Review Article

Open Access CrossMark

Optimal control of a spacecraft based on the combined criteria of quality related to the energy costs

Abstract

Solution of optimal control problem of spacecraft reorientation using the combined indicator (in sense of energy consumptions and time for rotation) and quaternion models is presented in analytical form. The minimized functional combines in a given proportion the contribution of control forces and the integral of rotary energy, as well as maneuver's duration. The construction of optimal turn control is based on a differential equation relating the attitude quaternion and angular momentum of a spacecraft. Analytical solution is obtained using the Pontryagin's maximum principle. The properties of optimal rotation are studied in detail. Formalized equations and computational formulas are written to construct the optimal rotation program. Analytical equations and relations for finding the optimal control are presented. Key relations that determine the optimal values of the parameters of rotation control algorithm are given. The made numerical experiments confirm the analytical conclusions. In the case of a dynamically symmetric solid body, the problem of spatial reorientation with minimum energy and time consumption is completely solved (in closed form).

Keywords: spacecraft attitude, quaternion, the combined criteria of optimality, maximum principle, optimal control, the boundary-value problem

Volume 11 Issue 2 - 2025

Levskii MV

Maksimov Research Institute of Space System, Russia

Correspondence: Levskii MV, Maksimov Research Institute of Space System, Russia

Received: May 20, 2025 | **Published:** June 25, 2025

Introduction

The problem of optimal reorientation of a solid (in particular, a spacecraft) from its initial angular position into a requested angular position is investigated in detail. Main difference of the proposed solution is new quality index based on which the optimal control is formed. Huge number of works study the problems of controlling the angular position of solid in various formulations and using a wide range of solution methods.[1–27] Some authors propose a synthesis of optimal control based on analytical design of optimal regulators,[1] others construct the program motion on basis of inverse problems of dynamics and obtain smooth controls to provide the properties of the programmed trajectory to a polynomial of a given degree with coefficients calculated by the values of phase variables at the boundary points of trajectory.[2] The issues of optimal control are of special attention.[3–25] The methods of optimization are various. In the problems of solid body reorientation, many authors apply the Pontryagin's maximum principle,[8–25] including classical criteria of optimality (fast performance,[4-12] minimum energy consumption,[11,13,14] minimum fuel consumption,[13] etc.). Kinematic problems of a turn have been studied in more detail.[15–17] Optimal control problems in the dynamical formulation are of particular interest, but here we encounter certain difficulties when solving the boundary-value problem on turn. In certain special cases, boundary-value two-point problem is solved by the method of separation of variables.[13] Analytical solutions of optimal control problem of a turn remain practically important. However, it is extremely difficult to obtain them for bodies with arbitrary moments of inertia. Several solutions (including analytical ones) are known for rotations of spherically symmetric bodies[12,18] and dynamically symmetric bodies.[9–11,19–22] In the paper published earlier,[19] optimal angular momentum vector of axial symmetric solid rotates in body coordinate system around the symmetry axis of body with constant angular velocity, but remains invariable in modulus (absolute angular velocity vector of solid body rotates in body coordinate system around the symmetry axis with same constant angular velocity, remaining unchanged in modulus); however, the minimized functional used by the authors did not include controlling torques.

Below we solve the problem of optimal turn of a spacecraft using new quality indicator that combines time and energy costs, in a given proportion, integral of rotation energy and contribution of control forces to perform a turn (in terms of energy costs). Phase variables are angular momentum of a spacecraft and attitude quaternion.

Equations of motion and formulation of optimal control problem

Spacecraft's rotary motion is described by the following equation[12]

$$\dot{\mathbf{L}} + (I^{-1}\mathbf{L}) \times \mathbf{L} = \mathbf{M} \quad (1)$$

and kinematical equation[12]

$$2\dot{\Lambda} = \Lambda \circ (I^{-1}\mathbf{L}) \quad (2)$$

where $\mathbf{L}$ is angular momentum of a spacecraft; $\mathbf{M}$ is the control torque; I is the inertia tensor of a spacecraft; Λ is the normalized quaternion[12] specified a motion of the related basis relative to inertial basis ($\|\Lambda\| = 1$), « $\circ$ » means multiplication of quaternions.[11–20] Control of a spacecraft around its center of mass is carried out by changing of the torque $\mathbf{M}$. We assume that spacecraft angular momentum $\mathbf{L}$ is zero in initial and final instant of time. Then, the boundary conditions for the controlled system (1)–(2) are:

$$\Lambda(0) = \Lambda_{in}, \;\; \mathbf{L}(0) = 0 \quad (3)$$

$$\Lambda(T) = \Lambda_f, \;\; \mathbf{L}(T) = 0 \quad (4)$$

where T is the time of ending the reorientation, and the quaternions Λ_{in} and Λ_f satisfy the condition $\|\Lambda_{in}\| = \|\Lambda_f\| = 1$.

In addition, we assume that rotary motion is regulated with use of attitude system that creates torques about three main central principal axes of inertia of a spacecraft. Control is optimal if the index

$$G=\int_0^T (M_1^2/J_1+M_2^2/J_2+M_3^2/J_3)dt + k_1 \int_0^T (L_1^2/J_1+L_2^2/J_2+L_3^2/J_3)dt + k_2 T \qquad (5)$$

is minimum, where $k_1>0$, $k_2>0$ are the constant positive coefficients ($k_1 \neq 0, k_2 \neq 0$); M_i are projections of control torque **M** onto the principal central axes of the spacecraft's inertia ellipsoid (these axes form the related basis); L_i are the projections of spacecraft's angular momentum **L** onto axes of the related basis; J_i are the main central moments of inertia of the spacecraft ($i=\overline{1,3}$).

The optimal control problem is formulated as follows: it is necessary to move the spacecraft from position (3) into position (4) in accordance with equations (1) and (2) with minimal sum (5) (the time T is not fixed). Optimal solution $\mathbf{M}(t)$ is a piecewise continuous function of time. Since Λ and -Λ reflect identical angular position in inertial coordinate system, we consider the problems when $\Lambda_f \neq \pm\Lambda_{in}$.

The adopted criteria of quality allows determine the mode of spacecraft rotation, at which a spacecraft moves from its initial position Λ_{in} into prescribed final angular position Λ_f with minimal costs of the control resources and energy, and find the relevant control program. The formulated control problem differs from the problems considered earlier due to the form of the functional (5), with which the control variables cannot be unlimited large even in the absence of constraints on the control. Other difference consists in following: since no constraints are imposed on the control torque **M**, the desired rotational maneuver is implemented at any Λ_{in} and Λ_f, and any values J_1, J_2, J_3, k_1 and k_2. Since optimization is based on the combination of quadratic criteria of quality and duration of maneuver T (in given proportion, with the prescribed coefficient of proportionality), then optimal value T_{opt}, relative to that the sum (5) increases with increasing as well as decreasing of the time T, exists. The coefficient k_1 determines how flat the change in the modulus of angular momentum will be during an optimal turn. The value of k_2 determines maximal modulus of control torque, and relation k_2/k_1 determines maximal kinetic energy of spacecraft rotation during reorientation maneuver.

Solving the reorientation problem and determination of optimal control

We solve the considered problem (in formulation (1)-(5)) as dynamic optimal rotation of a solid,[12] where torques M_i are control functions (because they are included in the indicator (5) minimized). Optimal control is found by the maximum principle,[29] using the following variables r_i[18]:

$$r_1=(\lambda_0\psi_1+\lambda_3\psi_2-\lambda_1\psi_0-\lambda_2\psi_3)/2,$$

$$r_2=(\lambda_0\psi_2+\lambda_1\psi_3-\lambda_2\psi_0-\lambda_3\psi_1)/2, \quad r_3=(\lambda_0\psi_3+\lambda_2\psi_1-\lambda_3\psi_0-\lambda_1\psi_2)/2$$

where ψ_j are the conjugate variables, corresponding to the components λ_j quaternion of attitude Λ ($j=\overline{0,3}$).

Optimal functions r_i and the vector **r** formed by r_i satisfy the equations[11,12,18]

$$\dot r_1=L_3r_2/J_3-L_2r_3/J_2,\quad \dot r_2=L_1r_3/J_1-L_3r_1/J_3,$$

$$\dot r_3=L_2r_1/J_2-L_1r_2/J_1,\quad \dot{\mathbf{r}}=\mathbf{r}\times(I^{-1}\mathbf{L}) \qquad (6)$$

(symbol × означает means the vector product of vectors).

Also, we introduce conjugate variables ϕ_i, corresponding to the projections of the spacecraft angular momentum L_i, and write the Hamilton-Pontryagin function

$$H=-k_2 - k_1(L_1^2/J_1+L_2^2/J_2+L_3^2/J_3) - M_1^2/J_1-M_2^2/J_2-M_3^2/J_3+\phi_1(M_1+(1/J_3-1/J_2)L_2L_3)+$$

$$+\phi_2(M_2+(1/J_1-1/J_3)L_1L_3)+\phi_3(M_3+(1/J_2-1/J_1)L_1L_2)+L_1r_1/J_1+L_2r_2/J_2+L_3r_3/J_3$$

The equations for ϕ_i are[29]

$$\dot\phi_i=-\frac{\partial H}{\partial L_i}\quad (i=\overline{1,3})$$

Therefore, the coupled system has the form:

$$\dot\phi_1=2k_1L_1/J_1+L_3\phi_2(1/J_3-1/J_1)+L_2\phi_3(1/J_1-1/J_2)-r_1/J_1$$

$$\dot\phi_2=2k_1L_2/J_2+L_1\phi_3(1/J_1-1/J_2)+L_3\phi_1(1/J_2-1/J_3)-r_2/J_2 \qquad (7)$$

$$\dot\phi_3=2k_1L_3/J_3+L_2\phi_1(1/J_2-1/J_3)+L_1\phi_2(1/J_3-1/J_1)-r_3/J_3$$

When composing the Hamilton-Pontryagin function, the constraint $\|\Lambda\|=1$ is not taken into account due to the equality $\|\Lambda(0)\|=1$, as previously mentioned. The vector **r** is stationary relative to the inertial basis and $\|\mathbf{r}\|$=const≠0 (the constancy of the modulus $|\mathbf{r}|$ follows from the properties of equations (6)).[12,18] The solution $\mathbf{r}(t)$ of system (6) is determined by the initial Λ_{in} and final Λ_f positions of the spacecraft. Optimal function $\mathbf{r}(t)$ is calculated through the quaternion $\Lambda(t)$:[12]

$$\mathbf{r}=\tilde\Lambda\circ\mathbf{c}_E\circ\Lambda, \text{ and } \mathbf{c}_E=\text{const}=\Lambda_{in}\circ\mathbf{r}(0)\circ\tilde\Lambda_{in}$$

where $\mathbf{r}(0)\neq 0$ (otherwise $r_1=r_2=r_3\equiv 0$ и and further it makes no sense to solve the control problem), $\tilde\Lambda$ is the quaternion conjugate to the quaternion Λ.[12]

The problem on finding optimal control has been reduced to solving the system of differential equations (1), (2), (6), (7) with simultaneous maximization of the function H at each current instant t and fulfillment of boundary conditions (3), (4). Hamiltonian H is the quadratic function relative to M_i, and the necessary conditions for the extremum $\partial H/\partial M_i=0$ determine its maximum. Therefore, optimal solution is

$$M_i=J_i\phi_i/2 \qquad (8)$$

We introduce the notation $r_0=\|\mathbf{r}(t)\|=\text{const}\neq 0$ and pass to the normalized vector $\mathbf{p}=\mathbf{r}/\|\mathbf{r}\|$, for which $p_i=r_i/r_0$. For the vector **p** and its components p_i the following equations are satisfied

$$\dot{\mathbf{p}}=\mathbf{p}\times(I^{-1}\mathbf{L}),\quad \dot p_1=L_3p_2/J_3-L_2p_3/J_2,$$

$$\dot p_2=L_1p_3/J_1-L_3p_1/J_3,\quad \dot p_3=L_2p_1/J_2-L_1p_2/J_1 \qquad (9)$$

The problem on finding the optimal rotation of the spacecraft consists in solving a system of equations of angular motion (1), (2), the coupled system of equations (7) and equations (9) with the equalities $r_i=r_0p_i$ in the presence of law (8) for the control torques M_i. The desired optimal solution satisfies the following relations:

$$\phi_i=a(t)p_i/J_i \qquad (10)$$

$$L_i=b(t)p_i \qquad (11)$$

where $a(t)$, $b(t)$ are the scalar functions of time ($b(t)\geq 0$ over the entire time period $t\in[0,T]$).

Consistent substitution of (10) into (7), bearing in mind (11) and $r_i=r_0p_i$, confirms that solution (10), (11) is valid for the system of differential equations (1), (7)-(9) (relations (11) directly follow from system (1), (8), (9) taking into account (10)). From (7), (9), (10) we see that optimal functions $a(t)$, $b(t)$ satisfy the dependence

$$\dot{a}(t) = 2k_1 b - r_0 \qquad (12)$$

From the equations (8)-(11) we see that control torque **M** is collinear the line which is immobile in inertial coordinate system during optimal rotation. Therefore, for zero boundary conditions $\mathbf{L}(0)=\mathbf{L}(T)=0$, solution of system (1), (7)-(9) describes a motion, when spacecraft's angular momentum **L** has the fixed direction in inertial coordinate system (vectors **M** and **L** are collinear), and this solution is unique. Angular momentum $\mathbf{L}(t)$ (i.e. solution of the system (1), (7)-(9) taking into account $r_i = r_0 p_i$) satisfies (11); substitution of (11) in the equation of motion (1) confirms that it satisfies the necessary condition of optimality (9) in the presence (8), (10).

For optimal function $b(t)$, differential equation $\dot{b} = a/2$ follows from complex analysis of the equations (1), (8)-(11), or

$$b(t) = \frac{1}{2}\int_0^t a(t)dt$$

Taking into account last equality and condition (12), we write $\ddot{a} = k_1 a$ for $a(t)$, and analytical solution

$$a(t) = C_1 \exp(-t\sqrt{k_1}) + C_2 \exp(t\sqrt{k_1}) \qquad (13)$$

where C_1, C_2 are some constants (they depend on time of a turn).

Since $\mathbf{L}(0)=0$ and $\mathbf{L}(T)=0$, then $b(0)=b(T)=0$ and

$\dot{a}(0) = \dot{a}(T) = -r_0$, whence $r_0 = \sqrt{k_1}(C_1 - C_2)$. Accordingly, the function $b(t)$ has the following form for optimal rotation

$$b(t) = (\dot{a}(t) + r_0)/(2k_1) =$$

$$\left[C_2 \exp(t\sqrt{k_1}) - C_1 \exp(-t\sqrt{k_1}) + C_1 - C_2\right] \Big/ (2\sqrt{k_1}) \qquad (14)$$

Time of finish of turn process T is not fixed, and the Hamiltonian H does not depend on time in explicit form. Therefore, optimal control must satisfy the condition $H = \text{const} = 0$ at any time $t \in [0, T]$ (over the entire control interval).[30] At the ends of optimal trajectory (at initial and final instant) $\mathbf{L}(0)=0$ and $\mathbf{L}(T)=0$, and function H is (taking into account (8))

$$H(0) = H(T) = -k_2 \square \ (J_1\phi_1^2 + J_2\phi_2^2 + J_3\phi_3^2)/4 +$$

$$(J_1\phi_1^2 + J_2\phi_2^2 + J_3\phi_3^2)/2 = (J_1\phi_1^2 + J_2\phi_2^2 + J_3\phi_3^2)/4 - k_2$$

or $H(0) = H(T) = M_1^2/J_1 + M_2^2/J_2 + M_3^2/J_3 - k_2 = 0$, whence

$$a^2(0) = a^2(T) = 4k_2/(p_1^2/J_1 + p_2^2/J_2 + p_3^2/J_3).$$

For motions corresponding to the equations (9), (11), the condition

$p_1^2/J_1 + p_2^2/J_2 + p_3^2/J_3 = \text{const}$ is satisfied. To check this statement, it is enough to differentiate the left-hand side of this equality with respect to time and make sure that the resulting derivative is equal to zero after substitution $\dot{p}_i$ using formula (9) and then L_i using (11). We have obtained key and very important property of optimal motion of a spacecraft: the proportion between the square of the modulus of the angular momentum of the spacecraft and the kinetic energy of rotation E is constant value (at any instant within entire time interval $[0, T]$).

$$E = b^2(p_1^2/J_1 + p_2^2/J_2 + p_3^2/J_3)/2, \quad E/\square\mathbf{L}\square^2 =$$

$$(p_{10}^2/J_1 + p_{20}^2/J_2 + p_{30}^2/J_3)/2 = \text{const}$$

(the dependence $b^2 = \square\mathbf{L}\square^2$ directly follows from (11)); $p_{i0} = p_i(0)$.

Thus, optimal function $a(t)$ must satisfy the following requirements:

$a(0) = 2\sqrt{k_2}/C$; $a(T) = -2\sqrt{k_2}/C$, $a(T/2)=0$

($\tilde{N} = \sqrt{p_{10}^2/J_1 + p_{20}^2/J_2 + p_{30}^2/J_3}$), и $\dot{a}(0) = \dot{a}(T) = -r_0$

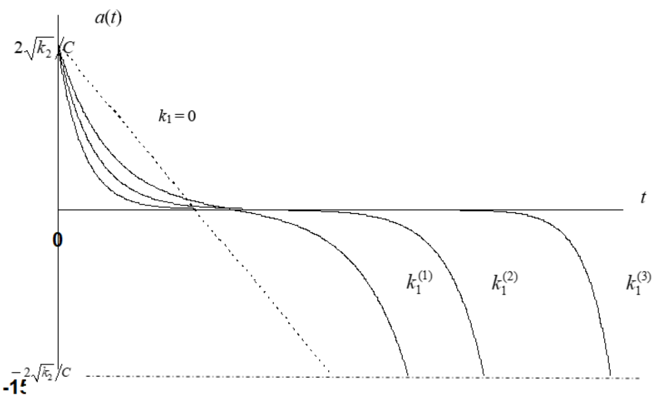
Last equality follows from (12), if we take into account $b(0)=b(T)=0$ due to the boundary conditions $\mathbf{L}(0)=0$ and $\mathbf{L}(T)=0$ for optimal turn. The property $a(T) = -a(0)$ follows from the necessary optimality condition $H(0)=H(T)=0$ (taking into account (8), (10)). From solution (13), we can write

$$\dot{a}(t) = \sqrt{k_1}(C_2 \exp(t\sqrt{k_1}) - C_1 \exp(-t\sqrt{k_1}))$$

At the ends of control interval: $\dot{a}(0) = \sqrt{k_1}(C_2 - C_1)$, $\dot{a}(T) = \sqrt{k_1}(C_2 \exp(T\sqrt{k_1}) - C_1 \exp(-T\sqrt{k_1}))$. Equating $\dot{a}(0)$ and $\dot{a}(T)$, we write the equation $C_2(\exp(T\sqrt{k_1}) - 1) = C_1(\exp(-T\sqrt{k_1}) - 1)$, hence $C_1 = -C_2\exp(T\sqrt{k_1})$. From (14) follows $b(0)=0$, $b(T)=0$. Then, we see that $a(T) = -a(0)$ and $a(T/2)=0$, since we have $a(0) = C_1 + C_2$, $a(T) = C_1\exp(-T\sqrt{k_1}) + C_2\exp(T\sqrt{k_1}) = -C_2 - C_1$, $a(T/2) = C_2(\exp(T\sqrt{k_1}/2) - \exp(T\sqrt{k_1})\exp(-T\sqrt{k_1}/2)) = 0$ based on (13). But $r_0 = \sqrt{k_1}(C_1 - C_2) = C_1\sqrt{k_1}(\exp(-T\sqrt{k_1}) + 1) > 0$, whence $C_1 > 0$, $C_2 < 0$ (and $|C_1| > |C_2|$).

If $k_1 = 0$ (the hypothetical case), then $a(t)$ is linear function of time $a(t) = 2\sqrt{k_2}(1 - 2t/T)/$.

The closer the coefficient k_1 is to zero the closer $b(t)$ is to a quadratic function of time. With increasing of k_1, function $b(t)$ is more remote relative to parabolic change. Figure 1 shows the nature of the change in optimal functions $b(t)$ and $a(t)$ (here $k_1^{(2)} > k_1^{(1)} > 0$, $k_1^{(3)} >> k_1^{(2)}$); the dotted line corresponds to the case $k_1 \to 0$. Due to the equation $\ddot{a} = k_1 a$, for greater value of k_1, derivative $\dot{a}$ greater under fixed value a. The values $a(0)$ and $a(T)$ are fixed (they do not vary if k_1 changes), $a(0) = 2\sqrt{k_2}/C$, $a(T) = -2\sqrt{k_2}/C$ и $a(T/2)=0$. The greater value of k_1, the further greater speed the function $a(t)$ approximates to zero, starting from the point $a(0) = 2\sqrt{k_2}/C$. Accordingly, the function $b(t)$ moves to a segment with $\dot{b} \approx 0$ more fast. With decreasing of b_{max}, time of finishing the reorientation maneuver evidently increases. Thus, the greater the coefficient k_1, the further the function $b(t)$ approaches to a piecewise continuous function of time dependence, which can be approximated by function including the following segments: rotation with c $\dot{b} \approx \sqrt{k_2}/C$, further $b \approx \text{const}$ and then $\dot{b} \approx -\sqrt{k_2}/C$. With unlimited magnification k_1 and k_2, maximal energy of rotation E_{max} approaches to the level $E_0 = k_2/(2k_1)$. If k_2 increases, then duration of a turn T decreases (if k_1 is invariable).



Citation: Levskii MV. Optimal control of a spacecraft based on the combined criteria of quality related to the energy costs. *Int Rob Auto J.* 2025;11(2):33‒38. DOI: 10.15406/iratj.2025.11.00298

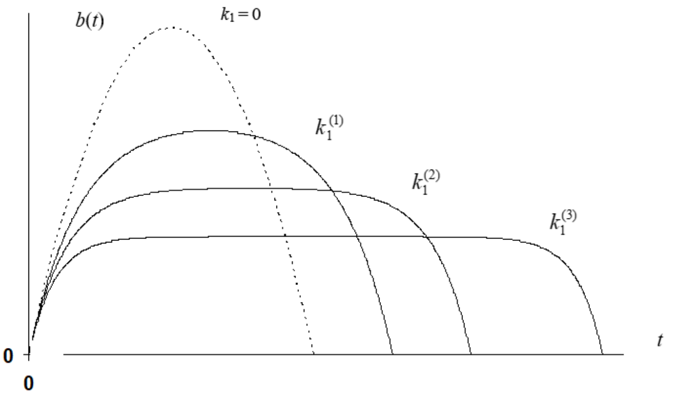


Figure 1 The form of optimal functions $a(t)$ and $b(t)$.

Taking into account the expressions (8), (10), (11) and optimal values of C_1 and C_2 for the functions $a(t)$, $b(t)$, optimal control and optimal rotation are determined by the laws:

$$M_i = C_1[\exp(-t\sqrt{k_1}) - \exp((t-T)\sqrt{k_1})]\, p_i/2 \qquad (15)$$

$$L_i = C_1[1+\exp(-T\sqrt{k_1}) - \exp(-t\sqrt{k_1}) - \exp((t-T)\sqrt{k_1})]p_i \big/ (2\sqrt{k_1}) \qquad (16)$$

where p_i are the components of the vector $\mathbf{p}$, that satisfies the equations(9), $\mathbf{p} = \tilde{\Lambda} \circ \Lambda_{in} \circ \mathbf{p}_0 \circ \tilde{\Lambda}_{in} \circ \Lambda$, $C_1 = 2\sqrt{k_2}\,/\,(C(1-\exp(-T\sqrt{k_1})))$, and $C_2 = 2\sqrt{k_2}\big/C - C_1$.

Solution of optimal turn problem is described by the equations (10), (11), (15), (16); control functions M_i and projections of angular momentum L_i change according to the formulas (15), (16). The desired vector $\mathbf{p}_0$ is found after solving the two-point boundary-value problem on turn. The programmed value $\mathbf{M}$ and the quaternion Λ are related by formula

$$\mathbf{M} = a(t)\tilde{\Lambda} \circ \Lambda_{in} \circ \mathbf{p}_0 \circ \tilde{\Lambda}_{in} \circ \Lambda/2$$

in which $a(t)$ changes according to (13).

For optimal control, property of symmetry is characteristic (primarily, for functions $a(t)$ and $b(t)$), and we have the following regularities:

$$a(0) = -a(T) > 0\,,\; b(t) \geq 0\,,\; a(T-t) = -a(t)\,,\; b(T-t) = b(t)$$

$$\int_0^{T/2} |a(t)|\,dt = \int_{\grave{O}/2}^{T} |a(t)|\,dt\,, \quad \int_0^{T/2} b(t)dt = \int_{\grave{O}/2}^{T} b(t)dt$$

$$\Lambda \circ \mathbf{M}(T-t) \circ \tilde{\Lambda} = -\Lambda \circ \mathbf{M}(t) \circ \tilde{\Lambda}\,,\; \Lambda \circ \mathbf{L}(T-t) \circ \tilde{\Lambda} = \Lambda \circ \mathbf{L}(t) \circ \tilde{\Lambda}$$

$$\max_{0 \le t \le T} |\mathbf{M}(t)| = \sqrt{k_2}\big/C\,,\; L_{\max} = \max_{0<t<T} \sqrt{L_1^2 + L_2^2 + L_3^2} = |\mathbf{L}(T/2)|$$

$$L_{\max} = (r_0 + \sqrt{r_0^2 - 4k_1k_2/C^2})/(2k_1) \text{ or } L_{\max} = (\sqrt{(\tilde{N}_1 - \tilde{N}_2)^2/4 - k_2/C^2} + (C_1 - C_2)/2)/\sqrt{k_1}$$

From (11) it follows that $\mathbf{p}$ is the unit vector of angular momentum $\mathbf{L}$. Optimal functions $L_i(t)$, $\phi_i(t)$, $p_i(t)$ meet requirements (10), (11), where $p_i(t)$ is the solution of system (9). Optimal control is determined by expressions (15), the vectors $\mathbf{M}$ and $\mathbf{L}$ are collinear at any time $t \in [0\,,\,T\,]$. Control torque $\mathbf{M}$ changes its direction to the opposite at time $t = T/2$ (modulus $|\mathbf{L}|$ has its maximum, $|\mathbf{L}(T/2)| = L_{\max}$).

Results and discussion

The given solution differs from all known ones; both the control functions and phase variables are smooth functions of time during optimal turn. The considered problem differs from other problems with the combined criteria of optimality in the type of quality functional, which includes not only the control and phase variables.[21–23] The presence of an integral of rotation energy in the minimized functional leads to a limitation of kinetic energy during the rotation, and control variables are limited even in an absence of control constraints due to the presence of controlling torques. Time factor limits duration of a turn.

For example, we consider a turn of 180° in position corresponding to the quaternion Λ_f with elements $□_0 = 0$; $□_1 = 0.7071068$; $□_2 = 0.5$; $□_3 = 0.5$. In the initial position, the directions of the same axes of the fixed and inertial bases coincide, and $\mathbf{L}(0) = \mathbf{L}(T) = 0$. We determine the optimal control program for transferring the spacecraft from state (3) to state (4) under condition that kinetic energy of spacecraft rotation does not exceed 10 J. Numerical solution on the problem of the controlled turn (in formulation (1)–(5)) is given for case when $k_1 = 0.002$ s^{-2} and $k_2 = 0.04$ W/s, and inertial characteristics of a spacecraft are accepted as follows: $J_1 = 63559$ kg m^2, $J_2 = 192218.5$ kg m^2, $J_3 = 176809$ kg m^2.

Visual illustration of spacecraft motion during optimal rotation when costs (5) is minimum with the desired reduction in rotational energy is shown on Figures 2–5 (as results of mathematical modeling). After solving the boundary-value problem on turn from position $□(0) = □_{in}$ to position $□(T) = □_f$, we obtain $\mathbf{p}_0 = \{\, 0.49535062\,;\; -0.11725655\,;\; 0.86074309\,\}$. Accordingly, maximal control torque is $\mathbf{M}(0)$= 70.2 Н□м. The boundary-value problem of a turn can be solved by the method of successive approximations[31] or method of iterations,[8] as we used in control system[32] (or by more modern methods).[33] Value of constant $r_0 = 6.2768$ W. Angular momentum has maximum $L_{\max} = 1561.9$ N m s at time $t = 135.6$ s. Rotational energy during a turn has maximal value $E_{\max} = 9.907$ joules. Figure 2 shows graphs of changes in the projections of the angular momentum on the axis of the fixed coordinate system $L_1(t)$, $L_2(t)$, $L_3(t)$ (projections L_i are given in N m s). Figure 3 illustrates the change in quaternion elements $\Lambda(t)$ during the optimal maneuver ($\lambda_0(t)$, $□_1(t)$, $□_2(t)$, $□_3(t)$ reflect the current attitude of a spacecraft). The behavior of the components $p_1(t)$, $p_2(t)$, $p_3(t)$ is demonstrated in Figure 4 (values p_i, as well as λ_j, are dimensionless quantities), and the projection p_1 changes slightly. The change in the modulus of angular momentum during optimal turn is given on Figure 5. Evidence that OX is the longitudinal axis of the spacecraft, is the fact that L_1 is of constant sign, and the nature of its change repeats the behavior of the angular momentum module (unlike L_2 and L_3). For optimal control variables p_i and $□_j$ are smooth functions of time; L_i are smooth functions of time (except for $t = 0$ and $t = T$).

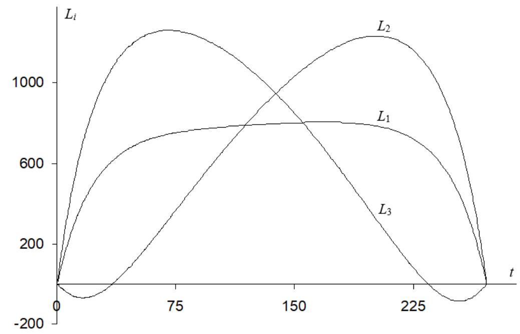


Figure 2 Changing the components of spacecraft angular momentum during a turn.

Citation: Levskii MV. Optimal control of a spacecraft based on the combined criteria of quality related to the energy costs. *Int Rob Auto J.* 2025;11(2):33–38. DOI: 10.15406/iratj.2025.11.00298

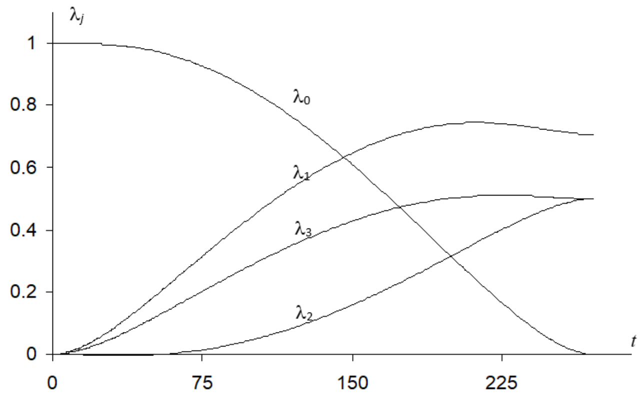


Figure 3 Changing the elements of the attitude quaternion $\Lambda(t)$ during a turn.

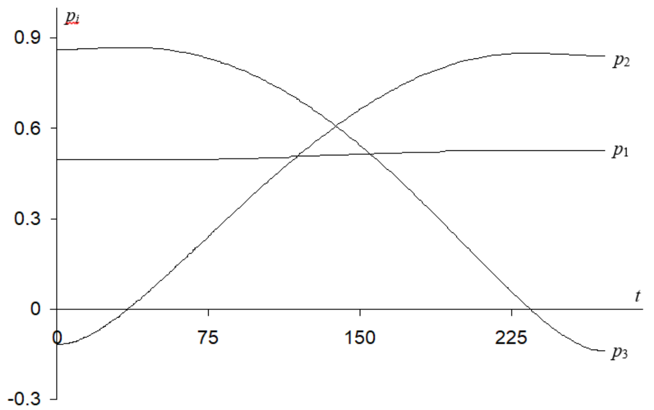


Figure 4 Behavior of the functions $p_1(t), p_2(t), p_3(t)$ during optimal turn.

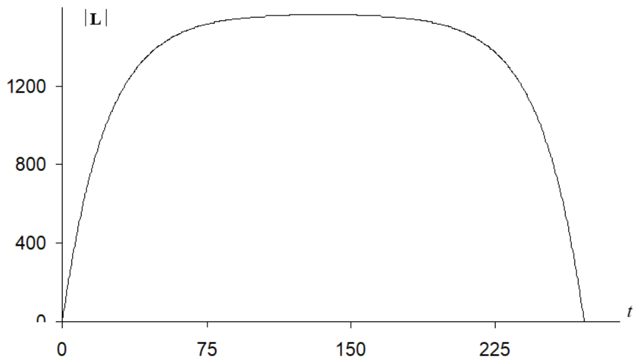


Figure 5 Change in angular momentum modulus during optimal control.

Data of mathematic simulation confirm that for any k_1 and k_2 kinetic energy of rotation $E(t) \le k_2/(2k_1)$. Presence of control torque in the minimized index (5) provides to the following: firstly, a narrow-mindedness of control and, secondly, to the fact that angular momentum is smooth function of time. However, the larger k_2, the greater the control torque at the beginning and at the end of a turn and the longer will be rotation period when $\mathbf{M}$ is close to zero and $|\mathbf{L}| \approx$ const. The smaller the coefficient k_2 the smaller the maximum control torque $|\mathbf{M}(0)|$.

In practice of space flight, it is necessary to take into account the conditions (as the restrictions for spacecraft motion): $|\mathbf{M}| \le M_{max}$ and $E(t) \le E_{per}$. This specified requirements will be satisfied if $k_2 \le C^2 M_{max}^2$ and $k_1 = k_2/(2E_{per})$ (or $k_1 \ge C^2 M_{max}^2/(2E_{per})$), because $k_2 = C^2|\mathbf{M}(0)|^2$.

Conclusion

In this research work, new control method of spacecraft attitude on a basis of quaternion variables is presented; the used criteria of optimality is new and specific: indicator of quality of control process combines, in a given proportion, time and energy costs as the sum of an integral of rotation energy and contribution of controlling forces to perform a turn (in terms of energy consumption). The solved problem is very topical. Significance and importance of the executed investigations consist in the fact that chosen criterion of optimality minimizes kinetic energy of rotation within the given interval of time. The obtained method is differs from all other known decisions. Main difference consists in new form of minimized functional which allows to turn spacecraft with minimal rotation energy if maneuver time is given. This useful quality is advantage of presented control mode because it significantly saves the controlling resources and increases the possibilities of spacecraft control. An example and results of mathematical modeling that confirm the practical feasibility of the developed method for attitude control of a spacecraft are given.

Acknowledgments

No financial exists.

Conflicts of interest

No conflicts of interest exist.

References

1. Sinitsin LI, Kramlikh AV. Synthesis of the optimal control law for the reorientation of a nanosatellite using the procedure of analytical construction of optimal regulators. *J Phys Conf Ser.* 2021;1745:012053.
2. Velishchanskii MA, Krishchenko AP, Tkachev SB. Synthesis of spacecraft reorientation algorithms using the concept of the inverse dynamic problem. *J Comput Syst Sci Int*. 2003;42(5):811–818.
3. Junkins JL, Turner JD. *Optimal spacecraft rotational maneuvers*. 1986.
4. Reshmin SA. Threshold absolute value of a relay control when time–optimally bringing a satellite to a gravitationally stable position. *J Comput Syst Sci Int*. 2018;57(5):713–722.- 320 c.
5. Scrivener S, Thompson R. Survey of time–optimal attitude maneuvers. *J Guidance Control and Dynamics*. 1994;17(2):225–233.
6. Zhou H, Wang D, Wu B, et al. Time–optimal reorientation for rigid satellite with reaction wheels. *International Journal of Control*. 2012;85(10): 1–12.
7. Reshmin SA. The threshold absolute value of a relay control bringing a satellite to a gravitationally stable position in optimal time. *Dokl Phys*. 2018;63(6):257–261.
8. Levskii MV. Pontryagin's maximum principle in optimal control problems of orientation of a spacecraft. *J Comput Syst Sci Int*. 2008;47(6):974–986.
9. Shen H, Tsiotras P. Time–optimal control of axi–symmetric rigid spacecraft with two controls. *AIAA J Guidance Control and Dynamics*. 1999;22(5):0682–694.
10. Molodenkov AV, Sapunkov YaG. Analytical solution of the minimum time slew maneuver problem for an axially symmetric spacecraft in the class of conical motions. *J Comput Syst Sci Int*. 2018;57(2):302–318.
11. Branets VN, Chertok MB, Kaznacheev YuV. Optimal turn of a rigid body with a single axis of symmetry. *Kosm Issl*. 1984;22(3):352–360.

12. Branets VN, Shmyglevskii IP. *Use of quaternions in the problems of rigid body attitude*. 1973.

13. Aipanov SA, Zhakypov AT. The method of separation of variables and its application to the problem of a spacecraft's optimal turn. *Cosmic Res*. 2020;58(1):53–63.

14. Strelkova NA. On optimal reorientation of a solid. *Problems of Mechanics of Controlled Motion. Nonlinear Dynamical Systems.* PGU, Perm. 1990. p. 115–133.

15. Levskii MV. Kinematically optimal spacecraft attitude control. *J Comput Syst Sci Int*. 2015;54(1):116–132.

16. Biryukov VG, Chelnokov YN. Construction of optimal laws of variation of the angular momentum vector of a rigid body. *Mechanics of Solids*. 2014;49(5):479–494.

17. Levskii MV. Optimal spacecraft terminal attitude control synthesis by the quaternion method. *Mechanics of Solids*. 2009;44(2):169–183.

18. Levskii MV. About method for solving the optimal control problems of spacecraft spatial orientation. *Problems of Nonlinear Analysis in Engineering Systems*. 2015;21(2):61–75.

19. Zelepukina OV, Chelnokov YN. Construction of optimal laws of variation in the angular momentum vector of a dynamically symmetric rigid body. *Mechanics of Solids*. 2011;46(4):519–533.

20. Molodenkov AV, Sapunkov YaG. Analytical solution of the optimal slew problem for an axisymmetric spacecraft in the class of conical motions. *J Comput Syst Sci Int*. 2016;55(6):969–985.

21. Molodenkov AV, Sapunkov YaG. Analytical quasi–optimal solution of the slew problem for an axially symmetric rigid body with a combined performance index. *J Comput Syst Sci Int*. 2020;59(3):347–357.

22. Sapunkov YaG, Molodenkov AV. Analytical solution of the problem on an axisymmetric spacecraft attitude maneuver optimal with respect to a combined functional. *Autom Remote Contr*. 2021;82(7):1183–1200.

23. Molodenkov AV, Sapunkov YaG. Analytical approximate solution of the problem of a spacecraft's optimal turn with arbitrary boundary conditions. *J Comput Syst Sci Int*. 2015;54(3):458–468.

24. Levskii MV. Control of the rotation of a solid (spacecraft) with a combined optimality criterion based on quaternions. *Mechanics of Solids*. 2023;58(5):1483–1499.

25. Levskii MV. Optimal control of the angular momentum for a solid (spacecraft) performing a spatial turn. *Mechanics of Solids*. 2023;58(1):63–77.

26. Quang M. Lam. Robust and adaptive reconfigurable control for satellite attitude control subject to under–actuated control condition of reaction wheel assembly. *Mathematics in Engineering, Science and Aerospace*. 2018;9(1):47–63.

27. Levskii MV. Special aspects in attitude control of a spacecraft, equipped with inertial actuators. *Journal of Computer Science Applications and Information Technology*. 2017; 2(4):1–9.

28. Gorshkov OA, Murav'ev VA, Shagaida AA. *Holl's and Ionic Plasma Engines for Spacecrafts*. 2008.

29. Pontryagin LS, Boltyanskii VG, Gamkrelidze RV, et al. *The mathematical theory of optimal processes*. Gordon and Breach: USA; 1986. p. 360.

30. Young LG. *Lectures on the calculus of variations and optimal control theory*. WB Saunders Company: USA; 1969.

31. Lyubushin AA. Modifications of the method of successive approximations for solving optimal control problems. *USSR Computat Math Math Phys*. 1982;22(1):29–34.

32. Levskii MV. System for control of spacecraft's spatial turn. RF Patent No.2006431. *Inventions Applications and Patents*. 1994.

33. Kumar S, Kanwar V, Singh S. Modified efficient families of two and three–step predictor–corrector iterative methods for solving nonlinear equations. *Journal of Applied Mathematics*. 2010;1(3):153–158.

Citation: Levskii MV. Optimal control of a spacecraft based on the combined criteria of quality related to the energy costs. *Int Rob Auto J.* 2025;11(2):33–38. DOI: 10.15406/iratj.2025.11.00298