Review Article

Shortest paths of Rubik's snake prime knots with up to 6 crossings and application to roller coaster design

Abstract

A Rubik's Snake is a toy that was invented over 40 years ago together with the more famous Rubik's Cube. It can be twisted to many interesting shapes including knots. Four blocks can form a trivial knot. Previously we have studied the shortest paths for Rubik's Snake prime knots with up to 5 crossings. In this paper we study how many blocks are needed to form prime knots with 6 crossings. There are three different types of such knots. The results are classified using the DT (Dowker-Thistlethwaite) code. We also apply our findings to roller coaster design by using the tube version of the Rubik's snake.

Keywords: Rubik's Snake, prime knot, shortest path

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Introduction

The Rubik's Snake is a toy invented by Prof. Rubik over 40 years ago.[1] It consists of right isosceles triangular prisms (called blocks) that, except for the first and last block, are connected to two other blocks at the centers of the square faces. The first design of Rubik's snake came with 24 blocks, which were twisted to form many solid bodies. (Bird, cat, caterpillar, frog, flower, spiral etc.) However, more complicated design needs more blocks.

The Rubik's Snake has been used for the study of protein folding[2] and also for the construction of reconfigurable modular robots.[3–5] There are more applications of robots presented in.[6,7] Some ideas in the study of Rubik's Snake such as the use of rotation matrix is also shared in the study of rigid Origami folding.[8,9] In previous papers that the first author published with others, some strategies have been given for the design of a Rubik's Snake,[10] and some mathematical problems concerning a Rubik's Snake have been studied.[11] Rotations that are not $0°$ (or $360°$), $90°$, $180°$, or $270°$ are mentioned in[10] but not much theoretical work is presented. On the other hand, there are quite some theoretical work but only concerned with integer multiple of 90 degree rotations in.[11] In,[12] Rubik's snakes with general rotation angles were studied with theoretical work presented. In,[13] theorems about Möbius, palindromic and periodic Rubik's snakes were proved.

Knot theory has been extensively studied.[14] However, how could a Rubik's Snake form a knot is not reported in the literature until our recent publication in which possible shortest Rubik's snake trefoil knot paths with 34 blocks and possible shortest tube version (with general rotation angles not limited to integer multiple of 90 degrees) of Rubik's snake trefoil knot path were presented.[15]

For a trivial knot, 4 blocks of a Rubik's snake can form a closed loop, which has the shortest path. In this paper, we study the shortest path for a Rubik's snake non-trivial prime knot with 6 crossings. Knots are classified according to their isotopy classes. For non-trivial prime knots up to 6 crossings, we have 3_1 (trefoil), 4_1, 5_1, 5_2, 6_1, 6_2 and 6_3.

The 3_1 (trefoil) knot was studied in our previous paper[15] and 4_1, 5_1 and 5_2 studied in our previous work.[16] We now focus on 6_1, 6_2 and 6_3. The idea is to use certain key patterns: 6_1 comes from certain 4_1 path with an extra hole as key pattern, 6_2 comes from a different 4_1 path with an extra hole as key pattern, and 6_3 based on certain 3_1 path with an extra hole as key pattern. These are combined with local searches.

The rest of the paper is organized as follows: in Section 2, we summarize our previous work on non-trivial prime knots up to 5 crossings. In Section 3, we consider prime knots with 6 crossings. All the results are verified by the DT code to make sure they are indeed the knot we claimed they are. In Section 4, we apply our findings to roller coaster design by using the tube version of the Rubik's Snake. We conclude in Section 5.

Prime knots with up to 5 crossings

The Rubik's Snake prime knots with up to 5 crossings with integer multiple of 90 degree rotations has been studied by us in the past. We just briefly summarize the results. In,[15] 36 solutions of possible shortest Rubik's Snake trefoil knots were given. Among those solutions, 6 of them are palindromic, the 30 non-palindromic solutions come in pairs and if we do not count the reverse order as a new solution, then effectively those are 15 non-palindromic solutions, making it 21 solutions overall in this sense. What have been verified are the following: there are no Rubik's Snake trefoil knots with less than 34 blocks that has period more than 1 (period 2 and 3 are verified[15] and trefoil knot cannot have period more than 3). There are no Rubik's Snake trefoil knots with less than 34 blocks that are palindromic. In the construction of the solutions, the key pattern appears to be optimal. Based on that, we believe there is no solution with less than 34 blocks. In,[16] we found possible shortest Rubik's Snake 4_1 knot with 46 blocks, 5_1 with 52 blocks and 5_2 with 56 blocks. For nontrivial knots other than 3_1, even exhausting period 2 or palindromic Rubik's snakes would be too expensive. The idea was to use certain key patterns combined with local searches.

Prime knots with 6 crossings

Now consider the knot 6_1. The shortest we found has 64 blocks shown in Figure 2 with its line representation in Figure 1:

[1, 1, 0, 1, 1, 3, 3, 1, 1, 0, 0, 1, 0, 3, 0, 0, 1, 2, 0, 0, 3, 0, 0, 1, 0, 3, 3, 2, 3, 0, 3, 3, 0, 2, 3, 0, 0, 0, 3, 0, 0, 1, 2, 0, 1, 1, 2, 1, 1, 1, 3, 0, 3, 3, 0, 3, 1, 0, 0, 0, 1, 0, 1, 1]
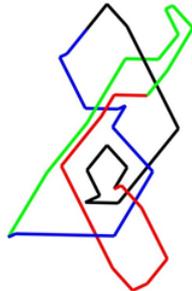

Figure 1 The line representation of a Rubik's snake 6_1 knot with 64 blocks.
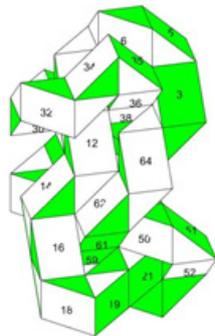

Figure 2 A Rubik's snake 6_1 knot with 64 blocks.

The path is based on the following 4_1 with an extra room in a hole to allow crossing another time. [1, 0, 1, 1, 3, 3, 1, 1, 0, 0, 1, 0, 3, 0, 0, 1, 2, 0, 0, 3, 0, 0, 1, 0, 3, 3, 2, 3, 0, 3, 3, 0, 2, 3, 0, 0, 0, 3, 0, 0, 1, 2, 0, 1, 1, 0, 0, 0]. Note that the shortest 4_1 we found has 46 blocks but does not have this extra room. This sequence with 48 blocks should be the shortest with such extra room needed for 6_1.

To better understand the construction, we observe that the 4_1 sequence has a common subsequence with the 2nd to the 46th of the 6_1. The green color in Figure 1 represents the first 16 blocks and the red color represents the 49th to the 64th blocks, and the black color is for the 33th to 47th. That means if we remove the red color and some green and black, then connect the green with black, we will get 4_1. That is exactly what happened.

The DT code of our 6_1 after the self-crossing is removed is [2, −10, −12, 14, −4, −16, 8, 6] (not the best reduction of DT but indeed corresponds to 6_1). The DT code for the 4_1 is [4, −8, 10, −2, 6] (not the best reduction of DT code but it indeed corresponds to 4_1). The two crossings saved are those two below between red and blue lines. To summarize, the construction of 6_1 is based on 4_1, going through a hole another time.

Now consider the knot 6_2. The shortest we found has 62 blocks shown in Figure 4 with its line representation in Figure 3:

[2, 1, 0, 0, 0, 3, 0, 0, 1, 2, 0, 1, 1, 1, 1, 1, 3, 3, 0, 0, 1, 1, 3, 3, 1, 1, 0, 0, 1, 0, 3, 1, 1, 0, 1, 3, 0, 0, 1, 0, 3, 3, 2, 3, 0, 3, 0, 0, 0, 2, 1, 0, 0, 1, 2, 0, 1, 1, 0, 1, 2, 1, 1]
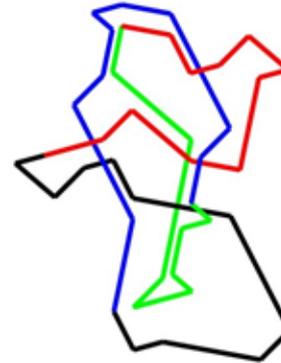

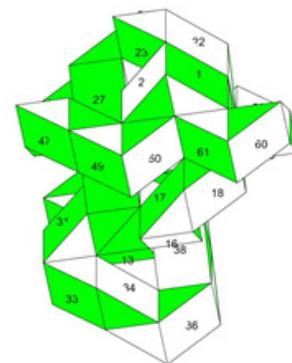**Figure 3** The line representation of a Rubik's snake 6_2 knot with 62 blocks.

Figure 4 A Rubik's snake 6_2 knot with 62 blocks.

The path is based on the following 4_1 with an extra room in a hole to allow crossing another time: [2, 3, 0, 0, 0, 3, 0, 0, 1, 2, 0, 1, 1, 1, 1, 1, 3, 3, 0, 0, 1, 1, 3, 3, 1, 1, 0, 0, 1, 0, 3, 1, 1, 0, 1, 3, 0, 0, 1, 0, 3, 3, 2, 3, 0, 3, 3, 0]. Note that the shortest 4_1 we found has 46 blocks but does not have this extra room. This sequence with 48 blocks should be the shortest with such extra room needed for 6_2.

To better understand the construction, we observe that the 4_1 sequence has a common subsequence with the 3rd to the 44th of the 6_2. The green color in Figure 3 represents the first 16 blocks and the red color represents the 49th to the 64th blocks, and the black color is for the 33th to 47th. That means if we remove the red color and some green and black, then connect the green with black, we will get 4_1. That is exactly what happened.

The DT code of our 6_2 is [4, 8, −12, 2, −14, −6, −10] (not the best reduction of DT but indeed corresponds to 6_2).

The DT code of the 4_1 is the standard [4, 6, 8, 2]. The three crossings difference between these DT codes are the two red-green crossings and the right and a red-blue crossing in the middle. To summarize, the construction of 6_2 is based on 4_1, going through a hole another time.

Finally consider the knot 6_3. The shortest we found has 64 blocks shown in Figure 6 with its line representation in Figure 5:

[0,0,1,0,0,3,2,3,3,2,0,0,1,0,1,2,1,1,3,0,0,1,0,0,2,3,0, 0,0,3,0, 1,2,1,0,1,0,1,2,1,0,3,0,0,3,0,0,1,3,0,0,1,0,0, 1,1,0,2,3,0,3,0,3,3]
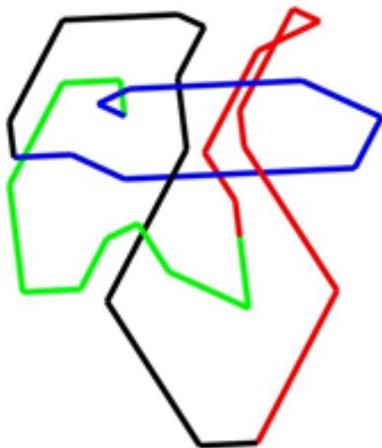

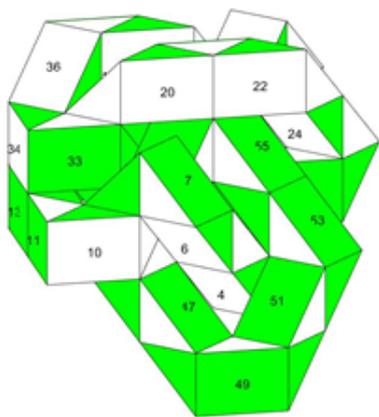**Figure 5** The line representation of a Rubik's snake 6_3 knot with 64 blocks.

Figure 6 A Rubik's snake 6_3 knot with 64 blocks.

The path is based on the following 3_1 with an extra room in a hole to allow crossing another time: [1,1,0,1,2,1,1,3,0,0, 1,0,0,2,3,0,0,0,3,0,1,2,1,0,1,0,1, 2,1,0,3,0,0,1,2,0]. Note that the shortest 3_1 we found has 34 blocks but does not have this extra room. This sequence with 36 blocks should be the shortest with such extra

room needed for 6_3. Also note that for the last hole if we go across the opposite way it would have been a 5_2 knot instead of a 6_3 knot.

To better understand this construction, we observe that this 3_1 sequence have a common subsequence with the 13th to the 48th block of the 6_3. The green color in Figure 5 represents the first 16 blocks and the red color represents the 49th to the 64th blocks. That means if we ignore the red part and most of the green part starting with the connection point to red, and then reconnect using a short path, we will get a trefoil knot 3_1. This is exactly what happened in that figure.

In addition to two self-crossings that can be easily removed, we could also remove two red-blue crossing because the red line is below the blue line for both crossings above. That gives a DT code of [4, 8, 10, 2, 12, 6] corresponding to 6_3 knot. The corresponding trefoil knot saves 3 crossings (two red-blue and one green-black) and has a DT code of [4,6,2].

To summarize, the construction of 6_3 is based on 3_1, going through a hole another time.

Just like the knots with less than 5 crossings, we found many other solutions for 6_1, 6_2 and 6_3 that tied for the shortest as well. The idea is that we could keep some key patterns and let the computer do a local exhaustive search for about 12 blocks.

Application to roller coaster design

In,[12] the tube version of the Rubik's Snake is introduced. The idea is that each block of the Rubik's Snake corresponds to a tube of the same shape. The connection is smooth with the first order derivative continuous. The roller coasters in parks typically has a path of just a trivial knot. It would be interesting to build a roller coaster with a non-trivial knot, especially with more crossing numbers than just a trefoil knot or a figure-8 knot.

Figure 7 shows the design of a roller coaster using the tube version of Rubik's snake 6_2 knot with the shortest path we found (62 blocks). The view angle is different from the earlier pictures but the snake sequence remains the same. The reason to choose the shortest path is to save the cost while keeping the complexity. Note that this is under a simplified model and in practice the design does not have to strictly follow the rule of having identical basic blocks.
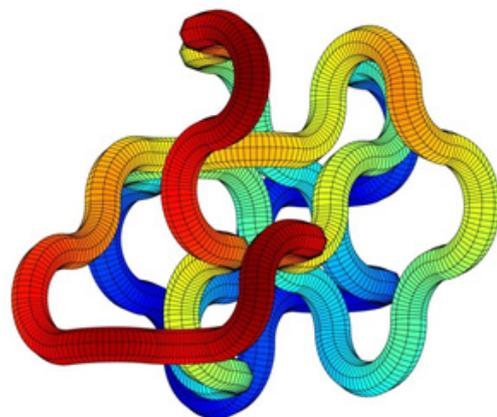

Figure 7 The tube version of Rubik's snake 6_2 knot with 62 blocks for roller coaster design.

Conclusion

Finding the shortest path for a Rubik's snake nontrivial knot is challenging. With each joint having 4 choices, the total number of attempts is too many to have an exhaustive search. By using certain key patterns combined with local searches, we found possible shortest Rubik's Snake 6_1 knot with 64 blocks based on a 4_1 knot with an extra hole that the path could go through, possible shortest 6_2 knot with 62 blocks based on a 4_1 knot with an extra hole that the path could go through, and possible shortest 6_3 knot with 64 blocks based on a 3_1 knot with an extra hole that the path could go through. All the results are verified by the DT code to make sure they are indeed the knots we claimed they are. It would be interesting future work to see if these can be proved to be optimal.

Acknowledgments

Conflicts of interest

The Authors declare that there is no conflict of interest.

References

1. Fenyvesi C. *Rubik's snake of infinite possibilities*. The Washington Post; 1981.

2. Iguchi K. A toy model for understanding the conceptual framework of protein folding: Rubik's magic snake model. *Mod Phys Lett B*. 1998;12(13):499–506.

3. Ding X, Lu S, Yang Y. *Configuration transformation theory from a chain-type reconfigurable modular mechanism-rubik's snake*. The 13th World Congress in Mechanism and Machine Science; 2011.

4. Zhang X, Liu J. *Prototype design of a rubik snake robot*. Mechanisms and Machine Science; 2016:36.

5. Liu J, Zhang X, Zhang K, et al. *Configuration analysis of a reconfigurable rubik's snake robot*. Proceedings of the Institution of Mechanical Engineers, Part C: Journal of Mechanical Engineering Science. 2019;233(9):3137–3154.

6. Yim M, Roufas K, Duff D, et al. Modular reconfigurable robots in space applications. *Autonomous Robots*. 2003;14(2-3):225–237.

7. Zhang X, Liu J, Feng J, et al. Effective capture of nongraspable objects for space robots using geometric cage pairs. *IEEE/ASME Transactions on Mechatronics*. 2020;25(1):95–107.

8. Hull TC, Belcastro SM. Modelling the folding of paper into three dimensions using affine transformations. *Linear Algebra and its applications*. 2002;348(1-3):273–282.

9. Tachi T. Simulation of rigid origami. *Origami*. 2009;4(08):175–187.

10. Li Z, Hou S, Bishop T. Computational design and analysis of a magic snake. *J Mech Rob*. 2020;12(5):054501.

11. Hou S, Chen Y, Li Z. Some mathematical problems related to the rubik's snake. *J Mech Rob*. 2021;13(1):014502.

12. Hou S, Atkins S, Chen Y. A rubik's snake with general rotation angles. *SCIREA Journal of Information Science and Systems Science*. 2021;5(6):123–135.

13. Hou S, Su J, Chen, Y. Palindromic, periodic and mobius rubik's snakes. *Int Rob Auto J*. 2021;7(3):84–88.

14. Adams C. The knot book: An Elementary introduction to the mathematical theory of knots. *American Mathematical Society, ISBN*. 2004;978-0-8218-3678-1.

15. Hou S, Su J. Shortest paths of trefoil knot designs using rubik's snakes. *Int Rob Auto J*. 2022;8(1):18–20.

16. Hou S, Su J. Shortest paths of rubik's snake prime knots up to 5 crossings. *Int Rob Auto J*. 2022;8(2):47–50.