

Shortest paths of Rubik's snake composite knots up to 8 crossings

Abstract

A Rubik's Snake is a toy that has been around for 40 years. It can be twisted to many interesting shapes. In particular a Rubik's snake can be twisted to form a knot. In this paper we study how many blocks are needed to form a composite knot with up to 8 crossings. Also, we improved a couple of previous results for Rubik's snake prime knots.

Keywords: Rubik's snake, composite knot

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Introduction

Researchers and amateurs alike have been enthralled by Prof. Rubik's adaptable puzzle toy, "Rubik's Snake",¹⁻³ for over 40 years. The right isosceles triangle prisms that make up Rubik's Snake are joined together to form blocks, and this system allows for a variety of configurations that let users build complex structures and shapes. 24 blocks were initially included in the toy, but more blocks are now needed because of the possibility of creating designs that are more intricate. The Rubik's Snake has been used as a tool for the study of protein folding^{4,5} and for the construction of reconfigurable modular robots.⁶⁻⁸ Some ideas in the study of Rubik's Snake such as the use of rotation matrix is also used in rigid Origami folding.^{9,10} In previous papers that the first author collaborated with others, strategies have been given for the design of a Rubik's Snake,¹¹ and some mathematical problems concerning a Rubik's Snake have been studied.¹² Rotations that are not 0°, 90°, 180°, or 270° are mentioned in¹¹ but not much theoretical work is presented. On the other hand,¹² has quite some theoretical work but is only concerned with integer multiple of 90 degree rotations. In,¹³ Rubik's snakes with general rotation angles were studied with theoretical work presented. In,¹⁴ theorems about palindromic, periodic and Möbius Rubik's snakes were proved. In,¹⁵ a general strategy for designing box shapes using a Rubik's snake was presented and a counting formula was derived. Knot theory¹⁶ is an interesting research area in mathematics that drew a lot of attention. It also has interesting applications in architectural design, for example, some work by Zaha.¹⁷ Knot classification tool is available.¹⁸ In,¹⁹⁻²¹ shortest path problems for Rubik's Snake prime knots are studied. These are the pioneering work that combines Rubik's snake with knots. Rubik's snake composite knot paths have not been studied in the literature. In this paper, we study the shortest path for a Rubik's snake composite knot with up to 8 crossings. The organization is as follows: in Section 2, we summarize our earlier work on the Rubik's Snake prime knots. Some results are needed for our discussion here. In section 3, we proved a theorem needed for some of our constructions. In Section 4 to 6, we present results for Rubik's Snake composite knots up to 8 crossings. We conclude in Section 7.

Prime knots

The Rubik's Snake prime knots with shortest path up to 6 crossings have been studied. We just briefly summarize the results. The 3_1 (trefoil) knot was studied in our previous paper¹⁹ and 4_1 , 5_1 and 5_2 studied in our previous work.²⁰ The prime knots with 6 crossings 6_1 , 6_2 and 6_3 were studied in our recent work.²¹ In this paper, in addition to the study on composite knots, we will make a couple of improvements for 4_1 and 5_2 .

A Rubik's snake prime knot we frequently use in this paper is the following palindromic 3_1 sequence from¹⁹: $\theta = [1, 0, 1, 2, 1, 0, 3, 0, 0, 1, 3, 0, 3, 3, 0, 3, 1, 1, 1, 3, 0, 3, 3, 0, 3, 1, 0, 0, 3, 0, 1, 2, 1, 0]$, where 0, 1, 2, 3 mean starting with a straight ruler and rotating a joint by 0°, 90°, 180°, or 270°.

A theorem about $[t, -t]$ sequence

We found a convenient tool to generate short path for composite knots based on our previous research on prime knots. The main idea is in this theorem and we will apply it to construct composite knots in the next section:

Theorem:

If the 1st and the $k+1$ th blocks of a Rubik's Snake has opposite entrance direction and opposite exit direction, and t is a sequence connecting the first $k+1$ blocks, then $[t, -t]$ is a closed loop.

Proof. If a rotation index is 0 or 2, and we change both the entrance and exit direction to negative of the original, then the exit direction of the next block is also changed to negative of the original. If a rotation index is 1 or 3, and we change both the entrance and exit direction to negative of the original, then the exit direction of the next block is unchanged. However, when we use $-t$ instead of t sequence, those 0 and 2 are unchanged but 1 and 3 interchanged. This will make the case of 1 or 3 also have the exit direction of the next block changed to negative of the original. It follows by induction that we will have k pairs of opposite vectors. Therefore the sum of $2k$ vectors has to be zero, forming a closed loop. Further, since the entrance and exit

directions of the first and $k+1$ th blocks are opposite respectively, the entrance and exit directions of the first and $2k+1$ th blocks must be the same respectively. Therefore the $2k$ blocks form a closed loop.

We are going to use this $[t, -t]$ pattern to construct some composite knots. This is very convenient, as the shift from the first block to the $k+1$ th block does not matter. What matters is to find a pair of blocks with opposite entrance directions and opposite exit directions. Note that the converse is not true. For example, $t = [2, 1, 0, 2, 1, 1, 2]$. Then $[t, -t]$ is a closed loop (though collision occurs). However, the exit directions of the 1st and 8th blocks are the same.

$3_1\#3_1$ knot and $4_1\#4_1$ knot

Our general strategy for constructing composite knot paths is to remove a part of a prime knot that is not knotted. We start with the simplest composite knot $3_1\#3_1$. The original sequence θ for a trefoil path is changed to $[0, 3, 0, 1, 2, 1, 0, 1, 0, 1, 2, 1, 0, 3, 0, 0, 1, 3, 0, 3, 3, 0, 3, 1, 1, 1, 3, 0, 3, 3, 0, 3, 1, 0]$ after a rotation redefining which one is the first block. Now we only use the first 28 of the 34 elements to remove a part that is not knotted. Now if we repeat the first 28 elements twice, it does not quite work. However, the condition in the theorem is met and by using $[t, -t]$ where t is the sequence with the first 28 elements, we get the $3_1\#3_1$ knot without crashing, which has $28*2=56$ blocks: $[0, 3, 0, 1, 2, 1, 0, 1, 0, 1, 2, 1, 0, 3, 0, 0, 1, 3, 0, 3, 3, 0, 3, 1, 1, 1, 3, 0, 0, 1, 0, 3, 2, 3, 0, 3, 0, 3, 2, 3, 0, 1, 0, 0, 3, 1, 0, 1, 1, 0, 1, 3, 3, 3, 1, 0]$

Note that instead of doing $[t, -t]$ we interchange 1 and 3. The reason is that in mod 4 sense, -2 is 2 and -1 is 3.

Figure 1 shows the $3_1\#3_1$ knot with 56 blocks.

Figure 2 shows the line representation corresponding to it that clearly reveals it is a composition of two trefoil knots.

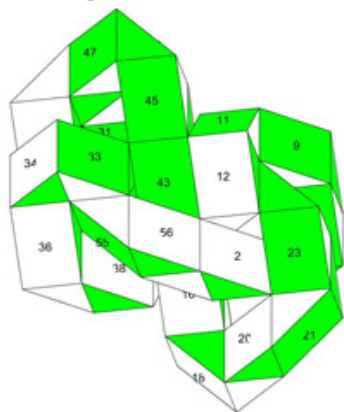


Figure 1 $3_1\#3_1$ knot with 56 blocks.

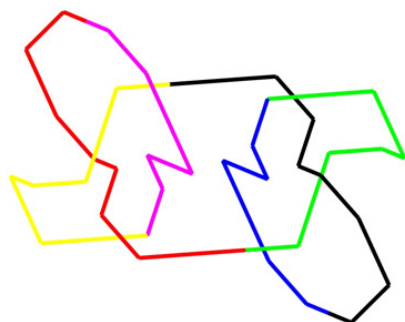


Figure 2 Line representation of $3_1\#3_1$ knot.

Note that if we are too greedy to use $t = [3, 0, 1, 2, 1, 0, 1, 0, 1, 2, 1, 0, 3, 0, 0, 1, 3, 0, 3, 3, 0, 3, 1, 1, 1, 3]$, even though we still have the condition in the theorem to ensure $[t, -t]$ is a closed loop, we have collisions. This also serves as a counter example that the theorem does not ensure no collision even if t part alone has no collision.

In²⁰ the shortest 4_1 path found at that time had 46 blocks. Motivated by the symmetry associated with 4_1 knot, we searched for period 2 sequence $[t, -t, t, -t]$ and found that the shortest path is improved to only 44 blocks. For example, $t = [0, 0, 0, 0, 1, 0, 1, 1, 0, 2, 1]$.

Figure 3 shows the improved 4_1 knot with 44 blocks.

Figure 4 shows the line representation corresponding to it that clearly reveals it is a 4_1 knot. The knot symmetry of 4_1 is also displayed.

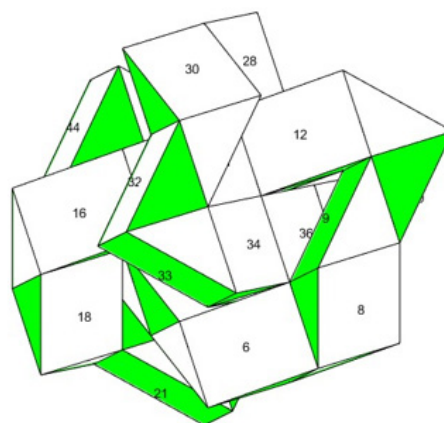


Figure 3 4_1 knot with 44 blocks that improved the previously published result.

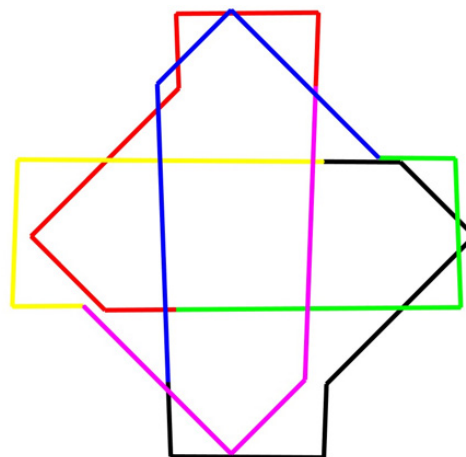


Figure 4 Line representation of 4_1 knot.

Now by using the $[t, -t]$ idea again, we found that 76 blocks are enough to make $4_1\#4_1$ knot: $[0, 0, 0, 1, 0, 1, 1, 0, 2, 1, 0, 0, 0, 0, 3, 0, 3, 3, 0, 2, 3, 0, 0, 0, 0, 1, 0, 1, 1, 0, 2, 1, 0, 0, 0, 1, 1, 0, 0, 0, 0, 3, 0, 3, 3, 0, 2, 3, 0, 0, 0, 1, 0, 1, 1, 0, 2, 1, 0, 0, 0, 0, 3, 0, 3, 3, 0, 2, 3, 0, 0, 0, 3, 3, 0]$

Figure 5 shows the $4_1\#4_1$ knot with 76 blocks.

Figure 6 shows the line representation corresponding to it that clearly reveals it is a composition of two 4_1 knots. We could also constructed a period two version of $4_1\#4_1$ with 76 blocks:

$[0, 0, 0, 1, 0, 1, 1, 0, 2, 1, 0, 0, 0, 0, 3, 0, 3, 3, 0, 2, 3, 0, 0, 0, 0, 1, 0, 1, 1, 0, 2, 1, 0, 0, 0, 0, 3, 0]$ repeated twice.

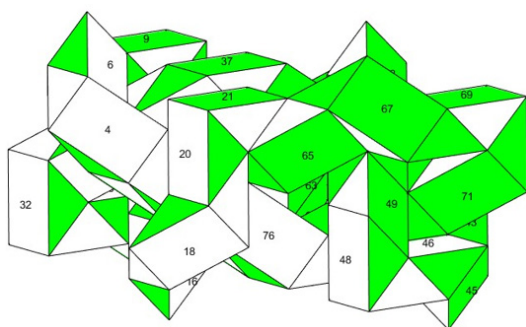


Figure 5 $4_1\#4_1$ knot with 76 blocks.

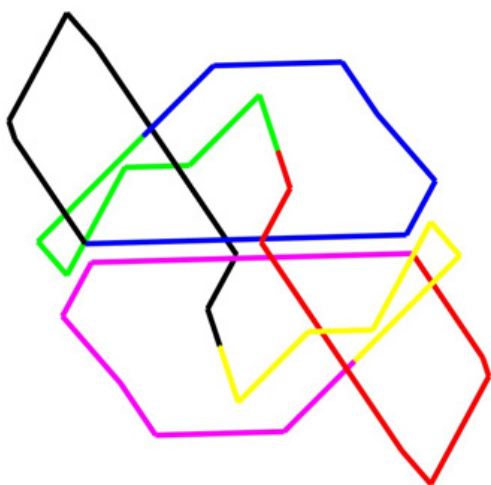


Figure 6 Line representation of $4_1\#4_1$ knot.

$3_1\#4_1$

The trick to get $3_1\#4_1$ is to reuse most of the first part of $3_1\#3_1$ and most of the reverse order of first part of $4_1\#4_1$: $[0, 1, 2, 1, 0, 1, 0, 1, 2, 1, 0, 3, 0, 0, 1, 3, 0, 3, 0, 3, 1, 1, 1, 3, 0, 3, 3, 0, 1, 0, 0, 0, 1, 2, 0, 1, 1, 0, 1, 0, 0, 0, 3, 2, 0, 3, 3, 0, 3, 0, 0, 0, 1, 2, 0, 1, 1, 0, 1, 0, 0, 0, 1, 1, 0]$

Figure 7 shows the $3_1\#4_1$ knot with 68 blocks.

Figure 8 shows the line representation corresponding to it that clearly reveals it is a composition of 3_1 and 4_1 knots.

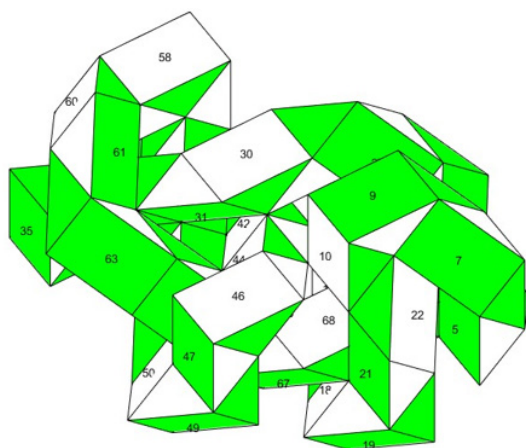


Figure 7 $3_1\#4_1$ knot with 68 blocks.



Figure 8 Line representation of $3_1\#4_1$ knot.

$3_1\#5_1$ and $3_1\#5_2$

These knots with 3_1 component have a similar construction. Recall that we had a sequence θ_i for palindromic 3_1 with 34 blocks in Section 2 cited from.¹⁹ First, let α_i be the sequence for the 5_1 knot with 52 blocks: $[1, 1, 3, 0, 0, 1, 0, 0, 2, 3, 0, 0, 0, 3, 0, 1, 2, 1, 0, 1, 0, 1, 2, 1, 0, 0, 3, 0, 0, 1, 3, 0, 0, 0, 3, 2, 0, 0, 1, 1, 1, 0, 1, 1, 0, 1, 0, 1, 1, 0, 1, 2]$

Note that this is not exactly the same 5_1 construction in.²⁰ As mentioned there, the solution is not unique and we choose this one for our purpose to construct the shortest composite knot. We construct $[\theta_{27}, \dots, \theta_{34}, \theta_{11}, \dots, \theta_{21}, 1, \alpha_{48}, \dots, \alpha_{52}, \alpha_{11}, \dots, \alpha_{42}, 0]$. This has 78 blocks. Then we could save 4 blocks by a shift to get $[\theta_{29}, \dots, \theta_{34}, \theta_{11}, \dots, \theta_{19}, 0, \alpha_{48}, \dots, \alpha_{52}, \alpha_{11}, \dots, \alpha_{42}, 0]$. This has only 74 blocks and is the shortest $3_1\#5_1$ we found.

We now take a different shortest 3_1 from [19]. $\theta = [1, 0, 1, 2, 1, 0, 3, 0, 3, 2, 3, 3, 2, 3, 0, 3, 1, 1, 1, 3, 0, 3, 2, 3, 3, 2, 3, 0, 3, 0, 1, 2, 1, 0]$

The shortest 5_2 sequence found in²⁰ had 56 blocks. Although it was verified to be optimal locally, we found that a non-local change could improve it to only 54 blocks below: $\alpha = [0, 3, 0, 0, 2, 1, 0, 3, 0, 0, 0, 3, 2, 0, 3, 3, 0, 0, 1, 2, 1, 1, 3, 0, 3, 3, 0, 3, 1, 0, 0, 3, 0, 0, 3, 2, 0, 0, 1, 0, 1, 1, 0, 2, 1, 0, 0, 0, 0, 1, 3, 3, 0, 3]$

Figure 9 shows the improved 5_2 knot with 54 blocks.

Figure 10 shows the line representation corresponding to it that clearly reveals it is a 5_2 knot.

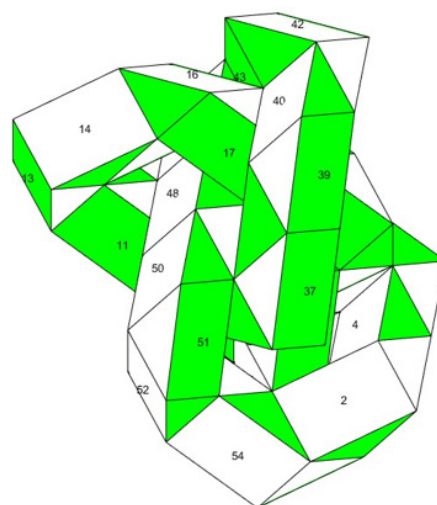


Figure 9 5_2 knot with 54 blocks that improved the previously published result.

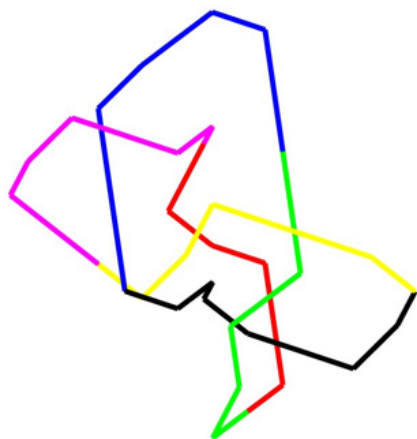


Figure 10 Line representation of 5_2 knot.

We construct $[\theta_{30}, \theta_{34}, \theta_{12}, \theta_{22}, 0, 0, 0, 0, \alpha_{12}, \alpha_{48}, 1, 0, 3, 2, 0, 0, 1, 0]$. This $3_1\#5_2$ has 76 blocks.

Figure 11, 12, 13 and 14 show the composite knots and their line representations constructed in this section.

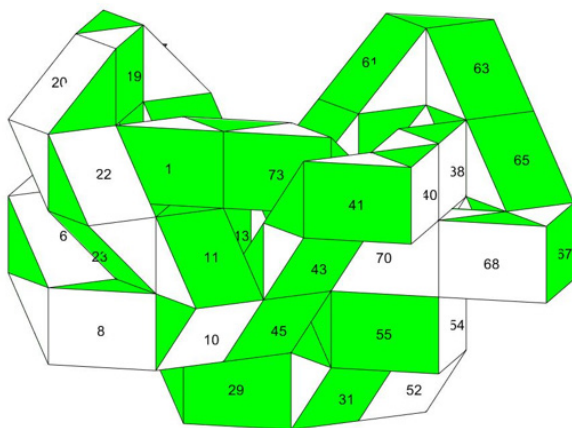


Figure 11 $3_1\#5_1$ knot with 74 blocks.

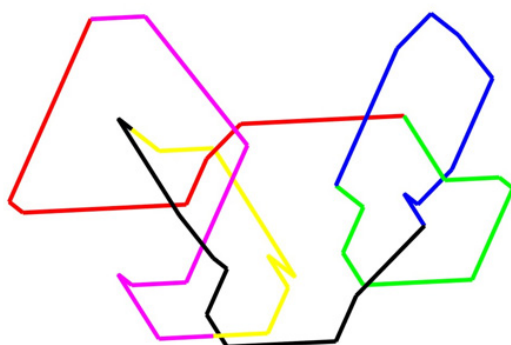


Figure 12 Line representation of $3_1\#5_1$ knot.

Table 1 A list of length of shortest paths found for composite knots based on prime knots

Shortest A	Shortest B	Shortest A#B
$3_1:34$	$3_1:34$	$3_1\#3_1: 34+34-12=56$
$4_1:44$	$4_1:44$	$4_1\#4_1: 44+44-12=76$
$3_1:34$	$4_1:44$	$3_1\#4_1: 34+44-10=68$
$3_1:34$	$5_1:52$	$3_1\#5_1: 34+52-12=74$
$3_1:34$	$5_2:54$	$3_1\#5_2: 34+54-12=76$

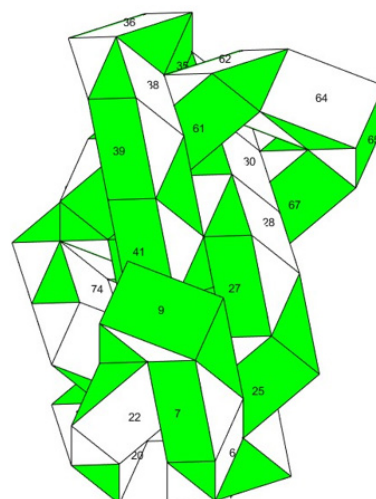


Figure 13 $3_1\#5_2$ knot with 76 blocks.

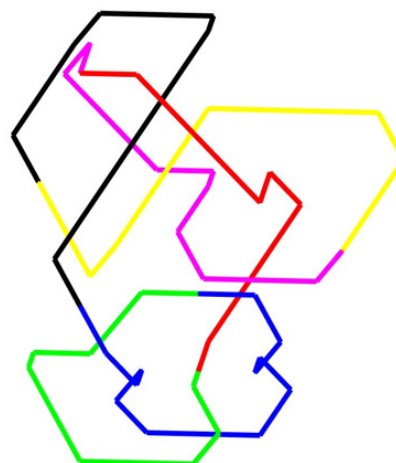


Figure 14 Line representation of $3_1\#5_2$ knot.

Conclusion

Finding the shortest path for certain Rubik's snake nontrivial knot is a challenging problem. With each joint having 4 choices, the total number of attempts is too many to have an exhaustive search. Based on our previous results for prime knots, we found possible shortest composite knots as follows: $3_1\#3_1$ with $34 + 34 - 12 = 56$ blocks, $4_1\#4_1$ with $44 + 44 - 12 = 76$ blocks, $3_1\#4_1$ with $34 + 44 - 10 = 68$ blocks, $3_1\#5_1$ with $34+52-12 = 74$ blocks, and $3_1\#5_2$ with $34 + 54 - 12 = 76$ blocks. We also improved the 4_1 knot from 46 to 44 blocks and 5_2 knot from 56 to 54 blocks. It appears all our results are of the form $A + B - 12$, where A and B are the smallest number of blocks we found for the corresponding prime knots, except $3_1\#4_1$. Table 1 summarized the above results. As before, we verified that no local improvement could be made. Although there is no rigorous proof, the results look reasonable that they might be the shortest paths.

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Conflicts of interest

Authors declare that there is no conflict of interest.

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