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Torus knot designs using a Rubik's snake

Abstract

A Rubik's Snake is a toy that was invented over 40 years ago together with the more famous Rubik's Cube. It can be twisted to many interesting shapes including knots. In this paper, we study Rubik's Snake torus knot designs. The general solutions are given for all torus knots T(2,n), T(3,n) and T(4,n). Some more challenging constructions are also provided.

Keywords: torus knot designs, Rubik's snake

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Introduction

The Rubik's Snake is a toy that was invented by Prof. Rubik in 1981.[1] It consists of right isosceles triangular prisms (what we are calling blocks) that, except for the firstand last block, are connected to two other blocks at the centers of the square faces.

The Rubik's Snake has been used as a tool for the studyof protein folding[2] and for the construction of reconfigurable modular robots.[3–5] More applications of robots can be found in.[6,7] In previous papers that the first author collaborated with others, strategies have been given forthe design of a Rubik's Snake,[8] and some mathematical problems concerning a Rubik's Snake have been studied.[9] General rotation angles are mentioned in[8] and more theoretical work was presented in.[10] On the other hand,[9] has quite some theoretical work but is only concerned with integer multiple of 90 degree rotations. In Hou S, et al.,[11] several theorems about palindromic, periodic and Möbius Rubik's snakes were proved. In, Hou S, et al.,[12] the design for general box shapes using a Rubik's snake was presented with a counting formula derived.

Knot theory[13] is an interesting research area that attracted a lot of mathematicians. It also has interesting applications in architectural design, for example, some work by Zaha.[14] Knot classification tool is available.[15] The Rubik's Snake prime knots up to 6 crossings and compositeknots up to 9 crossings have been studied in the past. The 3_1(trefoil) knot was studied in our previous paper[16] and 4_1, 5_1 and 5_2 studied in our previous work.[17] The prime knotswith 6 crossings 6_1, 6_2 and 6_3 were studied in our recent work.[18] The result for 4_1 was improved from 46 blocks to 44 blocked by using symmetry in our paper on compositeknots up to 8 crossing[19] and the result for 5_2 was improved from 56 blocks to 54 blocks by a non-local change inthe same paper. The result for 5_1 knot was improved from 52 to 50 blocks by a non-local change and the result for 6_1 wasimproved from 64 to 60 blocks using symmetry in our mostrecent work on composite knots with 9 crossings.[20]

In this paper, we study non-trivial torus knot designs using a Rubik's Snake. A (p,q)-torus knot is obtained by looping a string through the hole of a torus p times (the meridian direction) with q revolutions (the longitude direction) beforejoining its ends, where p and q are coprime. We denote it as $T(p,q)$. $T(p,q)$ and $T(q,p)$ are the same knot (equivalent).Our main idea is to reply on the theory developed in[9] that a Rubik's snake closed loop with integer multiple of 90 degree rotations can only have period 1,2,3 and 4. Therefore $T(2,n)$, $T(3,n)$ and $T(4,n)$ have the potential to have an easygeneral formula to realize using a Rubik's snake. The purpose of this paper is not to look for the shortest path. Rather, it is to provide some general construction formulas so that all torus knots $T(2,n)$, $T(3,n)$ and $T(4,n)$ can be designedusing a Rubik's snake and some other torus knots can have an interesting periodic design as well. The organization is as follows. In Section 2, we will present the Rubik's snake sequence formula for $T(2,n)$ knots. In Section 3, it will be for $T(3,n)$ and Section 4 for $T(4,n)$. In Section 5, we made some difficult constructions for $T(5,6)$, $T(5,8)$ and $T(6,7)$. We conclude in Section 6.

T (2, n)

For torus knots $T(2,n)$, n must be an odd number thatis at least 3. In, Hou S, et al.,[16] we found $T(2,3)$ (trefoil) knot with 34 blocks. In, Hou S, et al.,[17] we found $T(2,5)$ knot (5_1) with 52 blocks and then it was improved to 50 blocks in.[20]

For $T(2,7)$, by a similar method, we found the following sequence with only 66 blocks:

[0,0,0,3,1,0,1,1,0,1,2,1,0,1,1,0,2,1,0,0,3,0,0,1,2,0,0,3,0,3,3,0,2,3,0, 0,0,0,3,0,0,1,3,0,3,0,0,0,1,2,1,0,3,0,0,0,3,1,0,0,3,0,3,0,1,3]

For $T(2, 9)$, it could be made period 3. We search for palindromic sequence with period 3 and found the shortest having such pattern with 84 blocks. For example [1, 0, 0, 3, 1, 0, 1, 0, 1, 0, 0, 3, 2, 0, 1, 0, 2, 3, 0, 0, 1, 0, 1, 0, 1, 3, 0, 0] repeated three times.

For $T(2, 2n+1)$ with integer n at least 5, it is easy to construct a universal formula that always works.The sequence is [1,1,2,1,0,0,0,1,0,0,0,0,0, $(-1)^{n+1}$, P, 0,0,0,0,0,0,0,1,3,0,0, Q] repeated twice where P is [0, 2, 0, 0] repeated $n-5$ times (if zero times then we simply do not insert P) and Q is [1, 0, 1, 2] repeated $2n-1$ times. The length of the sequence (after repeated twice) is $24n+2$.

The above general results can be shown by induction. Once we have a working pattern, for example for $T(2, 11)$, the next one (in this case $T(2, 13)$) is to make more twists inside and connect outside. The Q part (with two periods) serves as the twist part inside and the P part (with two periods) serves as the additional blocks to ensure connections outside. Each time n is increased by 1, both [$(-1)^{n+1}$, P] and Q will add a two-unit shift, therefore preserving the closed loop.

To demonstrate that the above general formula indeed works, figure 1 shows the Rubik's snake $T(2,13)$ knot usingthe general formula and figure 2 shows the line representation of it.

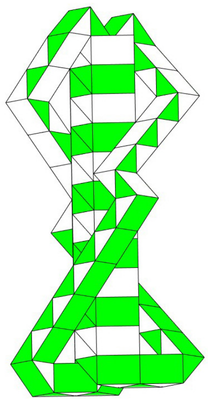

Figure 1 T(2,13).

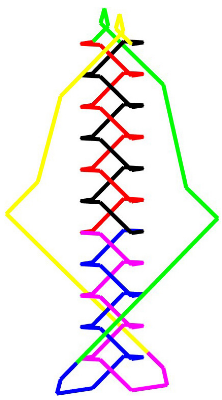

Figure 2 line representation for T(2,13).

T (3, n)

For $T(3,n)$ with n not a multiple of 3 (this is required for it to be a torus knot), the general formula is [1 repeated 3n-2 times, A, B repeated n-2 times, C]*3. Where $A = [3,3,1,1,3,1,1,1]$, $B = [1,1,1,3,1,1,1]$, $C = [1,1,1,3,1,1,3,3]$.The overall length is $(3n-2+8+7n-14+8)*3=30n$.

Again it can be shown by induction. To demonstrate that the above general formula indeed works, Figure 3 showsthe Rubik's snake $T(3, 8)$ knot using the general formula and figure 4 shows the line representation of it.

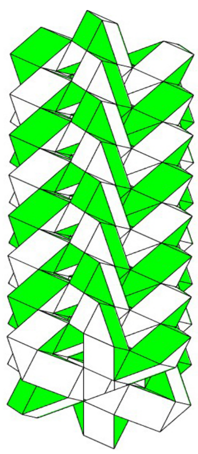

Figure 3 T(3,8).

Figure 4 line representation for T(3,8).

T (4, n)

For $T(4, n)$ with n at least 3 and odd (this is required for it to be a torus knot), first consider $T(4, 3)$. That is 8_{19} knot and we have the shortest with period 4 as $[0,0,1,1,0,1,0,0,3,0,0,1,0,1,1,0]$ repeated 4 times with only 64 blocks.

Then we have the general solution as follows with $A = [0,1,1]$, $B = [2,0,2,0]$, *reverse* $(B) = [0,2,0,2]$:

For $n = 4k + 1$ with positive integer k, [A repeated n-3 times, 0,0,0,1,3,0,0, B repeated (n-5)/4 times, 0,0,1,0,0,reverse(B) repeated (n-5)/4 times,0,0,3,1,0,0]*4. The lengthis $(3*(n-3)+7+n-5+5+n-5+6)*4 = 20n-4$.

For $n = 4k + 3$ with positive integer k, [A repeated n-3 times, 0,0,0,2,1,0,0,0, B repeated (n-7)/4 times, 0,0,1,0,0, reverse(B) repeated (n-7)/4 times,0,0,0,1,2,0,0]*4. The lengthis $(3*(n-3)+8+n-7+5+n-7+7)*4 = 20n-12$.

Again they can be shown by induction. To demonstrate that the above general formula indeed works, Figure 5 Shows the Rubik's snake T (4, 9) knot using the general formula and figure 6 shows the line representation of it.

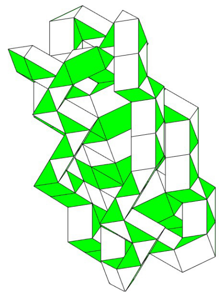

Figure 5 T(4,9).

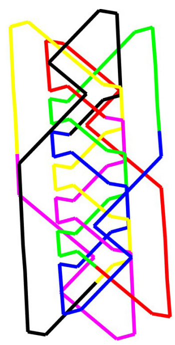

Figure 6 line representation for T(4,9).

T (5, 6), T (5, 8) and T (6, 7)

In this section, we discuss some tougher constructions of complicated torus knots.

Our idea of the construction is as follows. These torus knots can be viewed as period 3 (the first and third) or period4 (the second). We only use 0 or 1 in the snake sequence and we enforce a type II palindromic condition.[11] These can significantly reduce the computational cost needed for anexhaustive search.

For $T(5, 6)$, it can be regarded as period 3 since 6 is a multiple of 3 (we could break the symmetry to make it period 3 but no longer period 6 without changing the knot type).

Note that a Rubik's snake closed loop with integer multiple of 90 degree rotations can only have period 1,2,3,4 but not5 or 6. We could realize $T(5, 6)$ using a Rubik's snake with integer multiple of 90 degree rotations with period 3 but not 6. The computational cost is affordable to exhaust period 3 palindromic sequences with 168 blocks having 0 and 1 only. We came up with the sequence below with 168 blocks:

[1,0,0,0,1,0,0,1,0,1,0,0,0,0,1,0,1,0,1,1,1,1,1,0,1,1,0,1,1,1,0,1,1,0,1,1, 1,1,1,0,1,0,1,0,0,0,0,1,0,1,0,0,1,0,0,0]*3

By a local search, we could reduce to the following palindromic period 3 sequence with 138 blocks without changing the knot structure:

[1,0,1,0,0,1,0,0,0,1,1,0,0,0,3,0,3,0,0,1,0,1,1,1,1,1,0,1,0,0,3,0, 3,0,0,0,1,1,0,0,0,1,0,0,1,0]*3

Now a local improvement can reduce it to 132 blocks, still keeping period 3 and the knot structure but dropping the palindromic requirement:

[0,0,1,0,0,0,1,1,0,0,0,3,0,3,0,0,1,0,1,1,1,1,1,0,1,0,0,3,0,3,0,0,0,1,1,0, 1,1,0,0,0,3,0,3]*3

Figure 7 shows the Rubik's snake $T(5,6)$ knot and Figure 8 shows the line representation of it.

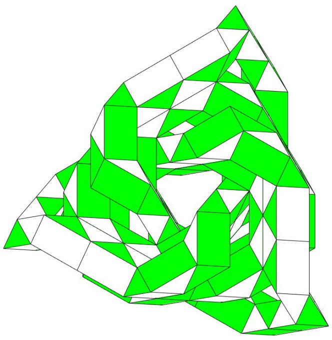

Figure 7 T(5,6).

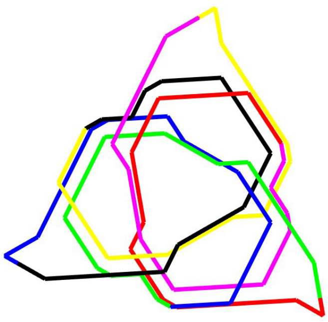

Figure 8 line representation for T(5,6).

For $T(5, 8)$, it can be regarded as period 4 since 8 is a multiple of 4. The computational cost is affordable to exhaust period 4 palindromic sequences with 0 and 1 only with 184 blocks. We came up with the sequence below with 184 blocks: [0,0,0,0,0,1,0,1,1,0,0,0,1,0,0,1,0,1,0 ,1,0,0,1,1,1,0,0,1,0,1,0,1,0,0,1,0,0,0,1,1,0,1,0,0,0,0]*4

Figure 9 shows the Rubik's snake $T(5,8)$ knot and Figure 10 shows the line representation of it.

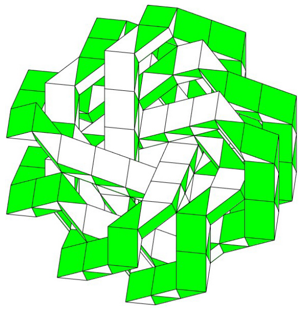

Figure 9 T(5,8).

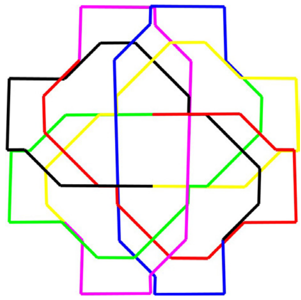

Figure 10 line representation for T(5,8).

For $T(6, 7)$, it can be regarded as period 3 since 6 is a multiple of 3. Because of the heavy burden of computational cost, we actually speed it up by enforcing the first few elements.

The motivation comes from some earlier experience with torus knots. For example, this is one of the sequences for $T(3, 5)$ (not the shortest for this knot) [1,0,0,0,0,0,1,0,0,1,0,0,0,1,0, 0,0,1,1,0,0,0,1,1,0,0,0,1, 0,0,0,1,1,0,0,0,1,1,0,0,0,1,0,0,0,1,0,0,1,0,0,0 ,0,0]*3

We use the first few elements [1,0,0,0,0,0,1,0,0,1,0,0,0,1,0,0,0] . This makes the computational power within reach to search for palindromic sequences with period 3 and rotations 0,1 only. We came upwith the sequence below with 258 blocks that is $T(6,7)$:

[1,0,0,0,0,0,1,0,0,1,0,0,0,1,0,0,0,0,1,0,1,0,1,0,0,0,0,0,0,1,0,1,0,0,1, 0,1,0,0,0,1,0,0,1,0,0,1,0,0,0,1,0,1,0,0,1,0,1,0,0,0,0,0,0,1,0,1,0,1,0,0,0, 0,1,0,0,0,1,0,0,1,0,0,0,0,0]*3

Figure 11 shows the Rubik's snake $T(6,7)$ knot and Figure 12 shows the line representation of it that demonstrates the torus knot structure.

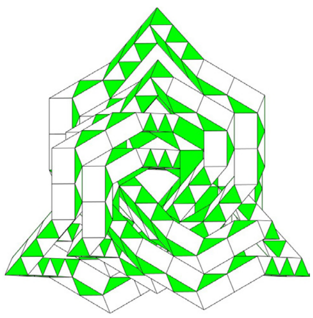

Figure 11 T(6,7).

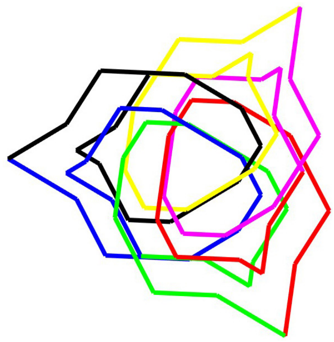

Figure 12 line representation for T(6,7).

Conclusion

We constructed all the torus knots $T(2, n)$, $T(3, n)$ and $T(4, n)$ using the Rubik's snake. We also constructed three more difficult torus knots $T(5,6)$, $T(5,8)$ and $T(6,7)$. Since the crossing number of $T(5, 7)$ is $(5-1)*7=28$ and it is the one with lowest crossing number we did not construct, (the difficulty is a Rubik's snake closed loop with integer multiple of 90 degree rotations cannot have period 5 or 7, if we completely break the symmetry the computational cost is not affordable) this means we constructed all torus knots with crossing number less than 28 using a Rubik's snake. The main idea of the construction is to try to use periodicity, palindromic feature and sometimes to limit the choice of rotations to simplify the search. Our intention is not to find the shortest path. Some sequences in this paper can be shortened without changing the knot.

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Conflicts of interest

Authors declare that there is no conflict of interest.

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