

Figure-eight knots in the general-angle Rubik's snake tube model

Abstract

For standard tubes defined in Rubik's snake blocks, this paper proves that a collision free nontrivial tube snake knot with angle word $t, -t, t, -t$ requires at least 16 blocks, and that the figure-eight knot 4_1 attains this bound. The argument combines total curvature, a complete certified cover of the 12 block closure locus, and an exact 16 block construction. We also give a 15 block 4_1 construction when removing the angle word constraint above. When the tube radius is small enough, we give an exact twelve-block figure-eight with the same alternating word that is collision free. Total curvature then shows that twelve is the exact minimum within the alternating class in this thin-tube regime.

Keywords: Rubik's snake, general rotation angles, figure-eight knot, shortest path

Volume 12 Issue 1 - 2026

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Received: October 15, 2026 | **Published:** October 27, 2026

Introduction

Originally introduced as a folding toy,¹ Rubik's snake consists of a serial chain of triangular-prism blocks joined at rotatable interfaces. Its simple local rules give rise to a large variety of spatial configurations, making it a useful setting for questions in geometry and mechanism design. The chain has been used as a conceptual model of protein folding² and studied as a reconfigurable mechanism,^{3,4} with related applications in modular robotics and geometric capture.^{5,6}

The mathematical study of the snake has progressed from representing configurations to designing them under geometric constraints. In⁷ the authors encoded three-dimensional configurations by one-dimensional rotation sequences and developed methods for following prescribed space curves and producing self-similar structures by embedding. In⁸ the authors established mathematical results for the traditional discrete-angle model. General rotation angles were then allowed in⁹ and matrix methods were used to study closure and repeated-angle configurations. Theorems on palindromic, periodic and Möbius snakes¹⁰ provided further tools for symmetric design, while box-shaped paths were studied with explicit counting formulas.¹¹ A comic book also makes the subject accessible to younger readers.¹²

Work on knotted configurations includes trefoils,¹³ other prime knots,^{14,15} composite knots,^{16,17} and torus knots.¹⁸ A recent paper¹⁹ improved several constructions through their local geometric structures. In particular, for the figure-eight knot,¹⁴ provided a 46-block construction which was improved to 44 blocks using the alternating word $(t, -t, t, -t)$ structure in.¹⁶ The joint angles are limited to integer multiple of 90 degree rotations. In,¹⁵ it also proposed tube-based roller-coaster layouts, with short paths motivated by the reduction of construction cost while retaining a prescribed knot structure. Similar motivation might be considered in affordable jewelry design.

In this paper, we address the figure-eight knot shortest path problem in a circular-tube model with general joint angles. Its solid geometry and angle constraints differ from those of the discrete-angle triangular-prism model, so their block counts are not directly comparable. With general angles allowed (infinitely many choices per joint instead of 4 choices per joint) and the tube block just a subset of the original snake block (less chance for collision), naturally we expect to have smaller block counts.

We combine Milnor's total-curvature theorem²⁰ with a complete exclusion of twelve-block closures to prove that the alternating word $(t, -t, t, -t)$ requires at least sixteen blocks for a nontrivial knot in the tube model with standard tube radius. An exact radical figure-eight construction attains this bound. We also give an exact fifteen-block figure-eight outside the alternating class, defined by rational parameters and uniquely isolated algebraic roots. The decimal sequences are provided only for the reader's convenience. With tube radius small enough, we provided twelve-block construction and showed it is the minimum in $(t, -t, t, -t)$ structure with such radius.

The rest of the paper is as follows: Section 2 specifies the model, Section 3 proves the standard-radius lower bound, and Sections 4 and 5 give the standard-radius constructions. Section 6 gives the twelve-block thin-tube construction. We conclude in Section 7.

Geometric model

In,⁹ the tube version of the Rubik's snake was introduced for general rotation angles. Normalize the shortest side length of a Rubik's snake block to be one. Each centerline block is a quarter circle of radius $R = 1/2$. Its tube consists of normal disks of radius $r = (\sqrt{2} - 1)/2$, the closest distance from the centerline to the rectangular face of the original Rubik's snake.

Tangents agree at joints, and the centerline is embedded. Adjacent blocks share their joint disk without interior overlap. For nonadjacent closed tube blocks B_i, B_j , collision freedom means

$$\text{int}(B_i) \cap \text{int}(B_j) = \emptyset.$$

Boundary contact is allowed. The two constructions below satisfy the stronger condition of strictly positive clearance between nonadjacent blocks. All lengths use the stated normalization. Sections 3–5 keep this standard radius fixed. Section 6 retains the same centerline geometry and transfer matrix but takes $0 < r \leq 1/100$.

Let θ_j denote an angle in radians, $d_j = 2\theta_j/\pi$ its value in units of 90 degrees, and $q_j = \tan(\theta_j/2)$ its half angle tangent. Thus

$$\theta_j = 2 \arctan q_j, \quad d_j = \frac{4}{\pi} \arctan q_j.$$

We follow the rotation convention of the knot constructions in [13,14]: 0 leaves the straight original Rubik's snake unchanged, 1 is a 90-degree turn to the right, and 3 is a 90-degree turn to the left. Positive general angles turn to the right. The transfer matrix below fixes the coordinate convention.

Combining rotation with translation, the one-block transfer matrix and accumulated motion are

$$T(q) = \begin{pmatrix} 0 & 1 & 0 & 1 \\ c & 0 & s & 0 \\ s & 0 & -c & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad c = \frac{1-q^2}{1+q^2}, \quad s = \frac{2q}{1+q^2},$$

$$G_0 = I_4, \quad G_{j+1} = G_j T(q_{j+1}), \quad G_j = \begin{pmatrix} Q_j & p_j \\ 0 & 1 \end{pmatrix}.$$

The upper-left three-by-three block gives the frame orientation, and the last column gives the position. The first arc is blow and subsequent arcs are its images under the preceding motions: $\gamma_0(u) = (1/2, -1/2, 0) + (-1/2, 0, 0) \cos u + (0, 1/2, 0) \sin u, 0 \leq u \leq \pi/2$.

The angle word includes the closing joint. Exact closure of both position and frame is the single condition $G_N = T(q_1) \cdots T(q_N) = I_4$. Angles are taken modulo 2π ; infinite half-angle tangents are represented in a projective way.

Exact minimum under the alternating constraint

Theorem 1. In the model above, among collision free nontrivial closed snakes with angle word $(t, -t, t, -t)$, where t is a finite angle word and minus acts entry wise, the minimum number of blocks is 16. An exact 16 block 4_1 attains this minimum.

If t has m entries, then $N = 4m$. Each block contributes $\pi/2$ to total curvature (the notation is TC) and the tangent continuous joints add no turning angle. Hence

$$\text{TC}(\Gamma) = \frac{N\pi}{2}.$$

The Fáry–Milnor theorem states that a nontrivial embedded closed curve has total curvature strictly greater than 4π [20]. Thus $N > 8$. The only remaining length below 16 is $N = 12$.

For twelve blocks, the first subword has three angles. Set $g = T(q_1)T(q_2)T(q_3)$ and let $S = \text{diag}(1, 1, -1, 1)$ be the homogeneous reflection matrix. Write $\mathcal{H} = gS = \begin{pmatrix} H & v \\ 0 & 1 \end{pmatrix}$, where H is

the three-by-three linear part and v is the translation.

The 3-by-3 matrix H is orthogonal with determinant -1 . Closure of the full grid motion requires the 4-by-4 matrix $\mathcal{H}^4 = I_4$. If H^2 is identity, the translation condition gives $2(I + H)v = 0$, hence $\mathcal{H}^2 = I_4$. The curve then retraces after six blocks and is not an embedded twelve block curve. Every remaining case satisfies

$$\text{spec}(H) = \{-1, i, -i\}, \quad \text{tr } H = -1.$$

Conversely, these eigenvalues imply $H^4 = I$ and $I + H + H^2 + H^3 = 0$, so the translation closes automatically as well. This handles position and frame, not merely endpoint position.

For finite nonzero half-angle tangents (u, v, w) , nonnegativity of the three quadratic discriminants gives $uw \geq 0$, $uv \leq 0$, $vw \leq 0$. Thus an overall mirror reflection puts them in the form $(A, -B, C)$, $A, B, C \geq 0$.

At zero or infinity we use the homogeneous equation directly; these boundary cases remain in the charts below. The closure equation is

$$F(A, B, C) = (A - B)^2 C^2 - 2(A + B)(AB + 1)C + (1 - AB)^2 = 0.$$

It is symmetric in A, B, C . Parametrize the first two magnitudes by

$$A = x^2 \text{ or } x^{-2}, \quad B = y^2 \text{ or } y^{-2}, \quad (x, y) \in [0, 1]^2.$$

Writing $A = a/d, B = b/e, C = c/f$, the third coordinate satisfies

$$Pc^2 + Qcf + Rf^2 = 0,$$

$$P = (ae - db)^2, \quad Q = -2(ae + db)(ab + de), \quad R = (de - ab)^2,$$

with discriminant

$$Q^2 - 4PR = 16abde(a^2 + d^2)(b^2 + e^2).$$

Its square root is $4xy\sqrt{(1+x^4)(1+y^4)}$. Four charts for the first two coordinates, each retaining both projective real root branches, give eight closed unit squares. The coefficients P, Q, R cannot all vanish. Roots are retained with multiplicity, and the projective representation includes zero, infinity, and double roots. Consequently these squares cover the entire closure locus. We sweep them rigorously below.

Bisection of the eight closed parameter squares gives a complete cover by 62,831 certified rectangles. Each local test holds throughout its rectangle, including the boundary. Interval endpoints are integer multiples of 10^{-40} , with outward rounding after arithmetic operations. The certification counts are listed below.

Certification route	Regions
Strict interior collision	6,252
Embedded isotopy to an unknot	10,438
Symmetry transfer within the root charts	44,715
One-maximum height function	397
Monotone polar angle	1,029
Total	62,831

We explain these row by row in details below.

A collision certificate places a common rational point strictly inside two nonadjacent tube blocks.

For the isotopy test, choose an exact algebraic reference closure (denoted with $*$) with a regular diagram of at most two crossings, and interpolate corresponding arc coefficients:

$$r_{j,s}(u) = (1 - s)r_j(u) + s r_j^*(u), \quad 0 \leq s \leq 1.$$

Interval bounds for rational Bezier control triangles separate nonadjacent subarcs; strict tangent projections handle neighboring subarcs. Every intermediate curve is therefore a closed embedding, so this local isotopy continuation preserves the unknot type. Intermediate curves need not remain rigid snake blocks.

Symmetry transfer is allowed only when the entire transformed rectangle is covered by earlier certified rectangles, giving an acyclic dependency order. Independent replay verifies every local certificate and reconstructs the bisection trees from all eight full squares, requiring both children at each split. No region remains unresolved. The counts describe certification routes, not distinct geometric configurations.

The height test certifies exactly two cyclic sign changes of the joint tangent heights (the coordinate along a fixed direction), hence

one maximum and one minimum. The test proves triviality for an embedded centerline.

The polar test certifies a strict angular-velocity sign and confines the first three projected arcs to an open half-plane through the axis. They turn through exactly a quarter revolution, so the full curve is a one-strand closed braid. The test also proves triviality for an embedded centerline.

Every twelve-block closure therefore has strict interior overlap or is an unknot when embedded. Boundary contact is allowed throughout. Together with the total-curvature bound, this proves the lower bound of sixteen blocks.

Exact sixteen block construction for $(t, -t, t, -t)$

Fix three rational numbers, the closure equation can determine the fourth one exactly within t part. We just need a choice that has no collision and has the figure-eight knot type. This can be done by a least squares search combined with finding nearby simple rational numbers for the first three. Also, as a solution we found below is so simple, an exhaustive search with small integers for numerator and denominator in fact also works.

$$q = \left(-\frac{5}{3}, -7, \frac{1}{3}, -\frac{378 + 50\sqrt{1598}}{2089} \right), \quad t_j = 2 \arctan q_j.$$

The complete angle word is exactly $(t, -t, t, -t)$, with half angle tangent word $(q, -q, q, -q)$. These are exact parameters, not rounded search values.

Set $D = -(378 + 50\sqrt{1598})/2089$. It satisfies $2089D^2 + 756D - 1844 = 0$.

Define the four-block motion and its composition with the reflection by

$$g_4 = T(q_1)T(q_2)T(q_3)T(q_4), \quad \mathcal{A} = g_4 S = \begin{pmatrix} A_4 & v_4 \\ 0 & 1 \end{pmatrix}.$$

Substitution of the exact parameters reduces $\text{tr } A_4 = -1$ to the quadratic above. The orthogonal matrix A_4 has determinant -1 and eigenvalues $\{-1, i, -i\}$. Hence $I_3 + A_4 + A_4^2 + A_4^3 = 0$ and $\mathcal{A}^4 = I_4$, proving exact position and frame closure. The spectrum belongs to the rotation-reflection, not to the four-block rigid motion alone.

Interval certification covers all $16 \cdot (16-1-2)/2 = 104$ nonadjacent tube pairs and proves a strict surface clearance greater than $1/100$. Each individual curvature radius exceeds the tube radius. Adjacent arcs lie on opposite sides of the common joint plane, except for the joint disk, so no internal overlaps are omitted.

Figure 1 shows the standard figure-eight form with numerical solution based on the exact construction. All blocks have the same shape and size. Each color consists of four blocks. Gauss code is checked to ensure the knot type. This exact construction attains the lower bound and proves Theorem 1.



Figure 1 The sixteen block construction viewed along its order four rotation-reflection axis. Red, yellow, blue, and green mark blocks 1 to 4, 5 to 8, 9 to 12, and 13 to 16, corresponding to $t, -t, t, -t$. Neighboring subwords differ by a quarter turn in projection; their spatial transformation also reverses the axial coordinate. This is not a pure spatial quarter turn. The tube has the standard radius.

For the reader's convenience, the four angles in units of 90 degrees are approximately $d \approx (-1.311917, -1.819331, 0.409666, -1.081928)$.

The full approximate angle word is $(d, -d, d, -d)$. These rounded values are for display only; the radical parameters above define exact closure.

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Exact fifteen block construction without the word constraint

Fix the first nine half angle tangents as exact rational numbers, and define the final six as the unique root of an integer polynomial closure system in the specified isolating box. This can give a closed fifteen block 4_1 with certified collision freedom at the standard tube radius with a least squares search.

The detailed construction corresponding to the above outline is rather technical and therefore omitted. The main purpose is to explain that when the $(t, -t, t, -t)$ pattern is not required, 16 is no longer the minimum for figure-eight knot with standard radius for tubes.

For the reader's convenience, the approximate angles in units of 90 degrees are

$$\begin{aligned} (d_1, \dots, d_5) &\approx (-1.466151, 0.715798, -1.054379, 1.287946, 1.872742), \\ (d_6, \dots, d_{10}) &\approx (-0.402306, 1.192737, -1.624474, -1.650348, 0.019758), \\ (d_{11}, \dots, d_{15}) &\approx (-1.276428, 1.567591, 1.730078, -0.715436, -1.987496). \end{aligned}$$

The rounding is used only for display. Figure 2 shows the construction. Each color represents one block. They all have the same shape and size. Gauss code is checked to ensure the knot type.



Figure 2 The fifteen block construction, with one color per block. The first nine half angle tangents are rational; the remaining six form the unique algebraic root in the specified isolating box. The image uses numerical approximations to this exact definition.

6 Twelve blocks suffice for thin tubes

The standard-radius exclusion in Section 3 does not exclude knotted centerlines whose tubes overlap only because of their thickness. We now give a twelve-block figure-eight in the same alternating class, with a smaller positive tube radius.

Theorem 2. For every fixed radius $0 < r \leq 1/100$, the minimum number of blocks for a nontrivial closed snake with word $(t, -t, t, -t)$ is twelve. This minimum is attained by a figure-eight knot with t consisting of three angles.

6.1 Exact parameters and closure

Write $t = (\alpha, \beta, \gamma)$. As in Section 3, write its half-angle tangents as $(A, -B, C)$, with $A, B, C \geq 0$. Choose

$$A = \frac{7}{8}, \quad B = \frac{20}{9}, \quad 9409C^2 - 94552C + 4624 = 0,$$

where C is the larger root. Explicitly,

$$C = \frac{94552 + \sqrt{94552^2 - 4 \cdot 9409 \cdot 4624}}{18818} \approx 9.999957277554.$$

Set $\alpha = (4/\pi) \arctan A$, $\beta = -(4/\pi) \arctan B$ and $\gamma = (4/\pi) \arctan C$. The complete twelve-angle word in unit of 90 degrees is (the superscript 2 means the sequence is repeated twice)

$$(\alpha, \beta, \gamma, -\alpha, -\beta, -\gamma)^2.$$

For the reader's convenience, in unit of 90 degrees, the above sequence is approximately

$$(0.915243, -1.461606, 1.873097, -0.915243, 1.461606, -1.873097)^2$$

The quadratic is the closure equation already derived in Section 3. Substituting the positive magnitudes $(7/8, 20/9, C)$ into that equation gives

$$F(7/8, 20/9, C) = \frac{9409C^2 - 94552C + 4624}{5184} = 0.$$

By the rotation-reflection calculation in Section 3, the three-block motion composed with the homogeneous reflection has linear eigenvalues $-1, i$ and $-i$. Its fourth power has identity linear part, and the sum $I + H + H^2 + H^3$ vanishes. The complete alternating word therefore closes exactly in both position and frame. Equivalently, the six-block motion is a half-turn with zero axial translation, and repeating it twice gives the identity.

Positive tube clearance and knot type

At radius $r = 1/100$, directed interval arithmetic certifies a centerline distance strictly greater than 0.021 for all $12 \cdot (12-1)/2 = 54$ nonadjacent arc pairs. Hence every corresponding pair of tube pieces has surface clearance strictly greater than 0.001. The rational arc parameter and its speed bound give a full-arc certificate.

Figure 3 displays this exact construction using numerical approximation. Locally each color represents one block. Gauss code is used to check the knot type.



Figure 3 The exact twelve-block thin-tube figure-eight at radius $1/100$, viewed along its rotation axis. Three colors repeat in block order through each of the four three-block subwords.

Minimality and scope

Every tangent-continuous chain of N quarter-circle blocks has total curvature $N\pi/2$. A nontrivial embedded knot has total curvature strictly greater than 4π [20], so $N > 8$. The alternating word has $N = 4m$; hence $N \geq 12$. The construction attains twelve and remains collision free when its radius is decreased. This proves Theorem 2 for every $0 < r \leq 1/100$. It does not establish twelve as the unrestricted figure-eight minimum, nor identify the largest radius admitting a twelve-block alternating figure-eight.

Conclusions

At the standard radius $r = (\sqrt{2} - 1)/2$, the minimum number of blocks for a nontrivial knot with the alternating word $(t, -t, t, -t)$ is exactly sixteen, attained by the explicit figure-eight construction. The exact fifteen-block example lies outside this class. It establishes an unrestricted upper bound of fifteen for the figure-eight knot in

this standard tube model. Both constructions have strictly positive nonadjacent surface clearance. For every fixed radius $0 < r \leq 1/100$, the constrained minimum drops to twelve and is attained by the new exact figure-eight. The different minima reflect different tube radii and are fully consistent.

Acknowledgment

The author thank Wen Li for an inspiring discussion and acknowledge ChatGPT's discussion and assistance with the preparation of this manuscript. S. Hou's research is partially supported by the Walter Koss Endowed Professorship. The title of the Professorship is made available through the State of Louisiana Board of Regents Support funds.

Conflicts of interest

There are no conflicts of interest.

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