Research Article

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An analytical form for a rigorous nonlinear model of the dynamics for systems with geometric constraints and its use by applying the results of control theory to a complete solution of stabilization problems

Abstract

For the first time, a single article presents not only a complete justification for a method of deriving an analytical form for a nonlinear dynamic model and a step-by-step algorithm of deriving it for systems with geometric constraints, but also the results of its effective application to a comprehensive consideration for stability and stabilization problems for a specific technical device with parallel kinematics (Ball and Beam). The rigor and generality of abstract theoretical methods for deriving an analytical model in the form of vector-matrix equations for the first time for non-free systems with geometric constraints created the possibility for a justified application of the general nonlinear stability theory results and mathematical control theory. In particular, using the theory of critical cases, sufficient conditions for the solvability of the genera stability problem for a given system configuration are obtained. For the general stabilization problem of configuration under incomplete information, a theorem on sufficient conditions for stabilization to asymptotic stability with respect to all variables is proved. The coefficients of the linear stabilizing control and the estimation system are determined by solving the corresponding linear-quadratic problems using Krasovskii's method. It is demonstrated how transient processes in actuators can be included in the analysis by significantly reducing the model's dimensionality. The entire process of using the proposed approach is applied to a complete solution to the stabilizing problem of two different equilibrium positions for the two different selection of dependent coordinate options for the Ball and Beam rig as a mechatronic system with a nonlinear geometric constraints. An additional voltage on the armature winding of the actuator commutator motor is used as a control. Both analytical and numerical modeling of the controlled dynamics of a specific rig variant are performed. Graphs of the transient processes are provided.

Keywords: mathematical dynamics model, geometric constraints, stability, stabilization, linear-quadratic problem

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Introduction

On the peculiarities of applying control theory for control problems in non-free systems

In many problems in modern engineering practice (in particular, the control of multi-link manipulators and other mechatronic systems with parallel kinematics,[1–6] certain constraints are imposed on the dynamics of an object, expressing certain nonlinear (continuously differentiable) dependencies between the parameters determining its configuration. As early as the 18th century, Lagrange noted that the solution to the problem, using these conditional equations, can be reduced to considering equations only for generalized coordinates (independent parameters in the minimum necessary number that uniquely determine the configuration of the object). However, he also pointed out that cases often arise where, to avoid unnecessary complication of the calculation, it becomes necessary and appropriate to retain a larger number of variables than the minimum required to determine the system's configuration. Thus, the need to study the dynamics of unfree mechanical systems with geometric constraints naturally arises. In studying the dynamics of systems with geometric constraints, In studying the dynamics of the systems with geometrical constraints, beginning with Lagrange and continuing[2–7] to the present day, differential equations with undetermined multipliers are used, to which constraint equations are added. The total number of equations becomes equal to the sum of the number of variables and the number of constraints. If the study does not involve determining the constraint reactions, these multipliers are eliminated from the resulting equations using various methods.

Since the last quarter of the 20th century, manipulators with geometric constraints have become one most widely used automated devices classes with advanced control loops (see for example).[8–12] The current level of development of programmable controller components, information technology, and the actuators technology creates the potential for implementing virtually any algorithm offered by control theory. However, modern results from mathematical control theory[13–16] have not yet found application in their control systems. This is primarily due to the lack of effective, rigorously validated methods for constructing nonlinear mathematical models that take into account the all component of the object dynamics under study, including actuators.

This situation, in our opinion, is due to the fact that intensive research into the methods development for the dynamics mathematical modeling for modern technical devices with geometric constraints began, driven by the needs of technical practice, only after this type devices had become widely used.[8–12,17–25]

The vast majority of authors these studies were completely unfamiliar with the rigorous analytical mechanics methods for non-free systems[3,6,7] and the general results of their application, published by this article authors both in terms of the abstract theoretical justification of the rigorous method[26–37] and its effective application to current problems of studying the nonlinear dynamics of specific modern technical devices with geometric constraints.[26,38–45] Until the last quarter of the twentieth century, this theoretical mechanics specific area had no technical application and was the study subject of by a very limited circle of theoretical specialists.

Therefore, this theoretical mechanics area was never included in standard engineering education. As a result, the needs engineering practice necessitated modeling the controlled dynamics of a specific, real, and widely used classes technical device, but there were no validated methods for such modeling.

Remark 1.1. The authors believe that an instructive this situation example is the history of modeling a widely used device, the Ball and Beam educational laboratory setup. In the first publications,[46–48] despite the obvious geometric constraints, attempts were made to apply Lagrange's equations of the second kind in independent coordinates, which are impossible to introduce in this device. Initially, attempts were made to apply Lagrange's equations of the second kind by linearizing the geometric constraint (a nonlinear trigonometric equation from the Pythagorean theorem for a constant-length lever connecting the drive output wheel and the right end of the chute). Moreover, in addition to the generally incorrect linearization of the constraint (see the example of such an operation), an incorrect model was then developed: the dependent coordinate was excluded, and its velocity remained a model variable. When the inadequacy of this model became clear, a rigorous nonlinear equation for the geometric constraint was first presented.[49] The possibility of modeling the dynamics of this setup using Lagrange's equations of the second kind was then considered. The existence of a procedure implementing this operation was formally established.[49] However, the presented proof clearly demonstrates the impossibility of its practical application.

Remark 1.2. It should be noted that the strictly proven existence of some abstract-theoretical operation that formally simplifies the study of a complex general problem does not always lead to practical simplification. A similar situation occurs in the theory of critical cases in non-linear stability theory,[50–52] when, in addition to roots with zero real parts, the characteristic equation of a first-approximation system necessarily has roots with negative real parts. Linear algebra reveals the existence of a non-singular linear change of variables, which results in the general equations of perturbed motion acquiring a so-called special form of critical case theory: the linear approximation for critical and non-critical variables is separated.

All general results for this class of critical cases are formulated in terms of new variables, for which there is no connection with the original ones. No one has even attempted to establish this connection. This is precisely why critical case theory has not found adequate application. Only A.M. Lyapunov conducted a full analysis with a linear change in the general case of a single zero root. Further, following Lyapunov's algorithm, the Aizerman-Gantmacher[53] substitution was proposed, which allowed one to obtain a special form in the simplest situation of studying the stability of equilibrium positions of non-holonomic systems with linear homogeneous constraints.[21–23]

Remark 1.3. A similar situation occurs when attempting to model the dynamics of systems with geometric constraints by solving an inverse kinematics problem.[20–25] Here, it becomes necessary to eliminate situations of Jacobian matrix degeneracy. Intensive research is underway on applying the Lie group apparatus to this problem. Formal considerations have demonstrated the effectiveness of this approach. However, as the authors themselves note.[25]: "Lie group integration methods, operating directly on the configuration space Lie group, are incompatible with standard formulations of the EOM, and cannot be implemented in existing MBS simulation codes without a major restructuring".

Consequently, to this day, the dynamics of systems of this class are traditionally modeled using Lagrange equations with constraint multipliers. The variables of such a model are all coordinates, all velocities (including dependent velocities) and unknown Lagrange multipliers. The dimensionality of the resulting model significantly exceeds the number of freedom degrees. Reducing its dimensionality by eliminating multipliers from consideration is accomplished either by a method involving differentiation of the BR constraint equations (this approach is discussed in more detail below) or by solving the inverse kinematics problem. Obviously, such methods can only model the dynamics of a specific technical device, and it is impossible to obtain an analytical form of a rigorous nonlinear mathematical model.

General results of mathematical control theory cannot be applied to the dynamic models of systems with geometric constraints obtained by such methods. In particular, it is impossible to estimate the minimum dimension of the control vector (thereby minimizing the number of actuators, and, in the case of incomplete state information [13-15], the number of sensors). It should be especially noted that practically in none of the published works on the study of the dynamics of systems with geometric connections does the mathematical model include a description of the dynamics of the actuator.

Remark 1.4. For over 50 years, the authors have been developing the application of fundamental results of analytical mechanics, nonlinear stability theory, and mathematical control theory to solving the most complex general stability and stabilization problems of steady-state (including non-isolated) motions of non-free mechanical systems with various constraints types ю From the very beginning, the method was focused on studying the stability and stabilization of motions whose stability is possible only in critical cases. Therefore, the main challenge was to develop a rigorous modeling method that, in addition to identifying a first approximation, also enables (necessary in such a situation) the analysis of the nonlinear terms structure. The basis of the developed approach is a method, created using analytical mechanics methods, for obtaining a general nonlinear the class under study of mathematical model forf systems in variables (Lagrange, Routh, Hamilton) that best correspond to the nature of the system.

It is this analytical form of the mathematical model, in the form of nonlinear vector-matrix equations first obtained by the authors (e.g., the Voronets equation in Routh variables, the equations of nonlinear dynamics in Routh variables in the absence of cyclic coordinates, etc.), that creates the possibility for effectively applying the results of the critical cases theory and the mathematical control theory. It should be noted that, unlike the abstract theory of critical cases, here, through the effective application and development of results from analytical mechanics (through a rational choice of equation forms and variable types), it was possible to obtain the necessary special form of the equations of disturbed motion in terms of the initial variables.

It was the generality and rigor of obtaining the analytical form of the nonlinear mathematical model that determined the effectiveness of applying the results of mathematical control theory with incomplete information. Using the qualitative theory of optimal processes, it became possible to estimate the minimum required dimensions of the control vector and the vector of measurement information about the state.[14] It should be noted that this rigorously justified approach has been effectively applied, in particular, at Lomonosov Moscow University and the Institute of Mechanics of Moscow State University by solve problems of stability and stabilization of non-holonomic systems.

Remark 1.5. However, stability and stabilization problems for systems with geometric constraints required rigorous analysis, since the results obtained for non-holonomic systems could not be transferred to these systems as a particular case under the condition of differential constraints integrability. The fact is that geometric constraints impose constraints not only on the initial perturbations (as in non-holonomic systems), but also restrict the dynamics throughout the entire operation of the system. Therefore, a new conditional stability problem arises here, distinct from the definition of conditional stability in the sense of Lyapunov.

It is the application of analytical mechanics results (the choice of variable types and equation forms for a nonlinear model) that obtained a special form of critical case theory in the initial variables. The analytical form of this rigorous nonlinear model serves in the general case as the basis for the subsequent application of abstract methods not only from nonlinear stability theory but also from modern mathematical control theory under incomplete information. In particular, it becomes possible for the first time to minimize our intervention in the behavior of a controlled object through a qualitative analysis of its behavior in the absence of control: thereby reducing resource costs for real-time generation and release of control actions and implementing control while simultaneously increasing the reliability of the control loop.

The authors' obtained abstract theoretical and applied, rigorously substantiated results have been published in dozens of papers and have been repeatedly presented and discussed, not only at conferences and seminars. These findings also included detailed personal discussions with leading experts, including academicians of the USSR Academy of Sciences and the Russian Academy of Sciences. This article presents, in the most concise form possible, both the methodological foundations and the algorithms for using the developed method, particularly in the modeling of nonlinear dynamics of parallel manipulators, in a form accessible to technical practitioners. Let us note once again that the practical application of the developed methods does not require the practicing engineer to have a complete understanding of their strict fundamental abstract-theoretical foundations. The algorithmic simplicity and efficiency of the application of the proposed approach are convincingly demonstrated by a detailed presentation of a complete study of the nonlinear dynamics of the Ball and Beam stand.[44–53]

Some historical background for the need to develop a previously developed general approach in relation to systems with geometric constraints

The first application of the general results of the abstract theoretical consideration of stability and stabilization problems published by the authors[26,27] based on the principle of reduction of nonlinear stability theory to modeling the dynamics of systems with geometric constraints was a complete rigorous study of the dynamics of the stand Ball and Beam The problem of stabilizing a given stand configuration by the time the authors began to consider it from the standpoint of nonlinear stability theory and mathematical control theory (including the model of a drive with a DC motor) had been investigated in many published papers, none of which succeeded in obtaining an adequate mathematical model. For this reason, the actual behavior of the rig under the action of control constructed by applying control theory to an inadequate model differed radically from its model behavior. Moreover, despite the fact that a complete rigorous solution to the problem was published quite a long time ago (which, apparently, remained unnoticed by technical specialists), research on the application of control theory results using less adequate mathematical models of its dynamics continues to this day (Figure 1).[54,55]

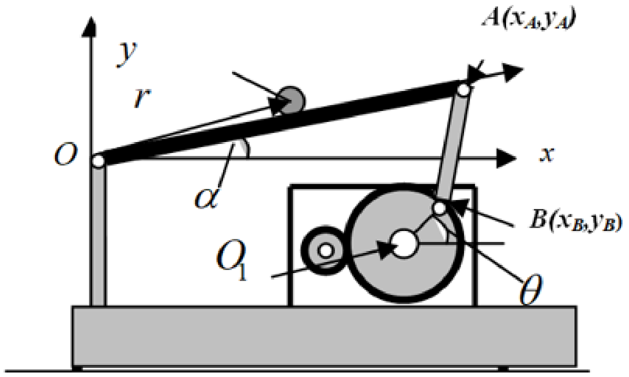

Figure 1 Scheme of the "Ball and Beam" stand.

In this system, the electric drive, due to the inclination of the chute OA, can roll the ball C to any predetermined position on the chute and stabilize this equilibrium. The chute is connected on one side at point O to a fixed supporting post, and on the other to a movable lever AB. The motion of the lever is controlled by a DC motor. A nonlinear geometric relationship is imposed on the system: the distance between points $A(x_A, x_B)$ and $B(x_B, y_B)$ is constant:

$$(\mathrm{L}(\cos\alpha - 1) + d(1-\cos\theta))^2 + (\mathrm{L}\sin\alpha + l - d\sin\theta)^2 = l^2 \tag{6}$$

($L = OA$ − length of the trough, $l = AB$ − length of the lever, d − radius of the output drive wheel).

Starting with,[46] in all studies of the dynamics of the Ball and Beam stand with a geometric constraint[46–48] (except,[49]), when constructing a mathematical model, immediately, starting with[46] and up to now,[54,55] they completely unjustifiably move from a nonlinear Equation (6) to a linear dependence $\alpha \approx \frac{d}{L}\theta$ and exclude the dependent coordinate from consideration. Despite the fact that the incorrectness of such an approach has already been discussed in detail earlier,[16–18] such a model is still used, and even referring to the article.[39] Therefore, the author considered it appropriate to once again present in detail an alternative algorithm for obtaining, using strict methods of analytical mechanics, non-free systems of complete nonlinear models, the dimensions of which are significantly smaller than those traditionally used.

Obviously, the equilibrium position is only possible at $\alpha=0$, when the trough is horizontal. Setting $\alpha = 0$ and assuming that $0 \le \theta \le 2\pi$, we obtain θ corresponding to all equilibrium positions of the system:

$$\theta_0 = 0; \theta_1 = 2\arcsin\left(\frac{l}{\sqrt{l^2 + d^2}}\right).$$

The corresponding configurations are shown in Figure 2 and Figure 3.

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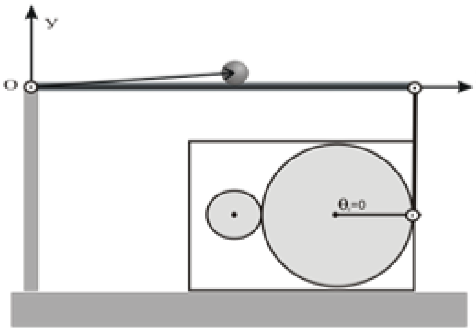

Figure 2 Zero equilibrium position.

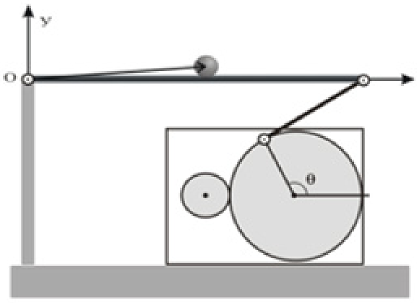

Figure 3 Non-zero equilibrium position.

Note that without the application of a constant torque generated by the drive motor, equilibrium does not exist in the Ball and Beam design considered here. The constant torque for a given equilibrium position is a non-zero non-conservative positional force, and the non-zero constant is also contained in the potential force. Consequently, in the equations in M.F. Shulgin's form, it is necessary to isolate the components dependent on the linear terms of the expansion of the kinematic constraint coefficient and compare them for two different equilibrium positions. These components vanish for a known equilibrium position when the angle of rotation of the drive wheel is zero. Then it is possible to use[56] the Lagrange equations of the second kind, excluding both the coordinate and velocity as dependent components using a linearized constraint. It is now necessary to establish the values of these components for a previously unknown equilibrium position of the Ball and Beam system and to demonstrate the incorrectness of the linearization of the constraint for studying this equilibrium position. For different equilibrium positions, the equations structure of the perturbed motion will be different.

Furthermore, for systems with geometric constraints, as in the Ball and Beam system, the dependent coordinates can be selected in different ways. The system models will therefore differ. In the Ball and Beam system, the redundant coordinate can be selected in two ways, and the Shulgin equations will have different forms. To obtain the same initial perturbed state with different choices of the dependent coordinate and to compare the control results (for the same equilibrium), it is necessary to match the initial perturbations of the angular variables due to the presence of constraint (5).

This paper (Section **6**) further details the potential for effectively applying the developed method to a complete study of the Ball and Beam setup's dynamics. A nonlinear model of the system will be constructed, a comparative analysis of the structure of first-approximation equations for two equilibrium positions and two options for selecting the redundant coordinate will be performed, and, based on this, stabilization problems will be solved and control actions compared as functions of time for both equilibrium positions.

To construct a complete model of the Ball and Beam system as a mechatronic system, it is necessary to add the equation of a controlled electric drive with a DC commutator motor with separate excitation. The control action is an additional voltage on the anchor winding of the drive motor. With the chosen method of implementing control actions, the mathematical model of the object represents an indirect control system, which, as is known,[15] significantly expands the capabilities of the control system. In the proposed stabilization method, the characteristic equation for the disturbed motion necessarily has zero roots equal to the number of geometric constraints. If the real parts of its remaining roots are negative, then a critical case occurs for any control method. Then, according to the theory of critical cases,[50–52] a linear substitution[53] must be performed in the equations of the disturbed motion, identifying variables—redundant coordinates—to which these zero roots correspond. This transformation generally alters the structure of the linear positional forces in the first approximation of the equations of the disturbed motion.

The linear controlled subsystem[27–29] does not include the critical variable and depends significantly on the choice of the redundant coordinate. The control action, as a linear function of phase variables, depends on the variables of the selected subsystem. Its coefficients will be determined by solving the corresponding linear-quadratic problems using N.N. Krasovskii's method.[16] A sufficient condition for the solvability of the stabilization problem for the selected linear controlled subsystem is the controllability condition (by virtue of N.N. Krasovskii's theorem on stabilization by the first approximation (Theorem 1, p. 509)). In a complete nonlinear system closed by the linear control found for the selected controlled subsystem, asymptotic stabilization will be ensured for all variables, including the excess coordinate, which corresponds to the zero root (according to Theorem 1[27,29]).

A general rigorous obtaining method the analytical form of a nonlinear dynamics mathematical model of non-free systems with geometric constraints in the form of multipliers-free vector-matrix equations in redundant coordinates.

In the analytical mechanics of non-free systems, general methods[3,6,7] have been developed that are much more effective than those traditionally used (but still not used by specialists in technical practice), significantly simplifying the description of dynamics. In our opinion, the most convenient of these is the transition to equations without constraint multipliers in redundant coordinates proposed by M.F. Shulgin.[7] Using the dynamics equations in M.F. Shulgin's form without considering the inverse kinematics problem in this case analytically eliminates Lagrange multipliers and dependent velocities from consideration. As a result, in the most general case, we obtain a rigorous analytical form of the nonlinear mathematical model (which is unattainable with traditional modeling methods).

It should be especially noted that, compared to Lagrange equations with multipliers, the dimensionality of this model is reduced by twice the number of geometric connections. It should be emphasized again that this approach generally eliminates the need to consider the inverse kinematics problem, which traditionally must be solved in one way or another.

Reducing the nonlinear mathematical dynamics model dimensionality in the analytical form for the mechanical component of an automatically controlled object with geometric connections allows us to include in the consideration the mathematical model of actuators and, thus, move on to strict mathematical modeling of controlled dynamics, in particular, manipulators with parallel kinematics.

Formal justification methods

Let us consider the most general case of a mechanical system with n the freedom degrees, the which state is described by coordinates $q_1, \ldots, q_{n+m}$ the which number is greater than the degrees freedom number, since m geometric constraints are imposed on the system

$$F(\mathrm{q}) = 0;\ F'(\mathrm{q}) = (F_1(\mathrm{q}), .., F_m(\mathrm{q})); \qquad (1)$$

$$\det\left\|\frac{\partial F_1(\mathrm{q}), .., F_m(\mathrm{q})}{\partial(\mathrm{q}_{n+1}, .., \mathrm{q}_{n+m})}\right\| \neq 0;$$

Let us assume that the kinetic energy has the most general form

$$T = T_2 + T_1 + T_0 = \frac{1}{2}\sum_{ik=1}^{n+m} A_{ik}(\mathrm{q})\dot{q}_i\dot{q}_k + \sum_{i=1}^{n+m} A_i(\mathrm{q})\dot{q}_i + T_0(\mathrm{q}); \qquad (2)$$

The system is under the action of potential forces with energy $\Pi(q)$ and non-potential forces Q_i related to the coordinates.

Kinetic and potential energies, as well as non-potential forces and functions in the expression of geometric relationships (1), are at least twice continuously differentiable in the region of space under consideration, so that the conditions of the theory of differential equations are satisfied, guaranteeing the existence and uniqueness of solutions to the equations of motion.

For the further consideration convenience, in accordance with the different nature of the dependence of kinetic (2) and potential energies, constraints (1) and non-potential forces on coordinates, we introduce matrices and vectors

$$q = \begin{pmatrix} q_1 \\ \vdots \\ q_{n+m} \end{pmatrix} = \begin{pmatrix} r \\ s \end{pmatrix};\ r = \begin{pmatrix} q_1 \\ \vdots \\ q_n \end{pmatrix};\ s = \begin{pmatrix} q_{n+1} \\ \vdots \\ q_{n+m} \end{pmatrix};\ F(\mathrm{q}) = \begin{pmatrix} F_1(q) \\ \vdots \\ F_m(q) \end{pmatrix};$$

$$Q(\mathrm{q}, \dot{\mathrm{q}}) = \begin{pmatrix} Q_1 \\ \vdots \\ Q_{n+m} \end{pmatrix};\ Q_r(\mathrm{q}, \dot{\mathrm{q}}) = \begin{pmatrix} Q_1 \\ \vdots \\ Q_n \end{pmatrix};\ Q_s(\mathrm{q}, \dot{\mathrm{q}}) = \begin{pmatrix} Q_{n+1} \\ \vdots \\ Q_{n+m} \end{pmatrix};$$

If we differentiate with respect to time the equations of geometric constraints imposed on a system with redundant coordinates, we obtain kinematic (holonomic) constraints in the form

$$\frac{\partial F}{\partial r}\dot{r} + \frac{\partial F}{\partial s}\dot{s} = 0; \qquad (3)$$

Or (to make it easier for practicing engineers to understand the meaning of the operations performed) in scalar form

$$\sum_{s=1}^{n+m} b_{ks}\dot{q}_s = 0;\ (\overline{k,m}) \qquad (1')$$

Here $b_{ks} = \dfrac{\partial F_k}{\partial q_s}$, *and* $\dfrac{\partial b_{ks}}{\partial q_r} = \dfrac{\partial b_{kr}}{\partial q_s}$, since the constraints are integrable and representable in the form (1).

Due to constraints (1), the conditions are imposed on the variations of coordinates

$$\frac{\partial F}{\partial r}\delta r + \frac{\partial F}{\partial s}\delta s = 0;\ \delta s = -\left(\frac{\partial F}{\partial s}\right)^{-1}\frac{\partial F}{\partial r}\delta r = B(\mathrm{q})\delta r;\ B(\mathrm{q}) = -\left(\frac{\partial F}{\partial s}\right)^{-1}\frac{\partial F}{\partial r};$$

$$\frac{\partial F}{\partial s} = \begin{pmatrix} \frac{\partial F_1}{\partial q_{n+1}} & \cdots & \frac{\partial F_1}{\partial q_{n+m}} \\ \cdots & \cdots & \cdots \\ \frac{\partial F_m}{\partial q_{n+1}} & \cdots & \frac{\partial F_m}{\partial q_{n+m}} \end{pmatrix};\ \frac{\partial F}{\partial r} = \begin{pmatrix} \frac{\partial F_1}{\partial q_1} & \cdots & \frac{\partial F_1}{\partial q_n} \\ \cdots & \cdots & \cdots \\ \frac{\partial F_m}{\partial q_1} & \cdots & \frac{\partial F_m}{\partial q_n} \end{pmatrix}$$

$\dfrac{\partial F}{\partial s}$ square matrix of dimension $m \times m$, $\dfrac{\partial F}{\partial r}$ rectangular matrix of dimension $m \times n$

$$\delta s - B(\mathrm{q})\delta r = 0; \qquad (3)$$

As equations of motion, we will use Shulgin's equations in redundant coordinates

$$\frac{d}{dt}\frac{\partial L^*}{\partial \dot{r}} = \frac{\partial L^*}{\partial r} = \overset{*}{Q}r - B'(\mathrm{q})\left(\frac{\partial L^*}{\partial s} - \overset{*}{Q}s\right);\ \dot{\mathrm{s}} = B(\mathrm{q})\dot{r} \qquad (4)$$

These equations are derived from the traditionally used equations obtained from the d'Alembert-Lagrange principle[2–7] with constraint multipliers λ

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - = \frac{\partial L}{\partial q} = Q + b'\lambda;\ b(\mathrm{q})\frac{\partial F}{\partial q}; \qquad (5)$$

Using (3) we can separate equations (5) for dependent and independent coordinates

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial r} = Q_r - B'\lambda; \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{s}} - \frac{\partial L}{\partial s} = Q_s - \lambda; \qquad (6)$$

Expressing the multipliers from the second equation (6)

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{s}} - \frac{\partial L}{\partial s} = Q_s - \lambda \qquad (7)$$

After substituting (7) for the first equation of system (6), we obtain

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = Q_r - B'\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{s}} - \frac{\partial L}{\partial s} - Q_s\right); \qquad (8)$$

Similar equations were obtained by Suslov GK[3] and Lurie AI.[6] Moreover, both of these researchers considered more complex systems, on which in addition to geometric constraints, non-integrable (non-holonomic) constraints were also imposed. Therefore, the modeling of the dynamics of such systems ended with equations of the form (8). Shulgin MF,[7] developing the analytical mechanics of non-holonomic systems, was able to simplify the mathematical model for the case of only geometric constraints.

To do this, dependent velocities $\dot{s}$ from equations (1) differentiated with respect to time

$$\frac{\partial F}{\partial r}\dot{r} + \frac{\partial F}{\partial s}\dot{s} = 0;$$

$$\dot{s} = B(\mathrm{q})\dot{r}\ ; \tag{9}$$

should be excluded from the kinetic energy and non-potential forces. Denoting the result of eliminating dependent velocities using (9) from the kinetic energy (2) through $T^{*}(\mathrm{q}_1,..,\mathrm{q}_{n+m,}\dot{\mathrm{q}}_1,..,\dot{\mathrm{q}}_n)$

$$T^{*} = T_2^{*} + T_1^{*} + T_0(q) = \frac{1}{2}\sum\nolimits_{pv=1}^{n} a_{pv}(\mathrm{q})\dot{q}_v\dot{q}_p + \sum\nolimits_{p=1}^{n} d_p(\mathrm{q})\dot{q}_p + T_0(\mathrm{q});$$

(2')

$$a_{pv}(\mathrm{q}) = A_{pv} + \sum\nolimits_{\mu=1}^{m} A_{\mu p}B_{\mu v} + \sum\nolimits_{\mu=1}^{m} A_{\mu v}B_{\mu p} + \sum\nolimits_{\mu\sigma=1}^{m} A_{\mu\sigma}B_{\mu p}B_{\sigma v}\,;\, d_p(\mathrm{q}) = A_p + +\sum\nolimits_{\mu=1}^{m} A_{\mu}B_{\mu p};\ ;$$

and from the acting non-potential forces through Q_i^{*}.

From a comparison of the corresponding derivatives $T^{*}(\mathrm{q}_1,..,\mathrm{q}_{n+m,}\dot{\mathrm{q}}_1,..,\dot{\mathrm{q}}_n)$ and $T(\mathrm{q}_1,..,\mathrm{q}_{n+m,}\dot{\mathrm{q}}_1,..,\dot{\mathrm{q}}_{n+m})$ taking into account the integrability of the constraints (9), in the general case, a nonlinear mathematical model of the system dynamics is obtained in the M.F. Shulgin equations form (4), in redundant coordinates. These equations can be considered as the P.V. Voronets equations[57] in the integrability case of the nonholonomic constraints.[7,58,59]

The obtained general analytical form (4) for the mathematical model of nonlinear dynamics allows, in the general case, to present the mathematical dynamics model for any systems with geometric constraints (under the action of arbitrary potential and non-potential forces that do not violate the conditions for the existence and uniqueness of the differential equations solutions) in the Shulgin equations form (4) in explicit vector form[27–33] without resolving the inverse problem of kinematics (cf.[8–10,18,20–24]).

Based on the above, we present the strictly justified method algorithm for obtaining an analytical form of a nonlinear dynamics mathematical model in the most general case for a non-free mechanical systems with geometric constraints:

Algorithm for obtaining an explicit analytical form of a nonlinear mathematical model of the dynamics of systems with geometric constraints in the general case:

1. Based on the geometric constraint equations (1), select m coordinates that will be considered redundant.

2. Differentiate the connection equations with respect to time as in (3);

3. Express the velocities of the dependent coordinates as $\dot{s} = B(q)\dot{r}$, according to (9).

3. Eliminate the dependent velocities $\dot{s}$ from the expressions for the kinetic energy and non-potential forces.

4. Determine the matrices of kinetic energy coefficients included in (2');

5. Write equations (4) (similar to how Lagrange equations of the second kind are written)

In normal form, the model in redundant coordinates (4) has a dimension of 2n + m, and compared to the dimension 2(n + m) + m of the traditional model (5) + (1), it decreases by 2m, i.e., by double the number of geometric constraints (1). Let us recall that the variables of the model (5)+(1) are, in addition to all coordinates and to all velocities (including dependent), and also undefined constraints multipliers.

It should be noted that the full analytical form (4) of the nonlinear dynamics model in systems with geometric constraints allows for the first time the general application of rigorous methods from the nonlinear stability theory and of the mathematical control theory. The achieved reduction in model dimensionality allows for the inclusion of transient processes in actuators, as will be demonstrated in Section 6.

Remark 3.1. *Explicit* ***scalar*** *form for the motion equations in redundant coordinates of M. F. Shulgin***.**

The vector form of the model is convenient for formulating abstract theoretical general results. When applying them to modeling the nonlinear dynamics of specific technical objects, one inevitably has to resort to scalar equations. Therefore, to better understand the conceptual simplicity of our method for technical practitioners, in the next section of this article, we will also present the results in scalar form.

We will represent[26] Shulgin's equations (4) in explicit scalar form:

$$a_{is}\ddot{q}_s + \frac{\partial a_{is}}{\partial q_j}\dot{q}_S\dot{q}_j + \frac{\partial a_{is}}{\partial q_\mu}B_{\mu\rho}\dot{q}_s\dot{q}_\rho - \frac{1}{2}\frac{\partial a_{s\rho}}{\partial q_i}\dot{q}_s\dot{q}_\rho - \frac{1}{2}\frac{\partial a_{s\rho}}{\partial q_\mu}B_{\mu i}\dot{q}_s\dot{q}_\rho +$$
$$+\left(\frac{\partial d_i}{\partial q_s} - \frac{\partial d_s}{\partial q_i} + B_{\mu s}\frac{\partial d_i}{\partial q_s} - B_{\mu i}\frac{\partial d_s}{\partial q_\mu}\right)\dot{q}_s + \frac{\partial W}{\partial q_i} + \frac{\partial W}{\partial q_\mu} = Q_i + B_{\mu i}Q_\mu; \quad \dot{q}_\mu = B_{\mu i}\dot{q}_i; \tag{4'}$$

To simplify the notation, here and below, summation is assumed over twice-repeating indices. The indices change as follows: $\gamma, v = \overline{1,n+m}$; $i, j, s, \rho = \overline{1,n}$; $\mu, k = \overline{n+1,n+m}$. Here $W(\mathrm{q}) = \Pi(\mathrm{q}) - T_0(\mathrm{q})$ is the changed (reduced) potential energy, Q_i, Q_μ denote the non-potential forces corresponding to the coordinates q_i, q_μ when they are introduced in excess.

Application of the developed mathematical model and nonlinear stability theory to the general stabilization problem for non-free systems

General stability problem for non-free systems

The problem of stable implementation of a given operating mode is one of the most important challenges in the control of modern technical devices. We will demonstrate the possibilities offered by the resulting nonlinear mathematical model for the substantiated application of rigorous results of nonlinear stability theory.

Let the system admit an equilibrium position

$$r = r_0 = const; s = s_0 = const; \quad (10)$$

Let us introduce disturbances,

$$r = r_0 + x; \dot{r} = x_1; s = s_0 + y (*)$$

compose the equations of the disturbed motion and single out the first approximation in them

$$\dot{x} = x_1;$$

$$\dot{x}_1 = Cx + Gx_1 + Hy + X^{(2)}(x, x_1, y); \qquad 11)$$

$$\dot{y} = B(x, y)x_1 = B(0)x_1 + B^{(1)}(x, y)x_1; B(0) = B(r_0, s_0);$$

Here the superscript in brackets denotes the order of the lowest terms in the expansion of the corresponding expression, and the matrices C,G,H are expressed in a known manner[26–36] through the kinetic energy coefficients (2), the constraints equations (1), and the coefficients in the potential and non-potential forces expansion.

After the replacement,[53]

$$y = z + B(0)x; (12)$$

the characteristic equation of the first approximation system

$$\dot{x} = \dot{x}_1;$$

$$\dot{x}_1 = (c + HB(0))x + Gx_1 + Hy + X^{(2)}(x, x_1, z + B(0)x); \quad (13)$$

$$\dot{z} = B^{(1)}(x, z + B(0)x)x_1;$$

will take the form

$$\begin{vmatrix} E_n\lambda & -E_n & 0 \\ -c - HB(0)E_n & \lambda - G & -H \\ 0 & 0 & E_n\lambda \end{vmatrix} = 0; \qquad (14)$$

Equation (14) for any equilibrium of a system with geometric constraints in the general case has *m* zero roots corresponding to the variable *z*. Its remaining roots are determined by the equation

$$\begin{vmatrix} E_n\lambda & -E_n \\ -c - HB(0)E_n & \lambda - G \end{vmatrix} = 0; \qquad (15)$$

If the real parts of all roots of equation (15) are negative, then the complete nonlinear system (13) has a special form of the equations of perturbed motion in the critical case of m zero roots.[50–52] The variables x, x_1 are noncritical, and the variable z is critical. According to Kamenkov's theorem,[52] if there exists a nonlinear replacement of noncritical variables after which there are no freely entering variables z in the equations for noncritical variables, and in the equation for z all nonlinear terms vanish after this replacement, then non-asymptotic stability holds in the complete nonlinear system. Based on Kamenkov's theorem, it was proved, that for non-free systems with constraints (1) in the complete nonlinear system (13) we obtain asymptotic stability, despite the presence of m zero roots.[29]

Remark 4.1. Note that Kamenkov's theorem in the general theory of critical cases guarantees non-asymptotic stability in the special (essentially special) case of the critical case with m zero roots. In this situation, the equilibrium under study is not isolated, but is located on a manifold whose dimension is equal to the number of zero roots. This is precisely the situation that occurs in the problem of equilibrium stability in nonholonomic systems with linear homogeneous constraints.[58–60] In the problem under consideration for systems with geometric constraints (1), the equilibria are isolated, so it was necessary to modify Kamenkov's theorem.

A thorough study of the structure of nonlinear terms after the replacement (13) was carried out. As a result, the following was proved

Theorem 1: *If the real parts of all roots in the characteristic equation (15) are negative, the equilibrium position under study in the complete nonlinear system (13) is asymptotically stable, despite the presence of m zero roots.*

Remark 4.2. ***Structure of the perturbed motion equations***.

To represent the process of obtaining the components of the first approximation coefficient matrices in the model (13) in scalar form, we will redefine the equilibrium (10):

$$q_i = q_{i0}, q_\mu = q_{\mu 0} \qquad (10')$$

Let us introduce disturbances, formulate the equations of the disturbed motion and identify the first approximation in them. To simplify the understanding of the origin of the components of the first approximation matrices, we will for now compose equations without taking into account non-potential forces.

$$q_{i0} = q_{i0} + x_i, q_\mu = q_{\mu 0} + y_\mu (**)$$

$$a_{is}(0)\ddot{x}_s + \left[\left(\frac{\partial d_i}{\partial q_s}\right)_0 - \left(\frac{\partial d_s}{\partial q_i}\right)_0 + B_{\mu 0}(0)\left(\frac{\partial d_i}{\partial q_\mu}\right)_0 - \left(\frac{\partial d_s}{\partial q_\mu}\right)_0 B_{\mu 0}(0)\right]\dot{x}_s +$$

$$\left[\left(\frac{\partial^2 W}{\partial q_i \partial q_s}\right)_0 + B_{\mu i}(0)\left(\frac{\partial^2 W}{\partial q_\mu \partial q_s}\right)_0 + \left(\frac{\partial W}{\partial q_\mu}\right)_0 \left(\frac{\partial B_{\mu i}}{\partial q_s}\right)_0\right]x_s + \left(\frac{\partial^2 W}{\partial q_i \partial q_k}\right)_0 y_k +$$

$$+B_{\mu i}(0)\left(\frac{\partial^2 W}{\partial q_k \partial q_\mu}\right)_0 y_k + \left(\frac{\partial B_{\mu i}}{\partial q_k}\right)_0 \left(\frac{\partial W}{\partial q_\mu}\right)_0 y_k = X_i^{(2)}(\dot{x}, x, y) \qquad (16)$$

Citation: Krasinskiy AY, Yuldashev AA. An analytical form for a rigorous nonlinear model of the dynamics for systems with geometric constraints and its use by applying the results of control theory to a complete solution of stabilization problems. *Int Rob Auto J.* 2026;12(1):98–120. DOI: 10.15406/iratj.2026.12.00323

The zero subscript at the bottom means that after obtaining the analytical form of the corresponding derivatives, the values of the variables at equilibrium () should be substituted into these expressions. We draw special attention to the fact that, based on the expressions $\left(\frac{\partial B_{\mu i}}{\partial q_s}\right)$, it is obvious that, in general, the equations of geometric connections cannot be linearized, since

$$B(\mathrm{q}) = -\left(\frac{\partial F}{\partial s}\right)^{-1}\frac{\partial F}{\partial r};\frac{\partial B_{\mu i}}{\partial q_s} = \frac{\partial}{\partial q_s}\left(-\left(\frac{\partial F}{\partial s}\right)^{-1}\frac{\partial F}{\partial r}\right);$$

Thus, the expression $\frac{\partial B_{\mu i}}{\partial q_s}$ is determined by **the second-order terms** in the decomposition of the constraints.

Let us add to these equations the differential equations with the selected first approximation

$$\dot{y}_k = B_{jk}(0)\dot{x}_k + B_{jk}^{(1)}(\mathrm{x},\mathrm{y})\dot{x}_k \quad (17)$$

If in the system (16), (17) we carry out a linear replacement[26–28]

$$z_k = y_k - B_{kj}(0)x_j \ (18)$$

then the equations of the constraints will take the form

$$\dot{z}_k = B_{kj}^{(1)}(x, z + B(0)x)\dot{x}_k \qquad (19)$$

Obviously, the variables z corresponds to zero roots of the characteristic equation.

Now, let's take into account, in addition to potential forces, the non-potential positional forces and velocity-dependent forces acting on the system. Identifying the first approximation in these forces and using summation over twice-repeating indices, we obtain

$$Q_\gamma = g_{\gamma j}\dot{q}_j + g_{\gamma\mu}\dot{q}_\mu + p_{\gamma j}q_j + p_{\gamma_\mu}q_\mu + \ldots$$

Excluding dependent velocities from these forces, and making the substitution (18), for the non-zero roots of the characteristic equation of the first approximation system under the action of non-potential forces we obtain

$$\det\left[a\lambda^2 + (\Gamma + D)\lambda + C_1 + C_1B + B'C_3 + B'C_{4B} + C^B + P\right] = 0 \qquad (20)$$

$$D = \left\| g_{ij} + g_{i\mu}B_{\mu i} + g_{\mu i}b_{\mu j} + g_{\mu i}b_{\mu k}B_{kj} \right\| \quad p = \left\| p_{ij} + p_{i\mu}B_{\mu i} + B_{\mu i}p_{\mu j} + B_{\mu i}p_{\mu k}B_{kj} \right\|$$

$$a = \left\| a_{ij}(0) \right\|;\Gamma = \left\| y_{is}(0) \right\|;C_1 = \left\| \left(\frac{\partial^2 w}{\partial q_i \partial q_s}\right)_0 \right\|;C_2 = \left\| \left(\frac{\partial^2 w}{\partial q_i \partial q_\mu}\right)_0 \right\|;C_4 = \left\| \left(\frac{\partial^2 w}{\partial q_\mu \partial q_\sigma}\right)_0 \right\|;C_3 = C_2^{'};(***)$$

$$\mathrm{C}^B = \left\| \left(\frac{\partial w}{\partial q_\mu}\right)_0 \left(\frac{\partial B_{\mu i}}{\partial q_s}\right)_0 + \left(\frac{\partial w}{\partial q_\mu}\right)_0 \left(\frac{\partial B_{\mu i}}{\partial q_k}\right)_0 B_{kj}(0) \right\|;\mathrm{B} = \left\| B_{\mu i}(0) \right\|$$

$$\gamma_{is}(0)\left[\left(\frac{\partial d_i}{\partial q_s}\right)_0 - \left(\frac{\partial d_s}{\partial q_i}\right)_0 + B_{\mu 0}(0)\left(\frac{\partial d_i}{\partial q_\mu}\right)_0 - \left(\frac{\partial d_s}{\partial q_\mu}\right)_0 B_{\mu i}(0)\right]$$

Section 6 provides a detailed description of how to obtain the components of these matrices for a specific Ball and Beam device.

General stabilization problem for non-free systems

To obtain such a root arrangement, asymptotic stability with respect to the variables x, x1 must be ensured. To do this, it is sufficient to introduce control into the second equation of the system (13), and then solve the stabilization problem of the linear controlled subsystem

$$\dot{\xi} = M\xi + Nu; \xi' = (x', x_1^{'}); \quad (16)$$

$$M = \begin{pmatrix} 0 & E_n \\ c + H\,B(0) & \mathrm{G} \end{pmatrix}; N = \begin{pmatrix} 0 \\ N_1 \end{pmatrix};$$

<u>*Remark 4.3.*</u> When formulating the problem, it was noted that control actions of a certain dimension may not necessarily be mechanical in nature and may be included not only in non-potential forces acting on both independent and dependent coordinates, but also in equations describing the dynamics of the actuator (for the stand Ball and Beam, the control is an additional voltage on the armature winding of the actuator motor).

The resulting analytical form of nonlinear dynamics makes it possible to consider a wide variety of methods for implementing control actions (see examples of such options.[32–34,61] Here, for simplicity, we will limit ourselves to the formal introduction of control $u(x, x_1)$ of a certain dimension l entering into the controlled subsystem with a matrix N of dimension $n \times l$, without discussing its structure.

By applying the obtained analytical form of the nonlinear model of the controlled dynamics of a system with constraints (1), we will not only prove sufficient conditions for the solvability of the equilibrium (10) stabilization problem by applying the introduced control. Using the results of mathematical control theory,[16] an explicit form of the stabilizing influences will be obtained in following.

Theorem 2. *If the pair (M, N) is controllable*

$$\text{rank}[N\ M\ N ... M^{2N-1}N] = 2n; \qquad (17)$$

then the equilibrium position (10) is stabilized to asymptotic stability with respect to all variables by applying a control of the form as a *linear function of only* x, x_1*:* rank

$$u = K_1 x + K_2 x_1; \quad (18)$$

Matrices K_1, K_2 *can be determined from the solution by Krasovskii's method of the linear-quadratic problem for subsystem (16), in particular with the simplest criterion*

$$\int_0^\infty (\xi'\xi + u'u)dt \to \min; \quad (19)$$

Proof: According to (14), the characteristic equation for the first-approximation system (13) has m zero roots corresponding to the variable z. Let us extract from the complete nonlinear system (13) a controlled linear subsystem (16) of lower dimension, which does not include critical variables. If condition (17) is satisfied, then there exists [16] a control u of the form (18) such that the real parts of all roots of subsystem

$$\dot{x} = x_1;$$
$$\dot{x}_1 = (\text{C}+\text{HB}(0))x + Gx_1 + N_1(\text{K}_1 x + \text{K}_2 x_1);$$

are negative.

Let such a control be found. Let us substitute it into the complete nonlinear system (13)

$$\dot{x} = x_1;$$
$$\dot{x}_1 = (\text{C}+\text{HB}(0))x + Gx_1 + Hy + N_1(\text{K}_1 x + \text{K}_2 x_1) + X^{(2)}(x, x_1, \text{z}+\text{B}(0)x);$$
$$\dot{z} = B^{(1)}(x, z+\text{B}(0)x)x_1; \qquad (20)$$

Moreover, the characteristic equation of the linearized system

$$\dot{x} = x_1;$$
$$\dot{x}_1 = (\text{C}+\text{HB}(0))x + Gx_1 + Hy + N_1(\text{K}_1 x + \text{K}_2 x_1);$$
$$\dot{z} = 0 + B^{(1)}(x, z+\text{B}(0)x)x_1;$$

has exactly m zero roots, and the real parts of the remaining roots

$$\begin{vmatrix} E_n\lambda & -E_n \\ -C - H\ B(0) - \text{N}_1 K_1 & E_n\lambda - G - \text{N}_1 K_2 \end{vmatrix} = 0$$

are negative. Thus, the complete nonlinear system (20) is reduced to a special form of critical case theory. Moreover, due to the structure of the nonlinear terms of the last m equations of system (20), a special case occurs. In this form, system (20) fully corresponds to the conditions of Theorem.1 on the asymptotic stability of equilibrium positions of systems with geometric constraints. The system dimension for determining the control is only *2n*, i.e. in the general case it is reduced by the geometric constraints number (the dependent coordinates number). In this case, information about only variables x, x_1 is sufficient to form the control (18).

The explicit analytical form (20) presence of the controlled dynamics nonlinear mathematical model provides grounds for applying the mathematical control theory results with incomplete information about the state for systems with geometric constraints.

Remark 4.4. The solvability of the linear-quadratic stabilization problem by Krasovsky's method is proven[16] for a quality criterion of general form,

$$\int_0^\infty (\xi'\text{R}\,\xi + u'Qu)dt \to \min$$

where R and Q are any square definite-positive matrices of the corresponding dimensions $2n \times 2n$ and $l \times l$.

In the case where the matrices *M* and *N* are constant, the Lyapunov function is sought in the form of a quadratic form $V_0 = \xi' V \xi$, where V is a constant symmetric matrix of coefficients. In this case, the Lyapunov function is determined from the algebraic matrix Riccati equation

$$VM + M'V - V(\text{NR}^{-1}\text{N}')\text{V} + \text{Q} = 0;$$

The corresponding linear control action u is calculated using the formula

$$\text{u} = -\text{R}^{-1}Q'V\xi$$

The matrices involved in the algebraic Riccati equation are explicitly defined by the formulas given earlier and can be calculated if expressions for the kinetic and potential energies, constraint equations, expressions for the non-potential forces, and certain characteristics of the control motors are given, provided that the model includes equations describing the dynamics of the drives (see Ball and Beam below).

Various numerical methods and their software implementations exist for solving the Riccati equation.

General stabilizing problem a given configuration with incomplete state information

In practice, the complete phase state of an object is usually unknown. Information on the system's operation is primarily determined by sensors, the number of which is typically limited. Measuring the values of some parameters can be fundamentally complex due to their nature.

Remark 4.5: This is precisely the situation that occurs in the control problem of a wheeled robot with deformable wheels. The phase state vector includes the current values of the deformation parameters for ach wheel in the contact patch with the rolling surface, which will inevitably also be included in the control vector. Obviously, measuring the current values of these phase state robot variables is, in principle, impossible. However, thanks to a phenomenological mathematical model[62,63] of tire deformation, it was possible to obtain an adequate estimate of the deformation parameters and, based on an estimate of the entire phase robot state, generate a control (additional voltage on the armature windings of the drive commutator motors with independent excitation), stabilizing the specified motion.[64]

But even if all phase variables are measurable in principle, the question arises of reducing the number of sensors to simplify system maintenance and potentially reduce its cost. It should also be noted that each measuring device is inertial, with its own transient process. Reducing the number of sensors simplifies the structure of the system and, thereby, increases its reliability. Therefore, a very important step in solving applied problems is the development of systems for estimating the phase state of the controlled object (observers) based on available measurement information, which will be used to formulate control. It should also be noted that in situations where it is necessary to continuously evaluate the vector of phase variables in real time, the adequacy of the mathematical model of the controlled object is even more important for improving the quality of the control system. With incomplete information, the mathematical model of the system is used throughout its entire operation since the state estimation system continuously compares the measured and corresponding model signals, the dynamics of a mathematical model constructed without taking into account the nonlinear terms in the constraint equations may differ radically from the behavior of a real object.

Remark 4.6: Obviously, reliability increases with the reduction in the number of actuators (the dimension of the control vector) and the number of measuring sensors (the dimension of the measurement vector). Mathematical control theory[14] provides the ability to estimate the minimum dimension of these vectors sufficient to satisfy the controllability (17) and observability () conditions: these numbers are determined by the number of nontrivial invariant polynomials of the matrix *M*. It's worth noting that conditions (), () are only sufficient for the solvability of stabilization problems using the proposed design method. In practice, a mathematical model of an object may contain subsystems that do not require control: the stability of a given operating mode is determined by the nature of the object itself. For example, the maximum angular velocity of a separately excited DC motor is determined by the actual voltage on the armature winding: the difference between the external voltage and the back-EMF. No additional control is required to stabilize the rotational speed. The corresponding term in the armature rotation equation can be interpreted as a dissipative force of a special structure, in the terminology of V.V. Rumyantsev.[65]

But for the maximum possible reduction in the dimension of control vector, in our opinion, an understanding of the principles of the object's operation at the level of engineering intuition is required.

Remark 4.7: To obtain a phase state estimation vector, the measurement information must be processed using a computing device: a system of differential equations () must be integrated, which may require some time, depending on the type of computing device. The problem of choosing the type of this device was discussed with N.N. Krasovskii back in March 1992. The discussion concluded that, for analog measuring sensors and an analog computer, the proposed approach was entirely adequate. However, if this operation were performed on a digital computing device, a delay in estimating the current state of the object would inevitably occur. Therefore, it is natural to introduce some delay into the object model. As a result, another problem arose in research, for which methods dating back to the work of N.N. Krasovskшш are successfully applied. This field is being effectively developed by his students and staff at the Institute of Mathematics and Mechanics of the Ural Branch of the Russian Academy of Sciences, which now bears his name. This problem is currently being addressed by increasing the speed of digital measurement data processing: both sensors and the devices that process their information are overwhelmingly digital.

For the sake of certainty, let us assume, that in the system (20) only the following measurement information about phase variables is available

$$\sigma = \Sigma\xi = \Sigma_1 x + \Sigma_2 x_1; \dim\sigma = s < 2n; \quad (21)$$

In order to obtain information about the full phase vector of the subsystem (16) using measurement (21), it is necessary to construct a state estimation system

$$\dot{\xi} = M\xi + \Lambda(\Sigma\xi - \sigma) + \mathrm{N}u; \quad (22)$$

Theorem 3. *If for systems (16) and (22) the pair (M, N) is controllable, and the pair (M, Σ) is observable, then there exists a linear control of the form*

$$\hat{u} = K_1\hat{x} + K_2\hat{x}_1 \qquad (23)$$

that stabilizes the equilibrium position (10) to asymptotic stability in all variables. The matrix Λ can be determined from the solution by Krasovskii's method of the dual linear-quadratic problem for the system

$$\dot{\mu} = M'\mu + \Sigma' w; w = \Lambda'\mu; \qquad (24)$$

with a quality criterion

$$\int_0^\infty (\mu' R_1 \mu + w' Q_1 \mathrm{w})dt \to \min$$

Proof: Satisfying the observability condition for the pair (M, Σ)

rank [Σ' M'Σ' ⋯ 〚M'〛^(2n-1) Σ']=2n; (25)

ensures controllability for (24). When condition (25) is satisfied, the matrix Λ can be determined from the solution of the linear-quadratic problem for (24) using Krasovskii's method[16] such, that the real parts of all roots of the system

$$\dot{\mu} = (\mathrm{M}' + \Sigma'\Lambda')\mu;$$

are negative.

The dynamics of a complete nonlinear system, closed by a stabilizing control, constructed based on the state estimate from the measurement (21) is described by the model

$$\dot{x} = x_1;$$
$$\dot{x}_1 = (\mathrm{C} + \mathrm{HB}(0))x + Gx_1 + Hy + N_1(\mathrm{K}_1\hat{x} + \mathrm{K}_2\hat{x}_1) + X^{(2)}(x, x_1, z + \mathrm{B}(0))x);$$
$$\dot{z} = B^{(1)}(x, z + \mathrm{B}(0)x)x_1; \qquad (26)$$
$$\dot{\hat{x}} = \hat{x}_1;$$
$$\dot{\hat{x}}_1 = (\mathrm{C} + \mathrm{HB}(0))\hat{x}' + G\hat{x}_1 + Hy + N_1(\mathrm{K}_1\hat{x} + \mathrm{K}_2\hat{x}_1) + \Lambda(\Sigma_1\hat{x} + \Sigma_2\hat{x}_1 - \sigma); \sigma = \Sigma\xi$$
$$\hat{u} = \mathrm{K}_1\hat{x} + \mathrm{K}_2\hat{x}_1$$

When conditions (17), (25) are met, the problem of stabilizing equilibrium (10) to asymptotic stability for all variables is solved by linear control (23), formed based on the assessment of the phase state by measuring (21). Since the conditions of controllability and observability for the pairs (M, U) and (M, Σ) are satisfied, there exist[13] matrices K_1, K_2, Λ, for which the real parts for all roots of the characteristic equation of the linear system

$$\dot{x} = x_1;$$
$$\dot{x}_1 = (\mathrm{C} + \mathrm{HB}(0))x + Gx_1 + Hy + N_1(\mathrm{K}_1\hat{x} + \mathrm{K}_2\hat{x}_1);$$
$$\dot{\hat{x}} = \hat{x}_1;$$
$$\dot{\hat{x}}_1 = (\mathrm{C} + \mathrm{HB}(0))\hat{x}' + G\hat{x}_1 + Hy + N_1(\mathrm{K}_1\hat{x} + \mathrm{K}_2\hat{x}_1) + \Lambda(\Sigma_1\hat{x} + \Sigma_2\hat{x}_1 - \sigma); \sigma = \Sigma\xi;$$
$$\hat{u} = \mathrm{K}_1\hat{x} + \mathrm{K}_2\hat{x}_1;$$

Are negative.

Thus, the complete nonlinear system (26), closed by the obtained control (23), satisfies all the conditions of *Theorem 1*. In this case, the control (23) is formed according to the vector of the phase state estimate obtained by processing the measurement (21) from the estimation system

$$\dot{\hat{x}} = \hat{x}_1;$$
$$\dot{\hat{x}}_1 = (\mathrm{C} + \mathrm{HB}(0))\hat{x}' + G\hat{x}_1 + Hy + N_1(\mathrm{K}_1\hat{x} + \mathrm{K}_2\hat{x}_1) + \Lambda(\Sigma_1\hat{x} + \Sigma_2\hat{x}_1 - \sigma); \sigma = \Sigma\xi;$$

Algorithm 3. Algorithm for Numerical Calculation of the Matrix of Stabilizing Effect Coefficients and Observer Gain Coefficients

1. Based on the geometric constraint equations (1), identify m coordinates that will be considered redundant.

2. Express the velocities of the dependent coordinates as $s^{\cdot} = B(q)\, r^{\cdot}$,

3. Eliminate the dependent velocities $s^{\cdot}$ from the expressions for the kinetic energy and non-potential forces.

4. Determine the matrices of kinetic energy coefficients included in (2') and the reduced potential energy $W(q)$;

5. Write equations (13) to determine all system parameters at equilibrium and program controls in accordance with the selected control method;

6. Enter disturbance vectors according to (*) or in scalar form (**);

7. Using formulas (***), calculate the matrices included in the linear controlled subsystem (16), depending on the selected option for introducing control actions.

8. Check the feasibility of the controllability condition (17) for the pair *(M, N),*

9. Check the feasibility of the observability condition (25) for the pair (*M, Σ*).

10. If the conditions (17) and (25) are met, define square matrices R and Q that define the quality criterion.

11. Find the matrix of stabilizing control coefficients $u = -R^{-1}Q'V\xi$, finding V from the corresponding matrix Riccati equation:

$$VM + M'V - V(\mathrm{NR}^{-1}\mathrm{N}')\mathrm{V} + \mathrm{Q} - 0;$$

12. Find the matrix $\Lambda = -Y'Q_1(\mathrm{R}_1')^{-1}$ of the observer coefficients, where the matrix Y is the solution of the corresponding matrix Riccati equation

$$YM' + MY - Y(\Sigma'(\mathrm{R}_1')^{-1}\Sigma)Y + Q_1 = 0;$$

Discussion

For reliable operation of a controlled technical device, it is very important to reduce the volume of measurement information and simplify the structure of the control loop. The choice of a convenient mathematical model that creates the possibility of minimal interference in the natural behavior of the object plays an important role. For systems with nonlinear geometric constraints, there are different forms of equations (with and without constraint multipliers)[2–7] and types of variables[2] (Lagrange, Hamilton or Routh). Due to the nonlinearity of the constraints, it is advisable to describe such systems in redundant coordinates and use the vector-matrix equations of M.F. Shulgin.[26–30] The use of Routh variables and the transition to cyclic impulses (Hamilton variables) over all controlled cyclic coordinates significantly simplifies[5] the analysis of the structure of a closed nonlinear system and the procedure for determining the coefficients of linear stabilizing controls. But this method is disadvantageous for implementing the found control in the case of incomplete information about the state, if the obtained control will depend on impulses, for example, in cases similar to that considered in Theorem 1.[28,29,34] Information about disturbances of cyclic impulses cannot be obtained directly by information sensors. In[33] the possibility of asymptotic stabilization over all positional coordinates and cyclic impulses with incomplete information is considered, where the measurement vector depends on the positional coordinates and velocities. However, the corresponding controllability and observability conditions cannot be satisfied for every system.

The use of Lagrange variables for all coordinates is convenient for obtaining information about the system's phase state due to the ability to directly measure disturbances in cyclic velocities [8]. However, these variables complicate the procedure for determining the stabilizing control and the analysis of the structure of the system's nonlinear equations of motion.

Using Routh variables, we can consider another approach to solving problems of stabilizing systems with cyclic coordinates: impulses (Hamilton variables) are introduced only along uncontrolled cyclic coordinates, while controls are applied along cyclic coordinates described by Lagrange variables. This eliminates the need to include the disturbances of impulses in the controlled subsystem and, accordingly, in the evaluation system when information about the state is incomplete. In accordance with the different dependence nature of the system parameters and the acting forces on different coordinates, we divide the independent coordinates vector into the following vectors

$$r = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}; \alpha = \begin{pmatrix} q_1 \\ \vdots \\ q_k \end{pmatrix}; \beta = \begin{pmatrix} q_{k+1} \\ \vdots \\ q_l \end{pmatrix}; \gamma = \begin{pmatrix} q_{l+1} \\ \vdots \\ q_n \end{pmatrix}; F(q) = \begin{pmatrix} F_1(q) \\ \vdots \\ F_m(q) \end{pmatrix};$$

Let us assume that the coordinates β, γ are cyclic by definition of Shulgin, that is, the following conditions are met:

1. the coordinates β, γ are not explicitly included in the expression of the function L, (but it clearly depends on $\dot{\beta}, \dot{\gamma}$), 2. the coordinates β, γ are not explicitly contained in the expressions of the finite constraints (1), i.e.

$$\frac{\partial L^*}{\partial \beta} = 0; \frac{\partial L^*}{\partial \gamma} = 0; \frac{\partial F}{\partial \gamma} = 0; \mathrm{Q}_\beta(\alpha, \mathrm{s}, \dot{\mathrm{q}}) = 0; \mathrm{Q}_\gamma(\alpha, \mathrm{s}, \dot{\mathrm{q}}) = 0; \quad (28)$$

Under these conditions, the constraints (1) are reduced to the form

$$\dot{s} = B_\alpha(\alpha, s)\dot{\alpha}; B_\alpha(\alpha, s) = -\left(\frac{\partial F}{\partial s}\right)^{-1}\left(\frac{\partial F}{\partial \alpha}\right)$$

As is known, in the cyclic coordinates presence it is more convenient[16,36,37] to model the dynamics in Routh variables. For this purpose, we introduce impulses and the Routh function

$$p_\beta = \frac{\partial T^*}{\partial \beta}; p_\gamma = \frac{\partial T^*}{\partial \gamma}; \mathrm{R} = T^{**}(\alpha, \mathrm{s}, \dot{\alpha}, p_\beta, p_\gamma) - \Pi(\alpha, \mathrm{s}) - p'_\beta \dot{\beta} - p'_\gamma \dot{\gamma}; \quad (29)$$

The dynamics mathematical model for systems with geometrical connections in Routh variables is

$$\frac{d}{dt}\frac{\partial R}{\partial \alpha} - \frac{\partial R}{\partial \alpha}\tilde{Q}_\alpha + \beta'_\alpha(\alpha, s)\left(\frac{\partial R}{\partial s} + \tilde{Q}_S\right); \dot{p}_\beta = 0; \dot{p}_\lambda = 0; \dot{\mathrm{s}} = \beta_\alpha(\alpha, s)\dot{\alpha}; \quad (30)$$

Under conditions (26) the system allows stationary motion

$$p_\beta = p_{\beta 0} = const; p_{\gamma 0} = const; \alpha = \alpha_0 = const; s = s_0 = const; \quad (31)$$

Further, implementing a similar algorithm, disturbances are introduced, composing equations of the disturbed motion with a distinguished first approximation and applying the results of the theory of critical cases and the theory of stability under constantly acting disturbances. Variants of application of stabilizing controls are considered both for positional coordinates and for the smallest possible part of cyclic coordinates [34]. It is with the latter variant that the division of the vector of cyclic coordinates into some two sub-vectors is associated, one of which remains uncontrolled.

The explicit analytical form of the nonlinear mathematical model of controlled dynamics (including the description of transient processes in actuators[26,27,38] enables the consideration of a wide variety of stabilizing control applications for positional coordinates (or their smallest possible portion), for redundant coordinates, and for the smallest possible portion of cyclic coordinates. The latter option involves dividing the cyclic coordinate vector into two sub-vectors, one of which remains uncontrollable. The proposed approach allows for the derivation of numerous sufficient conditions for the solvability of stabilization problems, enabling the effective study of the dynamics of complex modern technical devices such as the Delta robot,[40-45] wheeled robots with various deformable wheel designs, and others.

A complete nonlinear mathematical model of the controlled dynamics for the Ball and Beam system

As an example of the application of the developed methodology, we present the study of the dynamics of the Ball and Beam stand. When constructing all mathematical models of the Ball and Beam system, we will adopt the following basic assumptions:

- consider a plane problem;

- neglect the thickness of the electric drive gearbox wheel;

- take the radius R of the ball into account;

- assume the mass of the chute to be negligible;

- ignore friction;

- take into account the external resistance during rotation of the electric motor gearbox shaft;

- assume ideal flat joints;

- assume the link AB to be a thin, straight, weightless, and absolutely rigid rod.

We will take the kinetic and potential energy of the mechanical part of the system, including the ball and the rotor of the engine with a gearbox, in the form [26,30]:

$$T = \frac{1}{2}\left(m(r\dot{\alpha})^2 + m\frac{(r\dot{r})^2}{r^2-R^2} - 2mR\frac{r\dot{r}\dot{\alpha}}{\sqrt{r^2-R^2}} + J\left(\frac{r\dot{r}}{R\sqrt{r^2-R^2}} - \dot{\alpha}\right)^2 + J_0\dot{\theta}^2 \right), \qquad (32)$$

$$\Pi = mg(\sqrt{r^2 - R^2}\sin\alpha + R\cos\alpha),$$

where R is the radius of the sphere, m is its mass, J is its moment of inertia, and J_0 is the moment of inertia of the entire system, reduced to the motor. The expressions for kinetic and potential energies are compiled taking into account the dimensions of the sphere, which is necessary if we consider equilibrium and motion near the origin. The dynamics of a commutator DC motor with separate excitation is described by the Kirchhoff equation [17]:

$$L_a\frac{di_a}{dt} + R_a i_a + K_3\frac{d\theta}{dt} = k_1 e_v, e_b = K_3\frac{d\theta}{dt}; \qquad (33)$$

Where e_v is the output voltage of the amplifier supplying power to the motor's armature winding, e_b is the back-EMF voltage, k_1 is the power converter coefficient, k_3 is the motor constant, L_a is the armature winding inductance, R_a is its resistance, and i_a is the armature circuit current.

Only potential forces act on the system along the α and r coordinates:

$$Q_r^{\Pi} = -\frac{\partial\Pi}{\partial r} = -mg\frac{r}{\sqrt{r^2-R^2}}\sin\alpha; Q_\alpha^{\Pi} = -\frac{\partial\Pi}{\partial\alpha} = -mg\left(\sqrt{r^2-R^2}\cos\alpha - R\sin\alpha\right).$$

Only the non-potential force[33] $Q_\theta = k_2 i_\alpha - b_0\dot{\theta}$ is applied along the θ coordinate, where b_0 is the rotational resistance coefficient reduced to the motor, and k_2 is the motor's electromechanical constant.

By differentiating the constraint equation (6) with respect to time, we obtain the kinematic constraint equation. From this equation, we can express the velocity or, depending on which variant of the dependent coordinate we are considering. We will take the connection equation in the form (34) or (35).

$$\dot{\alpha} = B^{\mathrm{I}}(\alpha,\theta)\cdot\dot{\theta}, \qquad (34)$$

$$B^{\mathrm{I}}(\alpha,\theta) = \frac{d}{l}\frac{(\mathrm{L}\sin(\alpha-\theta) + (\mathrm{L}-\mathrm{d})\sin\theta + l\cos\theta)}{(d\sin(\alpha-\theta) + (\mathrm{L}-\mathrm{d})\sin\alpha + l\cos\alpha)},$$

$$\dot{\theta} = B^{\mathrm{II}}(\alpha,\theta)\cdot\dot{\alpha}, B^{\mathrm{II}}(\alpha,\theta) = 1/B^{\mathrm{I}}(\alpha,\theta). \qquad (35)$$

Let's write Shulgin's equations for the Ball and Beam system in two cases. In the first, we will consider the dependent coordinate to be the chute inclination angle α. In the second case, the dependent coordinate will be the wheel rotation angle θ. The equations of the disturbed motion will differ in each case.

Remark 6.1. From now on, we will call **zero** the equilibrium position corresponding to the coordinate values

$$r = r_0 = const, \alpha = \alpha_0 = 0, \theta = 0; \qquad (36)$$

The second position

$$r = r_0 = const, \alpha = \alpha_0 = 0, \theta = \theta_1 \qquad (37)$$

Let's call it **non-zero**. From the problem statemen $0 < r_0 < L$.

Citation: Krasinskiy AY, Yuldashev AA. An analytical form for a rigorous nonlinear model of the dynamics for systems with geometric constraints and its use by applying the results of control theory to a complete solution of stabilization problems. *Int Rob Auto J.* 2026;12(1):98‒120. DOI: 10.15406/iratj.2026.12.00323

Remark 6.2. If we use a simplified approach and use the linearized equation instead $\alpha = \frac{d}{\mathrm{I}}\theta$ of the nonlinear equation (5), the second (nonzero) equilibrium position is lost. In all published works (except our articles), only equilibrium (36) has been considered so far. In addition to the differences discussed below, there is one obvious one: to increase the angle α (the slope of the trough in the vicinity), the angle θ must be increased, while for (37), to increase the angle α, the angle θ must be decreased.

Remark 6.3. The control coefficients for the same equilibrium position will be different for different choices of the redundant coordinate. This is because, according to the introduced redundant coordinate, the sets of phase vector components included in the controlled subsystems are different. Thus, we obtain a total of four different stabilization problems: two equilibria, two options for introducing the redundant coordinate.

Case I. Dependent coordinate α

We use (36) to eliminate the dependent velocity $\dot\alpha$ from the kinetic energy (6). In this case, the Shulgin equations are written as:

$$\begin{cases} \dfrac{d}{dt}\dfrac{\partial L^{\mathrm{I}*}}{\partial \dot r} - \dfrac{\partial L^{\mathrm{I}*}}{\partial r} = 0; \\ \dfrac{d}{dt}\dfrac{\partial L^{\mathrm{I}*}}{\partial \dot\theta} - \dfrac{\partial L^{\mathrm{I}*}}{\partial \theta} - B^{\mathrm{I}}(\alpha,\theta)\dfrac{\partial L^{\mathrm{I}*}}{\partial \alpha} = Q_\theta; \end{cases} \quad (38)$$

To these equations it is necessary to add the engine equation (33), and also take into account the differentiated constraint equation (8). We obtain the system

$$\begin{cases} \left(m + \left(m + \dfrac{1}{R^2}\right)\dfrac{1}{R^2}\right)\left(\dfrac{r^2}{r^2 - R^2}\ddot r - R\dfrac{rB^{\mathrm{I}}}{\sqrt{r^2-R^2}}\ddot\theta - \dfrac{R^2 r}{(r^2-R^2)^2}\dot r^2\right) - \\ -\left(R\left(m+\dfrac{1}{R^2}\right)\dfrac{r\left(\dfrac{\partial B^{\mathrm{I}}}{\partial\alpha}B^{\mathrm{I}} + \dfrac{\partial B^{\mathrm{I}}}{\partial\theta}\right)}{\sqrt{r^2-R^2}} + mrB^{\mathrm{I}^2}\right)\dot\theta^2 = -\dfrac{mgr\sin\alpha}{\sqrt{r^2-R^2}}; \\ \left(\left(mr^2+J\right)B^{\mathrm{I}^2} + J_0\right)\ddot\theta - R\left(m+\dfrac{1}{R^2}\right)\dfrac{rB^{\mathrm{I}}}{\sqrt{r^2-R^2}}\ddot r + 2mrB^{\mathrm{I}^2}\dot r\dot\theta + \\ +\left(mr^2+J\right)B^{\mathrm{I}}\left(\dfrac{\partial B^{\mathrm{I}}}{\partial\alpha}B^{\mathrm{I}} + \dfrac{\partial B^{\mathrm{I}}}{\partial\theta}\right)\dot\theta^2 + R^3\left(m+\dfrac{1}{R^2}\right)\dfrac{B^{\mathrm{I}}}{\left(r^2-R^2\right)3/2}\dot r^2 = \\ = k_2 i_a - b_0\dot\theta - mgB^{\mathrm{I}}\left(\sqrt{r^2-R^2}\cos\alpha - R\sin\alpha\right); \\ L_a\dfrac{di_a}{dt} + R_a i_a + k_3\dot\theta = k_1 e_v;\ \dot\alpha = B^{\mathrm{I}}(\alpha,\theta)\dot\theta \end{cases} \quad (39)$$

The values of the parameters of the system (39) in the equilibrium position ($r = r_0,\ \alpha = 0,\ \theta = \theta_i$):):

$$\begin{cases} 0 = k_2 i_a^0 - mgB^{\mathrm{I}}(0,\theta_i)\sqrt{r_0^2 - R^2};\Rightarrow \\ R_a i_a^0 = k_1 e_v^0; \end{cases}$$

$$\begin{cases} i_a^0 = \dfrac{mgB^{\mathrm{I}}(0,\theta_i)\sqrt{r_0^2-R^2}}{k_2}; \\ e_v^0 = \dfrac{R_a i_a^0}{k_1} = \dfrac{mgR_a B^{\mathrm{I}}(0,\theta_i)\sqrt{r_0^2-R^2}}{k_1 k_2}; \end{cases}$$

let's introduce disturbances

$$\begin{cases} r = r_0 + x_1; \dot{r} = x_2; \theta = \theta_1 + x_3^{\mathrm{I}}; \dot{\theta} = x_4^{\mathrm{I}}; \\ i_a = i_a^0 + x_5; \alpha = x_6^{\mathrm{I}}; e_v = e_v^0 + u_i^{\mathrm{I}}; \end{cases} \quad (40)$$

Here, u_i^{I} is the control. The subscript on θ denotes the equilibrium position, while the superscript denotes the choice of the dependent coordinate.

Remark 6.4. For different choices of the redundant coordinate, not only the equations of motion but also some components of the phase vectors will differ. For the dependent coordinate α, we will denote the increments of these coordinates by the superscript I, and for the dependent coordinate θ, by the subscript II.

Having isolated the first approximation in the system of equations (39), we write it in normal form.

$$\dot{x}^{\mathrm{I}} = H_i^{\mathrm{I}} x^{\mathrm{I}} + S u_i^{\mathrm{I}} + X^{\mathrm{I}(2)}, \quad (41)$$

where $(x^{\mathrm{I}})' = (x_1, x_2, x_3^{\mathrm{I}}, x_4^{\mathrm{I}}, x_5^{\mathrm{I}}, x_6^{\mathrm{I}})'$ is the phase vector, $H_i^{I}$is the matrix, in whose non-zero elements are calculated using the formulas

$$h_{21} = -mg \frac{RB^{\mathrm{I}^2}}{\left(m\left(r_0^2 - R^2\right)B^{\mathrm{I}^2} + J_0\right)}; h_{23} = -\frac{mg}{r_0} \frac{RB^{\mathrm{I}}\left(r_0^2 - R^2\right)}{\left(m\left(r_0^2 - R^2\right)B^{\mathrm{I}^2} + J_0\right)} \frac{\partial B^{\mathrm{I}}}{\partial \theta};$$

$$h_{41} = \frac{-mg}{\sqrt{r_0^2 - R^2}} \frac{B^{\mathrm{I}} r_0}{\left(m\left(r_0^2 - R^2\right)B^{\mathrm{I}^2} + J_0\right)}; h_{43} = \frac{-mg\sqrt{r_0^2 - R^2}}{\left(m\left(r_0^2 - R^2\right)B^{\mathrm{I}^2} + J_0\right)} \frac{\partial B^{\mathrm{I}}}{\partial \theta};$$

$$h_{44} = \frac{-b_0}{\left(m\left(r_0^2 - R^2\right)B^{\mathrm{I}^2} + J_0\right)}; h_{45} = \frac{k_2}{\left(m\left(r_0^2 - R^2\right)B^{\mathrm{I}^2} + J_0\right)};$$

$$h_{46} = \frac{-mg\sqrt{r_0^2 - R^2}}{\left(m\left(r_0^2 - R^2\right)B^{\mathrm{I}^2} + J_0\right)} \frac{\partial B^{\mathrm{I}}}{\partial \alpha}; h_{54} = -\frac{k_3}{L_a}; h_{55} = \frac{R_a}{L_a}.$$

$$B^{\mathrm{I}} = B^{\mathrm{I}}(0, \theta_i), \frac{\partial B^{\mathrm{I}}}{\partial \alpha} = \left.\frac{\partial B^{\mathrm{I}}}{\partial \alpha}\right|_{(0,\theta_i)}, \frac{\partial B^{\mathrm{I}}}{\partial \theta} = \left.\frac{\partial B^{\mathrm{I}}}{\partial \theta}\right|_{(0,\theta_i)}, s = (0,0,0,0,s,0)', s = \frac{k_1}{L_a}$$

$X^{\mathrm{I}(2)}$ is a vector containing terms of the second order and higher.

In system (41), the matrix elements depend on the derivatives of the kinematic constraint coefficients (36), calculated for different equilibrium positions.

To identify the variables corresponding to the zero roots of the characteristic equation, we apply (according to the theory of critical cases) a linear substitution[26,33]:

$$x_6^I = B^I x_3^I + z^I. \quad (42)$$

Then the differentiated equation of connection (36, included in system (39), in the first approximation will take the form $\dot{z}^I = 0$, and the coefficients of the variable x_3^I will change as follows:

$$h_{23}' = h_{23} + h_{26}B^{\mathrm{I}} = \frac{mg}{r_0} \frac{B^{\mathrm{I}}\sqrt{r_0^2 - R^2}\left(m\left(r_0^2 - R^2\right)B^{\mathrm{I}^2} + J_0 + R\left(m + \frac{1}{R^2}\right)\sqrt{r_0^2 - R^2}\left(B^{\mathrm{I}}\frac{\partial B^{\mathrm{I}}}{\partial \alpha} + \frac{\partial B^{\mathrm{I}}}{\partial \theta}\right)\right)}{\left(m + \frac{1}{R^2}\right)\left(m\left(r_0^2 - R^2\right)B^{\mathrm{I}^2} + J_0\right)};$$

$$h_{43}' = h_{43} + h_{26}B^{\mathrm{I}} = -mg \frac{\sqrt{r_0^2 - R^2}}{m\left(r_0^2 - R^2\right)B^{\mathrm{I}^2} + J_0}\left(B^{\mathrm{I}}\frac{\partial B^{\mathrm{I}}}{\partial \alpha} + \frac{\partial B^{\mathrm{I}}}{\partial \theta}\right);$$

Now, according to,[26] the controlled subsystem should be chosen as

$$\dot{w}^{\mathrm{I}} = M_i^{\mathrm{I}} w^{\mathrm{I}} + N u_i^{\mathrm{I}} \qquad (43)$$

$$M_i^{\mathrm{I}} = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ h_{21} & 0 & h_{23} & h_{24} & h_{25} \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & h_{43} & h_{44} & h_{45} \\ 0 & 0 & 0 & h_{54} & h_{55} \end{pmatrix}, N = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ s \end{pmatrix}.;$$

not containing the critical variable z^I, where $(w^{\mathrm{I}})' = (x_1, x_2, x_3^{\mathrm{I}}, x_4^{\mathrm{I}}, x_5)$ is the phase vector. The controllability condition for system (43)

$$\operatorname{rank}\left(N\, M_i^{\mathrm{I}} N M_i^{\mathrm{I}^2} \cdot N\, M_i^{\mathrm{I}^3} \cdot N\, M_i^{\mathrm{I}^4} \cdot N\right) = 5 \quad (44)$$

completed. According to the method of N.N. Krasovskii,[16], the coefficients of the stabilizing control $u_i^I = K_i^I w^I$ are found by solving a linear-quadratic stabilization problem, and the integrand of the quality criterion is chosen in the for

$$\omega^{\mathrm{I}} = (x_1)^2 + (x_2)^2 + (x_3^{\mathrm{I}})^2 + (x_4^{\mathrm{I}})^2 + (x_5)^2 + u_i^{\mathrm{I}^2}. \quad (45)$$

Remark 6.5. The first approximation of equations (41) depends on the quadratic terms of the expansion of the geometric constraint equation (or the linear terms of the expansion of the kinematic constraint), which can affect the stability of the equilibrium position. The admissibility condition for the transition to linearized constraints is that the indicated terms vanish at the equilibrium position under consideration.

For the Ball and Beam system, the partial derivatives $\frac{\partial B^{\mathrm{I}}}{\partial \alpha}, \frac{\partial B^{\mathrm{I}}}{\partial \theta}, \frac{\partial B^{\mathrm{II}}}{\partial \alpha}, \frac{\partial B^{\mathrm{II}}}{\partial \theta}$ have the following values:

	Dependent coordinate α	**Dependent coordinate θ**
Zero equilibrium position	$B^{\mathrm{I}}(0,0) = \frac{d}{L}$, $\left.\frac{\partial B^{\mathrm{I}}}{\partial \alpha}\right\|_0 = \left.\frac{\partial B^{\mathrm{I}}}{\partial \theta}\right\|_0 = 0.$	$B^{\mathrm{II}}(0,0) = \frac{L}{d}$, $\left.\frac{\partial B^{\mathrm{II}}}{\partial \alpha}\right\|_0 = \left.\frac{\partial B^{\mathrm{II}}}{\partial \theta}\right\|_0 = 0.$
Non-zero equilibrium position	$B^{\mathrm{I}}(0,\theta_1) = \frac{d}{l} \cdot \frac{l^2+d^2}{d^2-l^2}$, $\left.\frac{\partial B^{\mathrm{I}}}{\partial \alpha}\right\|_{(0,\theta_1)} = \frac{2d^2 l}{L} \cdot \frac{(2dl-d^2-l^2)}{(l^2-d^2)^2}$, $\left.\frac{\partial B^{\mathrm{I}}}{\partial \theta}\right\|_{(0,\theta_1)} = 0.$	$B^{\mathrm{II}}(\alpha_1,\theta_1) = \frac{L}{d} \cdot \frac{d^2-l^2}{l^2+d^2}$, $\left.\frac{\partial B^{\mathrm{II}}}{\partial \alpha}\right\|_{(0,\theta_1)} = 2Ll \cdot \frac{(2dl-d^2-l^2)}{(l^2+d^2)^2}$, $\left.\frac{\partial B^{\mathrm{II}}}{\partial \theta}\right\|_{(0,\theta_1)} = 0.$

Remark 6.6.. Calculating these derivatives is one of the most labor-intensive operations. They must first be obtained in analytical form, and then their numerical values determined by substituting the values of the phase variables for the unperturbed motion into the analytical form. The problem of automating this procedure through the use of modern information technologies for processing symbolic information has been considered by the authors for a long time. In this area, there are both original officially registered software products[66-68] and software tools based on the application of software modules built into modern, widely used software environments. For example, when modeling the nonlinear controlled dynamics of Delta robot with three nonlinear geometric constraints, all calculations were performed using a software implementation written in the Python software environment. The following built-in libraries were used: scipy, numpy, matplotlib, and sympy.

Therefore, linearization of the geometric constraint equations is permissible when studying the zero equilibrium position, but when studying the stability of a non-zero position, quadratic terms must be taken into account. The results of the numerical calculations are presented below.

Case II. Dependent coordinate *θ*

In this case, all calculations are carried out similarly, so we present only the basic formulas and results.

Shulgin's equations can be written as:

$$\begin{cases} \frac{d}{dt}\frac{\partial L^{\mathrm{II}*}}{\partial \dot{r}} - \frac{\partial L^{\mathrm{II}*}}{\partial r} = 0; \\ \frac{d}{dt}\frac{\partial L^{\mathrm{II}*}}{\partial \alpha} - \frac{\partial L^{\mathrm{II}*}}{\partial \alpha} = B^{\mathrm{II}}(\alpha,\theta)\left(\frac{\partial L^{\mathrm{II}*}}{\partial \theta} + Q_\theta^*\right) \end{cases} ; \quad (46)$$

The complete mathematical model takes into account the constraint equation (35) and equation (33), which describes the dynamics of the electric drive.

Parameter values at equilibrium $(\mathrm{r} = \mathrm{r}_0, \alpha = 0, \theta = \theta_i)$:

$$\begin{cases} i_a^0 = \dfrac{mg\sqrt{r_0^2-R^2}}{k_2 B^{\mathrm{II}}(0,\theta_i)}\,; e_v^0 \dfrac{mgR_a\sqrt{r_0^2-R^2}}{k_1 k_2 B^{\mathrm{II}}(0,\theta_i)}\,; \end{cases}$$

Let's introduce the perturbations:

$$\begin{cases} r = r_0 + x_1\,; \dot r = x_2\,; \dot\alpha = x_3^{\mathrm{II}}\,; \dot\alpha = x_4^{\mathrm{II}}\,; \\ i_a = i_a^0 + x_5\,; \theta = \theta_i + x_6^{\mathrm{II}}\,; e_v = e_v^0 + u_i^{\mathrm{II}} \end{cases}$$

The first approximation of the system in normal form is similar to (43):

$\dot x^{\mathrm{II}} = H_i^{\mathrm{II}} x^{\mathrm{II}} + S u_i^{\mathrm{II}} + X^{\mathrm{II}(2)}$. (47)

Note that here, unlike (43), some of the components of the phase vector in (47) $x^{\mathrm{II}} = (x_1, x_2, x_3^{\mathrm{II}}, x_4^{\mathrm{II}}, x_5, x_6^{\mathrm{II}})$ correspond to other coordinates. In this case, the coefficients of the matrix H_i^{II}:

$$h_{21} = \frac{-mgR}{m\left(r_0^2 - R^2\right)B^{\mathrm{I}^2} + J_0}\,; h_{24} = \frac{-b_0 R B^{\mathrm{II}^2}\sqrt{r_0^2 - R^2}}{r_0\left(m\left(r_0^2 - R^2\right) + J_0 B^{\mathrm{II}^2}\right)}\,;$$

$$h_{23} = -mg\sqrt{r_0^2 - R^2}\;\frac{m\left(r_0^2 - R^2\right) + J_0 B^{\mathrm{II}^2} - R\left(m + \dfrac{J}{R^2}\right)\left(\dfrac{\sqrt{r_0^2-R^2}}{B^{\mathrm{II}}}\right)\dfrac{\partial B^{\mathrm{II}}}{\partial\alpha}}{r_0\left(m + \dfrac{J}{R^2}\right)\left(m\left(r_0^2 - R^2\right) + J_0 B^{\mathrm{I}^2}\right)}\,;$$

$$h_{25} = \frac{k_2 R B^{\mathrm{II}}\sqrt{r_0^2-R^2}}{r_0\left(m\left(r_0^2-R^2\right)+J_0 B^{\mathrm{II}^2}\right)}\,; h_{26} = \frac{mgRB^{\mathrm{II}}\left(r_0^2-R^2\right)}{r_0\left(m\left(r_0^2-R^2\right)+J_0 B^{\mathrm{II}^2}\right)}\frac{1}{B^{\mathrm{II}}}\frac{\partial B^{\mathrm{II}}}{\partial\theta}\,;$$

$$h_{41} = \frac{-mgr_0}{\sqrt{r_0^2-R^2}\left(m\left(r_0^2-R^2\right)+J_0 B^{\mathrm{II}^2}\right)}\,; h_{44} = \frac{-b_0 B^{\mathrm{II}}}{m\left(r_0^2-R^2\right)+J_0 B^{\mathrm{II}^2}}\,;$$

$$h_{43} = \frac{mg\sqrt{r_0^2-R^2}}{\left(m\left(r_0^2-R^2\right)+J_0 B^{\mathrm{II}^2}\right)}\frac{1}{B^{\mathrm{II}}}\frac{\partial B^{\mathrm{II}}}{\partial\alpha}\,; h_{45} = \frac{B^{\mathrm{II}} K_2}{m\left(r_0^2-R^2\right)+J_0 B^{\mathrm{II}^2}}\,;$$

$$h_{46} = \frac{mg\sqrt{r_0^2-R^2}}{\left(m\left(r_0^2-R^2\right)+J_0 B^{\mathrm{II}^2}\right)}\frac{1}{B^{\mathrm{II}}}\frac{\partial B^{\mathrm{II}}}{\partial\theta}\,; h_{54} = -k_3\frac{B^{\mathrm{II}}}{L_a}\,; h_{55} = \frac{R_a}{L_a}\,;$$

The controlled subsystem can be written in a form similar to (43):

$\dot w^{\mathrm{II}} = M_i^{\mathrm{II}} w^{\mathrm{II}} + N u_i^{\mathrm{II}}$. (48)

The components of the vector $w^{\mathrm{II}} = (x_1, x_2, x_3^{\mathrm{II}}, x_4^{\mathrm{II}}, x_5^{\mathrm{II}})$ the coefficients of the matrix M_i^{II} are distinct. For (48), the controllability condition[24] will also be satisfied:

$$\mathrm{rank}\left(N\, M_i^{\mathrm{II}} N M_i^{\mathrm{II}^2} \cdot N\, M_i^{\mathrm{II}^3} \cdot N\, M_i^{\mathrm{II}^4} \cdot N\right) = 5 \; (49)$$

The stabilizing control coefficients $u_i^{II} = K_i^{II} w^{II}$ are determined by solving the linear-quadratic stabilization problem, and the integrand in the quality criterion, similar to (45)

$$\omega^{\mathrm{II}} = (x_1^{\mathrm{II}})^2 + (x_2^{\mathrm{II}})^2 + (x_3^{\mathrm{II}})^2 + (x_4^{\mathrm{II}})^2 + (x_5^{\mathrm{II}})^2 + u_i^{\mathrm{II}^2}\,. \qquad (50)$$

Numerical determination of control laws

For the numerical calculations, we take the following parameter values:

$m = 0.064$ kg…ball mass; $R = 0.0254$ m…ball radius;

$L = 0.425$ m…trough length; $l = 0.34$ m………..arm length;

$d = 0.12$ m…drive wheel radius.

The stabilizing control coefficients are calculated using N.N. Krasovskii's method[16] for each equilibrium position for two choices of the dependent coordinate. To obtain transient process graphs, a system of differential equations describing the motion of the system closed by the determined control is solved.

Remark 6.7. The initial perturbations of the angles α and θ are, generally speaking, not free and must satisfy geometric constraint (5). To plot graphs of the same stabilization process in different mathematical models, it is necessary to take into account the constraints of the initial perturbations. The initial perturbations of the remaining coordinates were chosen arbitrarily and used for all graphs.

Remark 6.8. Since the stabilization process is being modeled for the same mechatronic system with corresponding initial perturbations, the graphs of changes in the coordinates of the ball's position, its velocity, and current are identical for the same equilibrium position and do not depend on the choice of the dependent coordinate. At the same time, the control, as a function of time, is also independent of the choice of the redundant coordinate, but changes only depending on the equilibrium position.

Case of the zero equilibrium position

1) Dependent coordinate – angle α.

Calculated optimal control:

$$u_0^{\mathrm{I}} = -1435 \cdot x_1 + 720 \cdot x_2 - 708 \cdot x_3^{\mathrm{I}} + 22 \cdot x_4^{\mathrm{I}} + 84.5 \cdot x_5.$$

Figures 4–9.

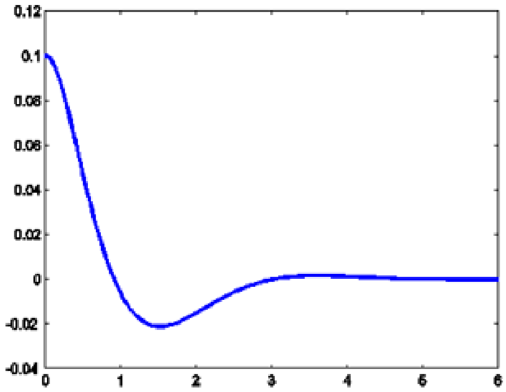

Figure 4 Graph x_1 (t).

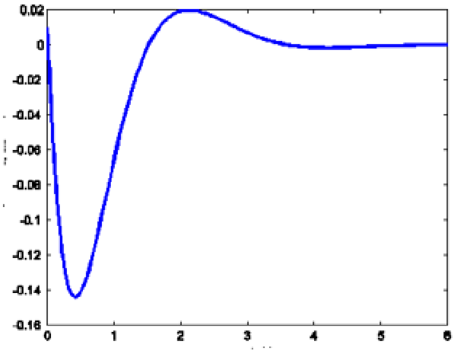

Figure 5 Graph x_2 (t).

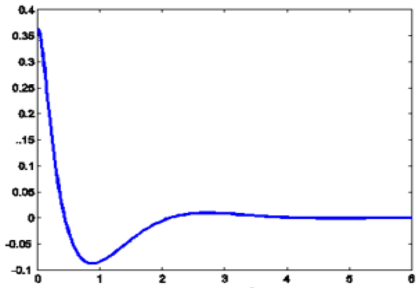

Figure 6 Graph x_3 (t)

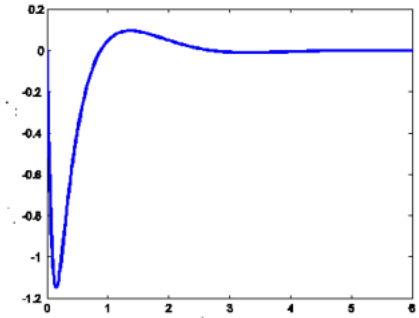

Figure 7 Graph x_4 (t).

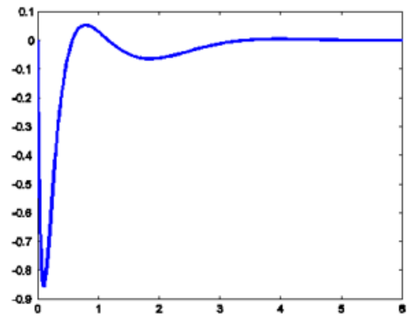

Figure 8 Graph $x_5(t)$.

Citation: Krasinskiy AY, Yuldashev AA. An analytical form for a rigorous nonlinear model of the dynamics for systems with geometric constraints and its use by applying the results of control theory to a complete solution of stabilization problems. *Int Rob Auto J.* 2026;12(1):98–120. DOI: 10.15406/iratj.2026.12.00323

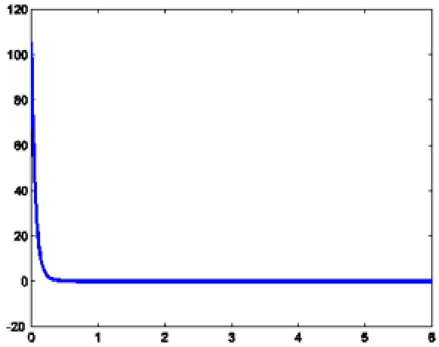

Figure 9 Graph $u_0^I(t)$.

2) The angle θ is chosen as the dependent coordinate.

Calculated optimal control:

$$u_0^{II} = 1434.4 \cdot x_1 + 720 \cdot x_2 - 2508.3 \cdot x_3^{II} - 79.2 \cdot x_4^{II} + 84.5 \cdot x_5.$$

Transient process graphs Figures 10–12

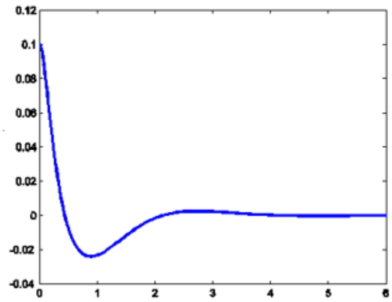

Figure 10 Graph of $x_3^{II}(t)$.

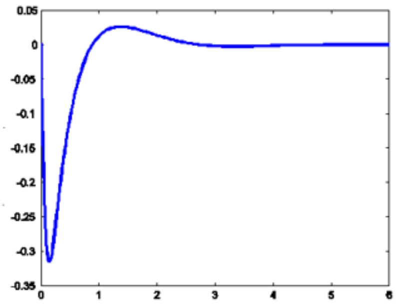

Figure 11 Graph of $x_4^{II}(t)$.

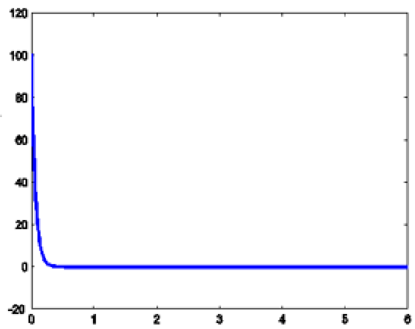

Figure 12 Graph of $u_0^{II}(t)$.

Case of a non-zero equilibrium position

1) The angle α is chosen as the dependent coordinate.

The calculated optimal control:

$$u_1^{I} = 1993 \cdot x_1 + 817 \cdot x_2 - 898 \cdot x_3^{I} - 32 \cdot x_4^{I} + 99 \cdot x_5.$$

The transient process graphs are shown in Figures 13–18.

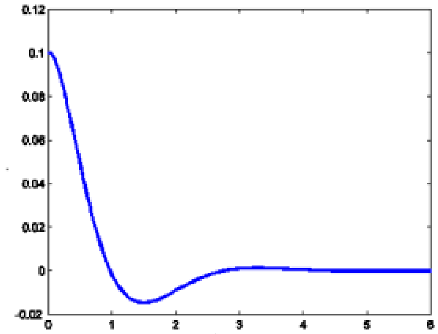

Figure 13 Graph $x_1(t)$.

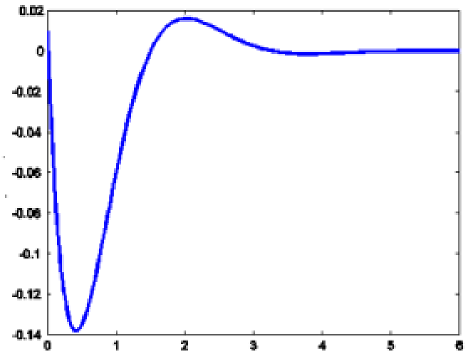

Figure 14 Graph $x_2(t)$.

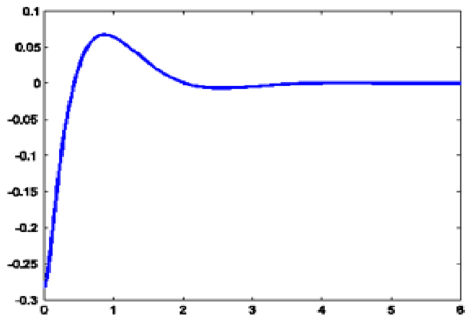

Figure 15 Graph of $x_3^I(t)$.

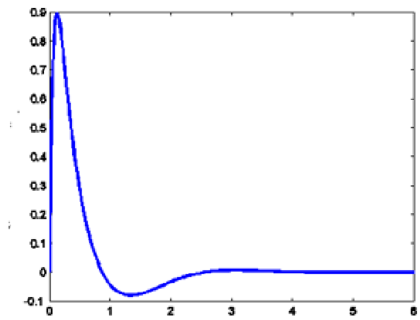

Figure 16 Graph of $x_4^I(t)$.

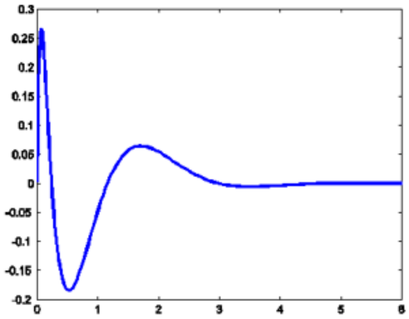

Figure 17 Graph of $x_5(t)$.

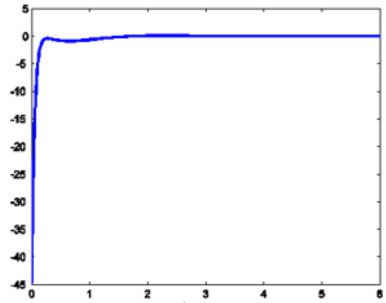

Figure 18 Graph of $u_1^I(t)$.

The angle θ is chosen as the dependent coordinate.

The calculated optimal control is:

$$u_1^{II} = 1986 \cdot x_1 + 857 \cdot x_2 - 2451.5 \cdot x_3^{II} - 88.4 \cdot x_4^{II} + 98 \cdot x_5.$$

The transient response graphs are shown in Figures 19–21.

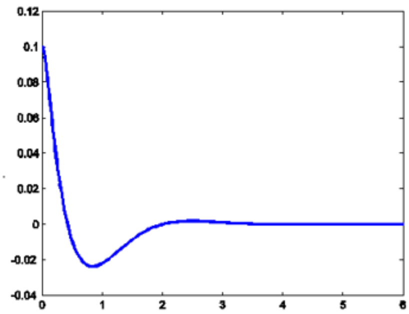

Figure 19 Graph of $x_3^{II}(t)$.

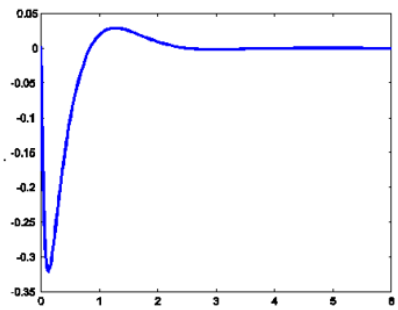

Figure 20 Graph of $x_4^{II}(t)$.

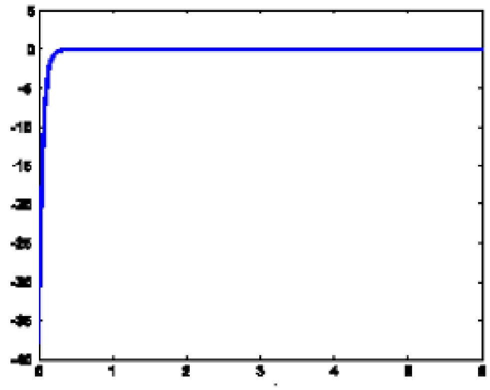

Figure 21 Graph of $u_1^{II}(t)$.

We note once again that the simulation for each equilibrium was performed with the same initial disturbances, regardless of the choice of the dependent coordinate. A special module has been introduced into the software for this purpose, ensuring not only the matching of initial disturbances with the constraint but also the preservation of these disturbance values when switching to a different choice of dependent coordinate. The presented graphs provide the basis for the following conclusions:

1. In all four problems, switching from one choice of dependent coordinate to another leaves the behavior of variables x_1, x_2, which describe the ball's motion along the groove, and variable x_5—the current in the actuator motor's armature winding—virtually unchanged.

2. The graphs for x_3, x_4 differ significantly, which is quite natural, since changing the choice of dependent coordinate also changes the choice of these variables: the composition of the phase vector of the controlled subsystem changes.

3. In controls u_1^I and u_1^{II}, the coefficients of x_1, x_2, and x_5 are practically identical, as are these controls themselves, as functions of time. The significant differences in the coefficients of x_3, x_{42} are already explained by the differences in the coefficients of different phase variables.

4. The slight difference between $u_i^I(t)$ and $u_i^{II}(t)$ is easily explained: they were determined by minimizing different performance criteria. It is obvious that the amplitude of the damped oscillations for the angle of inclination of the chute and the angle of rotation of the output drive wheel differs significantly.

5. The signs of the coefficients of x_1, x_2 in the zero- and non-zero-equilibrium controls confirm the *Remark 6.2.* With a positive initial disturbance in x_1, to return this variable to zero, the chute inclination angle should be increased. For zero equilibrium, this requires an increase in the output drive wheel rotation — the sign of x_1 in the control is positive. For a non-zero equilibrium, increasing the angle α requires turning the output drive wheel back and decreasing the angle θ: the sign of the coefficient of x_1 in the control for this equilibrium is negative.

Remark 6.9. We note another advantage of our method for describing nonlinear dynamics, which cannot be achieved with any of the traditionally used approaches and which is a direct consequence of the presence of an analytical model. Note that, given the same structure of the control coefficient matrices and of the measurement coefficient matrices the Riccati equations for the obtaining control, and for state estimation problems completely coincide if the matrices in the quality criteria are the same. In this way, it is possible to significantly reduce the computational complexity of generating a control signal, which can increase the reliability of the control loop operation with incomplete information about the state.

Conclusion

For the first time, a single article presents both a complete, rigorous justification of the developed general approach to applying methods of analytical mechanics, critical case theory from stability theory, and mathematical control theory to non-free systems (including systems with incomplete information), as well as algorithms for its practical application to the dynamics of a specific technical device. This research resulted in the creation of a general, rigorously tested method for nonlinear mathematical modeling of the controlled dynamics of non-free systems with an arbitrary number of geometric constraints, based on the qualified application of analytical mechanics in redundant coordinates with a rational choice of model variables.

The effectiveness of this method was tested and demonstrated using the example of solving a stabilization problem for the Ball and Beam setup configuration. We note once again that it was the complete solution of the stabilization problem for this device configuration with a single geometric constraint that served as the basis for the development of the general method presented in this article. A similarly effective application was demonstrated[40–45,69] in a delta robot—a manipulator with three geometric constraints, widely used in the real world. Modeling the dynamics of this robot has been the subject of numerous publications not only by technical specialists but also by specialists engaged in abstract theoretical research.[8–12,18–25]

Despite our numerous publications detailing the general methodology of rigorous approaches developed for the mathematical modeling of nonlinear controlled dynamics of systems with geometric constraints and their use in studying the dynamics of real technical devices, our results have not found application beyond technical practice. Unfortunately, this approach has gone completely unnoticed by specialists attempting to develop theoretical methods for mathematical modeling of the dynamics of modern technical devices with geometric constraints.

We note once again: in virtually all published studies on the modeling of specific technical devices, the analysis is based on Lagrange's equations with constraint multipliers. Moreover, when studying the delta robot's dynamics, completely irrational coordinates were used for the geometric constraint model (see[44]). Given this situation, in this article, the authors present a full justification of the developed method and a detailed description of the algorithm for its application and a detailed example of its specific application.

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None.

Conflicts of interest

There are no conflicts of interest.

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