varnothing_a^2\right) = \sqrt{\varnothing^2} \tag{31}$$

which, substituting into Eq. (6), we have that

$$\tilde{\pi}_a = \sqrt{\varnothing^2}\,\pi_a \tag{32}$$

Therefore, the gauge invariant Hamiltonian is written as

$$\tilde{H} = \frac{1}{2}\tilde{\pi}_a\tilde{\pi}_a - \frac{1}{2}\partial_i\tilde{\varnothing}_a\partial^i\tilde{\varnothing}_a \tag{33}$$

where $\tilde{\varnothing}_a$ and $\tilde{\pi}_a$ are given by Eqs. (5) and (8) respectively.

Conclusion

In this paper, we have used the so called extended GU formalism which, by gauging the original phase space variables of a constrained system, we can carry out the transformation (conversion) of a second−class system into a first-class one and thereby, a gauge invariant theory is obtained. In other words, considering the case of a system with two second−class constraints, one of the constraints will be chosen in order to shape the scaled gauge symmetry generator while the other will be discarded. The discarded constraint can be used to construct a series for the gauge invariant fields. Consequently, any functions of the gauge invariant fields are gauge invariant quantities. We apply our formalism to the O(N) non linear sigma model where new results are obtained a gauge invariant Hamiltonian was obtained in Eq. (33). As a perspective for future research, this extended GU formalism can also be used to study the non−Abelian version of the Chern−Simons theory.[13,20]

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Conflict of interest

Author declares that there is no conflict of interest.

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