Research Article

Shortest paths of Rubik's snake composite knots with 9 crossings

Abstract

A Rubik's Snake is an interesting toy that was invented over 40 years ago. It can be twisted to many interesting shapes, including non-trivial knots. In this paper we proposed a systematic way to study how many blocks are needed to form a composite knot with 9 crossings. We also improved some results from our previous papers.

Keywords: Rubik's snake, composite knots

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Introduction

In the interesting terrain of recreational mathematics, the Rubik's Snake exemplifies Erno Rubik's creativity.[1,2] The Rubik's Snake, a cryptic member of the Rubik's puzzle family, came into the hands of puzzle lovers who wanted to unravel not just the permutations but also the underlying geometry and topology. With its colorful, interlocking prismatic blocks, it provides a malleable, three-dimensional shape for exploration, captivating the interest of both casual players and dedicated mathematicians. Each twist and turn of the snake results in a variety of permutations, sometimes changing its appearance into a complicated knot. As we look deeper into the history of this engaging puzzle, we realize that it is more than just a hobby; it is also a useful tool for exploring mathematical concept.

The Rubik's Snake has been used as a tool for the study of protein folding[3,4] and for the construction of reconfigurable modular robots.[5–7] More applications of robots can be found in.[8,9] Some ideas in the study of Rubik's Snake such as the use of rotation matrix is also used in rigid Origami folding.[10,11] In previous papers that the first author collaborated with others, strategies have been given for the design of a Rubik's Snake,[12] and some mathematical problems concerning a Rubik's Snake have been studied.[13] Rotations that are not $0°$ (or $360°$), $90°$, $180°$, or $270°$ are mentioned in[12] but not much theoretical work is presented. On the other hand,[13] has quite some theoretical work but is only concerned with integer multiple of 90 degree rotations. In, Hou S[14] Rubik's snakes with general rotation angles were studied with theoretical work presented. In, Hou S[15] theorems about palindromic, periodic and Möbius Rubik's snakes were proved. In, Hou S[16] a general strategy for designing box shapes using a Rubik's snake was given and a counting formula was proved.

Knot theory[17] is an interesting research area that attracted a lot of mathematicians. It also has interesting applications in architectural design, for example, some work by Zaha.[18] Knot classification tool is available.[19] The Rubik's Snake prime knots up to 6 crossings and composite knots up to 8 crossings have been studied in the past. The 3_1 (trefoil) knot was studied in our previous paper[20] and 4_1, 5_1 and 5_2 studied in our previous work.[21] The prime knots with 6 crossings 6_1, 6_2 and 6_3 were studied in our recent work.[22] The result for 4_1 was improved from 46 blocks to 44 blocked by using symmetry in our paper on composite knots up to 8 crossing[23] and the result for 5_2 was

improved from 56 blocks to 54 blocks by a non-local change in the same paper.

As we explore the complexities of Rubik's Snake composite knots, our goal is not only to gain a better understanding of their mathematical properties, but also to contribute to the ongoing story at the intersection of knot theory, recreational mathematics, and the Rubik's Snake's ingenious design. Here we study the shortest path for a Rubik's snake composite knot with 9 crossings. The organization is as follows: In Section 2, we explained the idea of using local structures. In Section 3, we improved our previously published result for 5_1 knot and $3_1\#5_1$ knot. In Section 4, we presented an innovative method to improve our previous result for 6_1 knot. The results in Sections 3 and 4 will be used in later sections. In Section 5 to 7, we present results for Rubik's Snake composite knots with 9 crossings. We conclude in Section 8.

Local structures

In, Hou S[23] Rubik's snakes' shortest paths for composite knots up to 8 crossings were studied. However, some of the study was based on some inspirations rather than a systematic way. We take a different approach here. In, Hou S[20] a complete list of 6 shortest palindromic trefoil knots with 34 blocks were presented. From there, we could extract the following 6 local structures for the knotted part of 3_1:

$A_1 = [0,1,2,1,0,1,0,1,2,1,0,3,0,0,1,2,0,1,1,0,1,2,1]$

$A_2 = [0,1,2,1,0,1,0,1,2,1,0,3,0,0,1,3,0,3,3,0,3,1,1]$

$A_3 = [0,1,2,1,0,1,0,1,2,1,0,3,0,3,2,3,3,2,3,0,3,1,1]$

$A_4 = [0,1,2,1,0,1,0,1,2,1,0,3,0,3,3,0,2,3,0,0,1,2,1]$

$A_5 = [0,1,2,1,0,1,0,1,2,1,0,3,0,3,3,0,3,1,0,0,3,1,1]$

$A_6 = [0,1,2,1,0,1,0,1,2,1,0,3,0,3,3,1,1,3,3,0,3,1,1]$

Similarly, in, Hou S[23] the shortest 4_1 knot was improved to 44 blocks. There are 12 different solutions of the form $[t,-t,t,-t]$, where t is listed below:

$[0,0,0,0,1,0,1,1,0,2,1],[0,0,0,1,0,0,3,2,0,3,3]$

$[0,0,0,1,1,0,1,1,0,1,3],[0,0,1,0,1,2,1,0,0,1,2]$

$[0,0,1,0,3,2,3,0,0,1,2],[0,0,1,0,3,3,1,0,0,3,1]$

[0,0,1,1,0,0,3,1,0,1,1],[0,0,1,1,0,1,1,3,3,1,1]

[0,0,1,1,0,1,2,1,1,2,1],[0,1,0,1,1,3,3,0,3,1,1]

[0,1,0,1,2,1,0,3,3,2,3],[0,1,1,3,0,3,2,3,3,3,1]

Based on above, we have the following 12 local structures for the knotted part of 4_1:

$B_1 = [0, 0, 0, 1, 0, 1, 1, 0, 2, 1, 0, 0, 0, 0, 3, 0, 3,3, 0, 2, 3, 0, 0, 0, 0, 1, 0, 1, 1, 0, 2, 1, 0, 0]$

$B_2 = [0, 0, 0, 1, 0, 0, 3, 2, 0, 3, 3, 0, 0, 0, 3, 0, 0,1, 2, 0, 1, 1, 0, 0, 0, 1, 0, 0, 3, 2, 0, 3, 3, 0]$

$B_3 = [0, 1, 1, 0, 1, 1, 0, 1, 3, 0, 0, 0, 3, 3, 0, 3, 3,0, 3, 1, 0, 0, 0, 1, 1, 0, 1, 1, 0, 1, 3, 0, 0, 0]$

$B_4 = [0, 0, 3, 2, 0, 0, 1, 0, 1, 2, 1, 0, 0, 1, 2, 0, 0,3, 0, 3, 2, 3, 0, 0, 3, 2, 0, 0, 1, 0, 1, 2, 1, 0]$

$B_5 = [1, 2, 1, 0, 0, 3, 2, 0, 0, 1, 0, 3, 2, 3, 0, 0, 1,2, 0, 0, 3, 0, 1, 2, 1, 0, 0, 3, 2, 0, 0, 1, 0, 3]$

$B_6 = [0, 1, 1, 3, 0, 0, 1, 3, 0, 0, 1, 0, 3, 3, 1, 0, 0,3, 1, 0, 0, 3, 0, 1, 1, 3, 0, 0, 1, 3, 0, 0, 1, 0]$

$B_7 = [0, 1, 1, 0, 0, 3, 1, 0, 1, 1, 0, 0, 3, 3, 0, 0, 1,3, 0, 3, 3, 0, 0, 1, 1, 0, 0, 3, 1, 0, 1, 1, 0, 0]$

$B_8 = [0, 1, 1, 0, 1, 1, 3, 3, 1, 1, 0, 0, 3, 3, 0, 3, 3,1, 1, 3, 3, 0, 0, 1, 1, 0, 1, 1, 3, 3, 1, 1, 0, 0]$

$B_9 = [0, 1, 1, 0, 1, 2, 1, 1, 2, 1, 0, 0, 3, 3, 0, 3, 2,3, 3, 2, 3, 0, 0, 1, 1, 0, 1, 2, 1, 1, 2, 1, 0, 0]$

$B_{10} = [0, 1, 0, 1, 1, 3, 3, 0, 3, 1, 1, 0, 3, 0, 3, 3,1, 1, 0, 1, 3, 3, 0, 1, 0, 1, 1, 3, 3, 0, 3, 1, 1, 0]$

$B_{11} = [1, 2, 1, 0, 1, 0, 1, 2, 1, 0, 3, 3, 2, 3, 0, 3,0, 3, 2, 3, 0, 1, 1, 2, 1, 0, 1, 0, 1, 2, 1, 0, 3, 3]$

$B_{12} = [0, 1, 1, 3, 0, 3, 2, 3, 3, 3, 1, 0, 3, 3, 1, 0,1, 2, 1, 1, 1, 3, 0, 1, 1, 3, 0, 3, 2, 3, 3, 3, 1, 0]$

Other knots are more complicated and it is not realistic to make such lists. However, for $5_1, 5_2, 6_1, 6_2, 6_3$, we could at least provide one local structure for each.

When combining local structures, we need to search for some connection blocks and also keep in mind that we have 4 choices: the original sequence, the negative sequence (left hand becomes right hand), the reverse sequence, and the negative of the reverse sequence. For a composite knot *A#B*, there are 4 choices as mentioned above. For a composite knot *A#B#C*, there are 4 ∗ 4 = 16 choices.

We will apply the above method to the construction of possible shortest paths for composite knots in the next few sections.

Improving the 5_1 and $3_1#5_1$ knot paths

The shortest Rubik's Snake 5_1 knot we found in Hou S[21] has 52 blocks. We rotate the sequence and write as:

[1,1,1,3,3,0,2,1,1,2,1,0,1,2,1,1,3,0,0,1,0,0,2,3,0,0,0,3,0,1,2,1,0,
1,0,1,2,1,0,0,3,0,0,1,3,0,0,0,3,2,0,0]

Now we replace the subsequence [0,1,0,0,2,3,0] with [3,0,1,2,1,0,1] without changing the relative location and orientation of the starting and end blocks of this part.

By doing so, we make room for another part of the path. We now replace the subsequence [0,0,3,2,0,0] with [3,3,0,3] and improved the 5_1 path to only 50 blocks:

[1,1,1,3,3,0,2,1,1,2,1,0,1,2,1,1,3,0,3,0,1,2,1,0,1,0,0,3,0,1,2,1,0,1,
0,1,2,1,0,0,3,0,0,1,3,0,3,3,0,3]

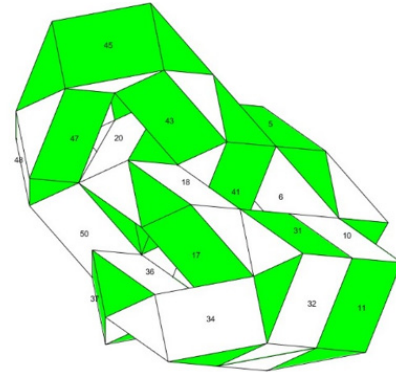Figure 1 and 2 show the Rubik's snake 5_1 knot with 50 blocks and its line representation.

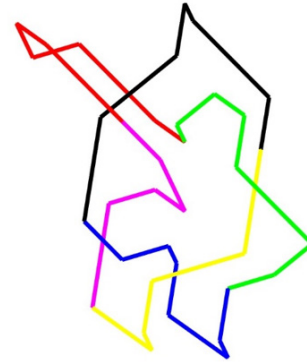**Figure 1** 5_1 knot with 50 blocks.

Figure 2 Line representation for 5_1 knot.

Since the 5_1 knot path is improved, the new result can improve the composite knot path $3_1#5_1$ in Hou S.[23] We use one of our 6 local structures for 3_1 combined with a local structure of the new 5_1. The new path has 72 blocks instead of 74. The notation "..." separates local knot structures and connections between local knot structures:

[0,1,2,1,0,1,0,1,2,1,0,3,0,0,1,2,0,1,1,0,1,2,1,...2, 1,0,...3,3,1,0,1,0,
3,2,3,0,3,0,0,1,0,3,2,3,0,3,0,3,2,3,0,0,1,0,0,3,1,0,1,1,0,1,3,3,...2,
3,3,3,1,0,0,3]

Figure 3 and 4 show the Rubik's snake $3_1#5_1$ knot with 72 blocks and its line representation.
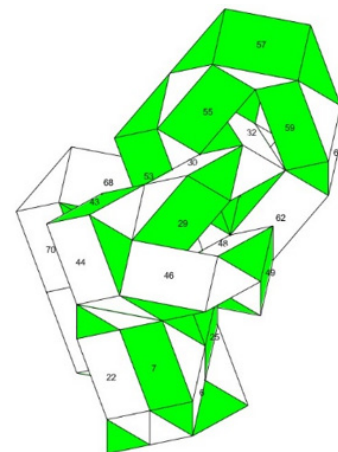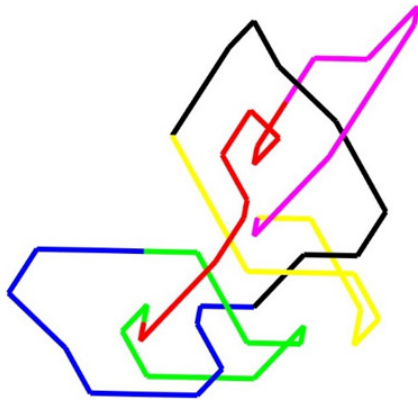

Figure 3 $3_1#5_1$ knot with 72 blocks.

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Figure 4 Line representation for $3_1\#5_1$ knot.

Improving the 6_1 knot path

The shortest Rubik's Snake 6_1 knot we found in[22] has 64 blocks. We verified that could not be improved locally. A new idea is needed to improve it.

Based on the structure of 6_1 knot, we could try to construct a period 2 Rubik's snake 6_1 knot. However, the computational cost is not acceptable for an exhaustive search. A trick is needed.

At the website,[19] knot diagrams of prime knots with a small crossing number are available. We could see that the 6_1 knot and 8_3 knot differ only by a local change. Also, 8_3 has nice knot symmetry so that we could use the pattern $[t,-t,t,-t]$ to construct.

We first use the pattern $[t,-t,t,-t]$ to construct a 8_3 knot, where $t = [0,0,0,0,0,1,0,1,3,0,3,3,0,3,3,3,1,0,1]$.

This is done by an exhaustive search with rotations 0,1,3 only to save computational cost. Note that 8_3 knot has this kind of symmetry while 6_1 knot does not. We keep periodic 2 structure and do local improvement. This can change the knot structure from 8_3 to 6_1. When we shrink to 60 blocks, we get a possibly shortest 6_1 with 60 blocks:

$[3,0,3,0,0,0,0,0,1,0,1,3,0,3,3,0,3,3,3,1,0,1,0,0,0,0,3,2,0,3]$ repeated twice.

We could also make a local change to remove the 2 and rotate the sequence to $[0,0,0,0,1,0,1,3,0,3,3,0,3,3,3,1,0,1,0,0,0,0,3,1,0,1,1,0,1,1]$ repeated twice.

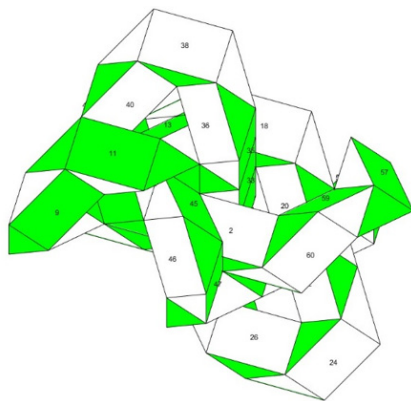Figure 5 and 6 show the Rubik's snake 6_1 knot with 60 blocks and its line representation.
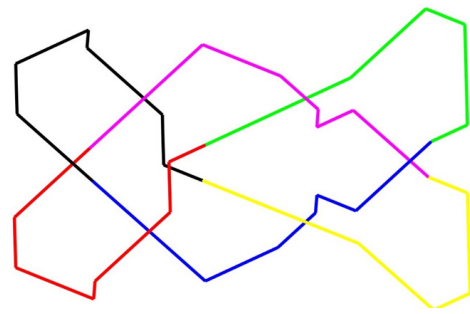

Figure 5 6_1 knot with 60 blocks.

Figure 6 Line representation for 6_1 knot.

We will use part of this sequence in the next section.

$3_1\#6_1, 3_1\#6_2, 3_1\#6_3$

We exhaust the 6 choices for the local 3_1 structure and search for the smallest number of connection blocks with a local structure of 6_1, 6_2 and 6_3, where the local structure of 6_1 comes from the previous section (the improved 6_1) and the other two come from.[22] The following are the shortest solutions we found (only one representative for each tied solutions, again the notation "..." separates local knot structures and connections between local knot structures):

$[0,1,2,1,0,1,0,1,2,1,0,3,0,0,1,2,0,1,1,0,1,2,1,...0,3,0,0,...0,0,0,1,0,$
$1,3,0,3,3,0,3,3,3,1,0,1,0,0,0,0,3,1,0,1,1,0,1,1,0,0,0,0,1,0,1,3,0,3,$
$3,0,3,3,3,1,0,1,0,0,...0,0,0,1,2,0,0,3]$

$[0,1,2,1,0,1,0,1,2,1,0,3,0,0,1,2,0,1,1,0,1,2,1,...1,3,0,3,0,1,0,...0,1,$
$1,3,3,1,1,0,0,3,3,1,1,1,1,1,1,0,2,1,0,0,3,0,0,0,1,2,1,1,2,1,0,1,1,0,2,$
$1,0,0,1,2,0,0,3,0,3,2,3,3,...0,1,0,0,3]$

$[0,1,2,1,0,1,0,1,2,1,0,3,0,0,1,3,0,3,3,0,3,1,1,...1,3,0,3,3,0,...0,0,1,$
$1,0,2,3,0,3,0,3,3,0,0,1,0,0,3,2,3,3,2,0,0,1,0,1,2,1,1,3,0,0,1,0,0,2,$
$3,0,0,0,3,0,1,2,1,0,1,0,1,2,1,1,0,3,0,0,...3,0,0]$

Figure 7,8,9,10,11 and 12 show the Rubik's snake knots constructed in this section and their line representations.

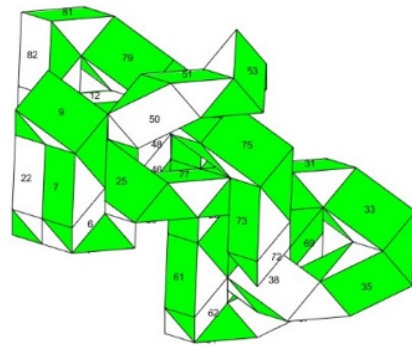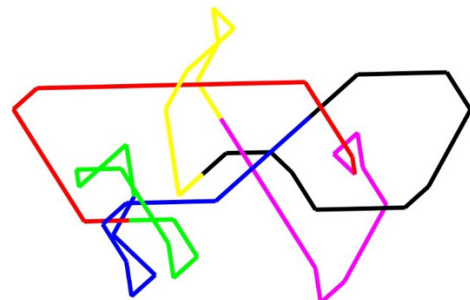**Figure 7** $3_1\#6_1$ knot with 84 blocks.
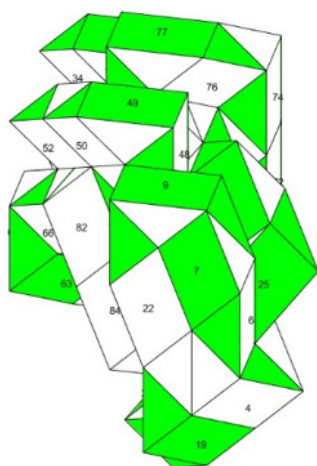

Figure 8 Line representation for $3_1\#6_1$ knot.

Figure 9 $3_1\#6_2$ knot with 84 blocks.

Figure 10 Line representation for $3_1\#6_2$ knot.

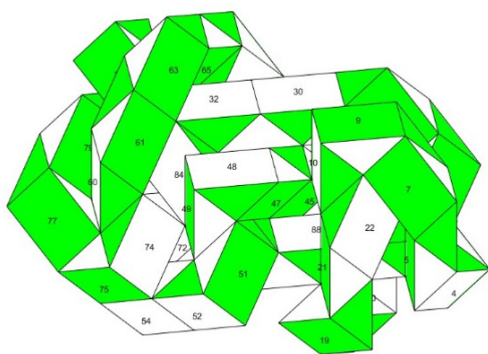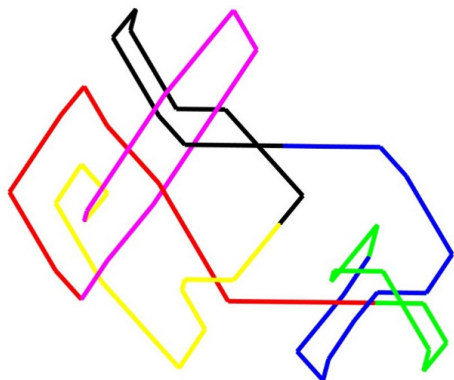**Figure 11** $3_1\#6_3$ knot with 88 blocks.

Figure 12 Line representation for $3_1\#6_3$ knot.

$4_1\#5_1, 4_1\#5_2$

By exhausting the 12 choices for local patterns related to shortest Rubik's snake 4_1 knots with the pattern $[t, -t, t, -t]$ and using local patterns for 5_1 and 5_2, we constructed $4_1\#5_1$ with 82 blocks and $4_1\#5_2$ with 86 blocks:

[0,0,0,1,0,1,1,0,2,1,0,0,0,0,3,0,3,3,0,2,3,0,0,0,0,1,0,1,1,0,2,1,0,0, ...0,0,3,0,...1,1,3,0,3,0,1,2,1,0,1,0,0,3,0,1,2,1,0,1,0,1,2,1,0,0,3,0, 0,1,3,0,3,3,0,3,1,1,...1,3,3,0,3,0]

[0,0,0,1,0,1,1,0,2,1,0,0,0,0,3,0,3,3,0,2,3,0,0,0,0,1,0,1,1,0,2,1,0,0, ...0,1,0,3,0,0,0,2,...1,0,3,0,0,0,3,2,0,3,3,0,0,1,2,1,1,3,0,3,3,0,3,1,0, 0,3,0,0,3,2,0,0,1,0,1,1,0,2,1,0,0,0,0,...0,1]

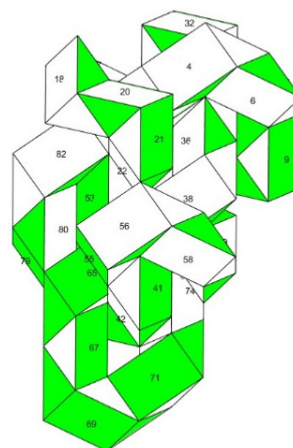Figure 13, 14, 15 and 16 show the knots constructed in this section and their line representations.

Figure 13 $4_1\#5_1$ knot with 82 blocks.

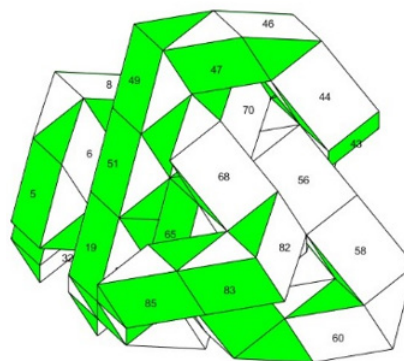**Figure 14** Line representation for $4_1\#5_1$ knot.

Figure 15 $4_1\#5_2$ knot with 86 blocks.

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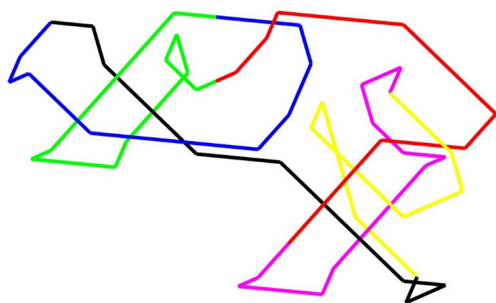
Figure 16 Line representation for $4_1\#5_2$ knot.

$3_1\#3_1\#3_1$ knot

The natural way to construct a $3_1\#3_1\#3_1$ knot with short path is to use one of the local structures of 3_1 and add a few blocks then repeat three times, hopefully to form a closed loop. By a simple search, we found the shortest has 84 blocks, for example $[0,1,2,1,0,1,0,1,2,1,0,3,0,0,1,2,0,1,1,0,1,2,1,0,3,3,0,3]$ repeated 3 times.

Figure 17 shows the $3_1\#3_1\#3_1$ knot with 84 blocks and period 3.
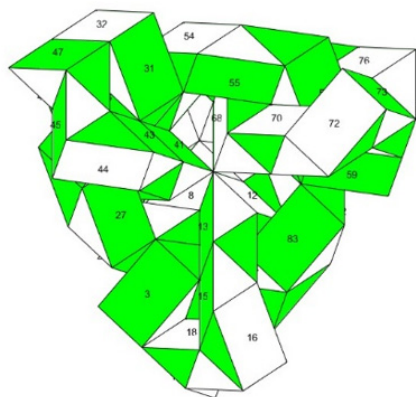

Figure 17 $3_1\#3_1\#3_1$ knot with 84 blocks and period 3.

Figure 18 shows the line representation.

Figure 18 Line representation of $3_1\#3_1\#3_1$ knot with 84 blocks.

However, if we break the symmetry, it allows us more degree of freedom. As mentioned in the previous section, for a composite knot $A\#B\#C$, there are $4*4=16$ choices. For each part using local 3_1 structure, there are 6 choices. $16*6*6*6$ is a small amount we could afford to exhaust and for each case we seek for the shortest closed loop connections.

We found the shortest path has 82 blocks. For example, let $t = [0,1,2,1,0,1,0,1,2,1,0,3,0,0,1,2,0,1,1,0,1,2,1]$.

Then $[t,0,3,t,1,0,0,3,t,1,0,0,2,0,0,3]$ is a $3_1\#3_1\#3_1$ that has 82 blocks.

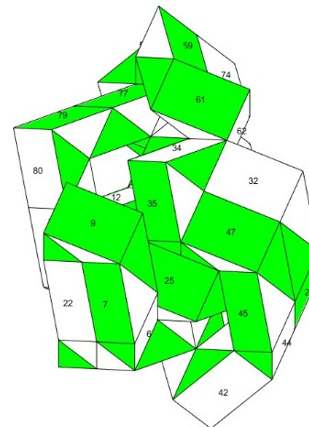Figure 19 shows the $3_1\#3_1\#3_1$ knot with 82 blocks.

Figure 19 $3_1\#3_1\#3_1$ knot with 82 blocks.
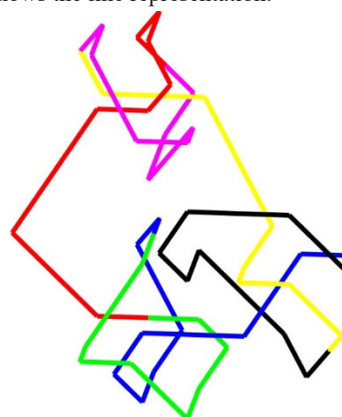
Figure 20 shows the line representation.

Figure 20 Line representation of $3_1\#3_1\#3_1$ knot with 82 blocks.

Conclusion

Finding the shortest path for certain Rubik's snake's nontrivial knot is a challenging problem. With each joint having 4 choices, the total number of attempts is too many to have an exhaustive search. Based on our previous results for prime knots, we found possible shortest composite knots as follows: $3_1\#6_1$ with $34+60-10=84$ blocks, $3_1\#6_2$ with $34+62-12=84$ blocks, $3_1\#6_3$ with $34+64-10=88$ blocks, $4_1\#5_1$ with $44+50-12=82$ blocks, $4_1\#5_2$ with $44+54-12=86$ blocks, $3_1\#3_1\#3_1$ with $34+34+34-20=82$ blocks. We summarize the results in Table 1. It is verified that no further local improvement could be made. Although there is no proof that these are optimal globally, the results look reasonable that they might be the shortest.

Table 1 A list of length of shortest paths found for composite knots with 9 crossings based on prime knots

Shortest A	Shortest B	Shortest composite
3_1: 34	6_1: 60	$3_1\#6_1$: 34+60-10=84
3_1: 34	6_2: 62	$3_1\#6_2$: 44+62-12=84
3_1: 34	6_3: 64	$3_1\#6_3$: 34+64-10=88
4_1: 44	5_1: 50	$4_1\#5_1$: 34+50-12=82
4_1: 44	5_2: 54	$4_1\#5_2$: 34+54-12=86
3_1: 34	3_1: 34	$3_1\#3_1\#3_1$: 34+34+34-20=82

Conflicts of interest

Authors declare that there is no conflict of interest.

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