Mini Review

On the methods of analytical mechanics for mathematical modeling of the dynamics of non-free systems and some variants of their application to the dynamics of parallel manipulators

Abstract

The level of development of actuator technology and the element base and software of control loops of modern automatic devices makes it possible to use in technical practice complex control algorithms based on the results of mathematical control theory, including with incomplete information about the state. The effectiveness of such algorithms is determined by the presence of an adequate mathematical model of the device under study. For such a widespread class of automatic devices as manipulators with parallel kinematics, the problem of developing methods for creating mathematical models that allow taking into account nonlinear effects, despite the efforts of numerous researchers, remains relevant. To a large extent, the complexity of modeling the dynamics of this class of automatic devices is associated with the presence of parallel kinematic chains, which leads to the need to apply the results of analytical mechanics of non-free systems. In this section of theoretical mechanics, several rigorous methods have been developed, as a result of which mathematical models of different dimensions are obtained with different levels of complexity of the algorithms for their derivation and study. This leads to the need for a critical analysis of both the methods themselves and their possible application in technical practice. In this paper, the main algorithms for obtaining mathematical models of the dynamics of systems with geometric constraints are presented in detail and some comparative analysis of their application to modeling the dynamics of manipulators with parallel kinematics is carried out.

Keywords: non-free system, geometric constraints, parallel kinematic chains, manipulators with parallel kinematics

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Introduction

Most modern technical devices are automatic. Therefore, for modern engineering, the development of methods for applying the abstract-theoretical results of various areas of science to technical practice is becoming an increasingly urgent problem. Perhaps the main stage, which largely ensures success in solving this problem, is to obtain an adequate nonlinear mathematical model of the device under consideration. The problem of mathematical modeling is especially acute in robotics. This is due not only to the widespread use of robots, but also to the fact that the developed software of the robot control loop can reliably provide the specified behavior of the device only if there is a mathematical model of its actuator with a rather complex kinematics.

This paper discusses the main methods of analytical mechanics of non-free systems, compares the dimensions of the resulting mathematical models and the complexity of their research and considers some results of their application to modeling the dynamics of manipulators with parallel kinematics as systems with geometric constraints.

The need to apply methods for modeling the dynamics of mechanical systems with geometric constraints, comparing the dimensions of models

Basic general methods of analytical mechanics of non-free systems with geometric constraints and comparison of the complexity of their application

To automate heavy, harmful, tedious and monotonous work in various industries, robotic manipulators with various types of actuators are widely used. Historically, the first type of these devices are sequential kinematics manipulators.[1–3] In such manipulators, the actuator is an open kinematic chain with serial connection of nodes. With the development of technology, manipulators with parallel kinematics are becoming more and more widespread. In parallel manipulators, at least two independent kinematic chains are closed in the final link.[1,2]

The presence of parallel kinematic chains increases the rigidity of the actuator, since in such manipulators the links, unlike sequential kinematics manipulators, work practically only in compression or tension. Due to this mode of operation, the mass of the manipulator links can be significantly reduced compared to the mass of a sequential kinematics manipulator of the same rigidity, which creates certain advantages for their practical use.[2–4]

However, at the same time, such a design of the actuator imposes severe restrictions on the freedom of movement of its links, which leads to non-linear geometric relationships for the coordinates of the manipulator nodes and the distances between the nodes. From the point of view of mechanics, the relationships limiting freedom should be considered as geometric constraints,[5–9] which significantly complicates the mathematical modeling of the dynamics of parallel manipulators.[10–13]

The latter circumstance is particular importance due to the fact that at the present stage of development of electronics and software in the control loops of mechatronic systems,[1,14] which include manipulators, it is possible to use algorithms based on the latest achievements of mathematical control theory, including with incomplete information.[15] A necessary condition for the implementation of this possibility is the availability of an adequate mathematical model of the device obtained by rigorous methods.

The adequacy of mathematical models of nonlinear systems is of particular importance in the case of incomplete information about the state. In such tasks, the mathematical model is used not only at the design stage of the object control system, but also during its entire operation[1,15] to obtain an estimate of the phase state of the object based on the processing of current measurement information. Obviously, not only the complexity of building a model, but also the complexity of its research is of great importance.

The presence of geometric connections leads to the need to apply the results of analytical mechanics of non-free systems,[5-9] and the most widely used are equations with constraints multipliers. The traditional approach to using equations with constraints multipliers for mathematical modeling of the dynamics of specific parallel manipulators requires the definition of explicit expressions for the constraints multipliers as functions of coordinates and velocities.

To obtain them, it is carried out,[12,13] similarly to[5] (and without reference to this work), a two-fold differentiation of geometric relationships in time. Next, in the obtained relations, instead of accelerations, substitute their expressions from the equations of motion of a system freed from constraints with the introduced reactions of constraints. Such a procedure complicates the study so much that an analytical consideration becomes practically impossible. Due to the complexity of the model, the adequacy of the simulation is proposed to be assessed by comparing the real behavior of the object with the results of computer simulation over a finite period of time.[12,13] Comparison of the results of computer simulation with the real dynamics of the object over a finite period of time in no way gives grounds to draw a reasonable conclusion about the adequacy of the model. As is known, by selecting the parameters of completely different models of the same dynamical system, it is possible to ensure an arbitrarily small difference in the values of its phase variables over a finite time interval.[16–18]

At the same time, it should be noted that even for computer simulation, the mathematical model is often simplified by refusing to take into account some of the parameters previously introduced into the model.[13]

In the analytical mechanics of systems with geometric constraints, a completely different rigorous approach to modeling their dynamics has long been known, which significantly reduces the dimension of the mathematical model and simplifies its study due to the exclusion of not only constraint multipliers, but also dependent velocities.[5,19,20] The overall dimension of the mathematical model is reduced compared to the above approach by twice the number of geometric connections, which not only allows the results of mathematical control theory to be applied with incomplete information about the state,[15] but also, based on the nonlinear stability theory,[21,22] to use, in particular, the method of N.N. Krasovskii to determine the control coefficients and parameters of the evaluation system.[23] Despite the fact that this approach has shown its effectiveness in solving problems of modeling the dynamics of a number of complex systems with geometric constraints,[24–28] it has not yet found wide application.

We process to a detailed consideration of the main methods for obtaining mathematical models of the dynamics of systems with geometric constraints.

General algorithm for obtaining equations of motion for non-free systems with constraint multipliers

Consider a mechanical system with n degrees of freedom, the configuration of which is determined by the parameters $q_1,...q_{n+m}$, the number of which exceeds the number of degrees of freedom n, due to the fact that m geometric constraints

$$f_s(q_1,...q_{n+m}) = 0;\ s = \overline{1,m} \tag{1}$$

are imposed on the system. Due to the fact that between $n+m$ parameters $q_1,...q_{n+m}$, there are m independent relations (1) m of these $n+m$ parameters are redundant coordinates.

We introduce vectors

$$q^1 = (q_1,...,q_{n+m});\ r^1 = (q_1,...,q_n);s^1 = (q_{n+1},...,q_{n+m});$$

dash means transpose.

$$f(q) = 0;\ f^1(q) = (f_1(q),...,f_m(q));$$

We assume that, due to the independence of the constraints, the condition

$$\frac{\partial(f_1,...,f_m)}{\partial(q_{n+1},...,q_{n+m})} = \frac{\partial f}{\partial s} \neq 0; \tag{2}$$

To study non-free mechanical systems, Lagrange created two methods: the method of indefinite multipliers and the method of generalized coordinates. The transition to generalized coordinates requires the elimination of redundant coordinates, which is associated with a complication of the study with a complex form of constraint equations.[8] Despite the fundamental possibility of elimination due to (2), the elimination of unnecessary dependent coordinates s from expressions (1) often leads to cumbersome formulas, especially when there are trigonometric functions in the constraint equations.[8,6]

Let $T(q_1,...,q_{n+m},q_1,...,q_{n+m})$, be the kinetic energy without regard to constraints, $\tilde{Q}i(q_1,...,q_{n+m},\dot{q}_1,...,\dot{q}_{n+m})$ (potential and nonpotential) forces referred to the coordinates q_i.

If we differentiate the equations of geometric constraints (1) with respect to time, we obtain kinematic (holonomic) constraints in the form

$$\frac{\partial f}{\partial q}\dot{q} = \frac{\partial f}{\partial r}\dot{r} + \frac{\partial f}{\partial s}\dot{s} = 0; \tag{3}$$

Then, multiplying each equation with number k by an indefinite multipliers λ_k from the d'Alembert–Lagrange principle, one can obtain the equations of motion of a system with redundant coordinates with constraints multipliers[5,6]

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}} - \frac{\partial T}{\partial q} = \tilde{Q} + \left(\frac{\partial f}{\partial q}\right)^1 \lambda; \tag{4}$$

The equations (4) together with the constraint equations (1) represent a complete nonlinear model of the dynamics of a non-free holonomic mechanical system with geometric constraints, obtained by rigorous methods of analytical mechanics. The model includes $n+m$ second-order differential equations and m algebraic equations (1). The unknowns to be determined are all $n+m$ coordinates, all $n+m$ velocities, and m constraints multipliers.

Application of equations of motion with constraint multipliers to model the dynamics of specific systems

The algorithm of one of the methods,[5–7,29] (pp. 354-359) for obtaining a mathematical model of the dynamics of a particular device includes the following steps:

1. The equations of constraints in the form (3) are once again differentiated by time, which leads to linear relations with respect to accelerations.

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2. Equations of motion (4) are resolved with respect to accelerations.
3. As a result of substituting the expressions obtained for the accelerations into the doubly differentiated equations of geometric constraints, we obtain a system of linear inhomogeneous algebraic equations with respect to the constraint multipliers, which is always uniquely resolvable due to the fact that the determinant of the matrix of coefficients at the multipliers is different from zero (a rigorous proof is given in.[5]).
4. The substituting the values of the constraint multipliers, determined from this algebraic system as functions of all coordinates, velocities and acting forces into equations (4) gives a mathematical model of the object under study.

Despite the rather obvious extreme complexity and high laboriousness of the described modeling method, it is this method that is used in numerous studies[11–13,30] with large lists of references) of the dynamics of manipulators with parallel kinematics. In this case, the coordinates that determine the configuration of the system are often rather unsuccessfully chosen (see the form of the equations of the constraints of the Delta robot.[13,28,30])

In analytical mechanics, alternative, much more efficient methods for determining bond factors have been developed quite a long time ago, which are still practically not used, despite the algorithmic simplicity and much lower complexity.

Some methods for removal of constraint multipliers from models of non-free systems dynamics with geometric constraints

From the equation (3) due to the condition (2) we can obtain the relation

$$\dot{s} = -\left(\frac{\partial f}{\partial s}\right)^{-1}\left(\frac{\partial f}{\partial r}\right)\dot{r} = B(q)\dot{r}; \tag{5}$$

which we rewrite in a form convenient for further transformations

$$-B(q)\dot{r} + \dot{s} = (-B(q)\;\; E_m)\begin{pmatrix}\dot{r}\\ \dot{s}\end{pmatrix} = 0;$$

taking into account which it is possible to rewrite the equation (4) in the form

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{r}} - \frac{\partial T}{\partial r} = \tilde{Q}_r - B^1(q)\lambda;$$
$$\frac{d}{dt}\frac{\partial T}{\partial \dot{s}} - \frac{\partial T}{\partial s} = \tilde{Q}_s + \lambda;$$

From the last equation, the Lagrange multiplier is determined

$$\lambda = \frac{d}{dt}\frac{\partial T}{\partial \dot{s}} - \frac{\partial T}{\partial s} - \tilde{Q}_s;$$

which, after substitution into the first equation, leads to a mathematical model of the system dynamics with constraints (1):

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{r}} - \frac{\partial T}{\partial r} = \tilde{Q}_r - B^1(q)\left(\frac{d}{dt}\frac{\partial T}{\partial \dot{s}} - \frac{\partial T}{\partial s} + \tilde{Q}_s\right); \tag{6}$$

In a slightly different form, similar equations without constraints multipliers were obtained in.[6] Although Lurie A.I. assumes the solvability of the kinematic constraint equations with respect to dependent velocities, he does not write them in the form (5). Therefore, obtained by Lurie A.I. the equations of motion (7.10.9) without multipliers have a more complex form than the above (6).[6] The problem of elimination of factors was also considered by Suslov G.K.[7]

Moreover, both Lurie A.I. and Suslov G.K. more complex systems were considered, on which, in addition to geometric constraints, non-integrable kinematic ones were also imposed. Perhaps that is why further operations to simplify the equations, which Shulgin M.F. did,[8] were not performed.

Reducing the dimension of the mathematical model of the dynamics of systems with geometric constraints. Equations in the redundant coordinates in the form M.F. Shulgin's, free from constraint multipliers

Consider the method of further reduction of the dimension of the mathematical model and simplification of its study, proposed by Shulgin M.F[8]

We considered the exclusion the dependent velocities using constraint equations (5) from the kinetic energy $T(q_1,...,q_{n+m},\dot{q}_1,...,\dot{q}_{n+m})$ and forces $\tilde{Q}_i(q_1,...,q_{n+m},\dot{q}_1,...,\dot{q}_{n+m})$ and denote the resulting expression for kinetic energy as $T^*(q_1,...,q_{n+m},\dot{q}_1,...,\dot{q}_n)$ and for forces as $Q_i(q_1,...,q_{n+m},\dot{q}_1,...,\dot{q}_n)$

Taking into account the integrability of kinematic constraints, from (6) one can obtain the equations of motion of the system in redundant coordinates in the form of M.F. Shulgin.[8]

$$\frac{d}{dt}\frac{\partial T^*}{\partial \dot{r}} - \frac{\partial T^*}{\partial r} = Q_r - B^1(q)\left(\frac{d}{dt}\frac{\partial T^*}{\partial \dot{s}} - \frac{\partial T}{\partial s} + Q_s\right); \tag{7}$$

The variables of the nonlinear mathematical model of the system dynamics with $n+m$ coordinates , on which m geometric constraints (1) are imposed, obtained in the form of m second-order differential equations (7) and m first-order differential equations (5), are all $n+m$ coordinates, only n independent speeds.

The research algorithm using this method includes the following actions:

i. By differentiating the equations of geometric constraints with respect to time, differential equations () are obtained, resolving which, with respect to dependent velocities, equations of kinematic constraints (5) are obtained;
ii. Using (5), dependent speeds are excluded from the kinetic energy and acting forces;
iii. Compiled differential equations in the form (7).
iv. The overall dimension of the model is reduced in comparison with the model (1), (4) by $2m$ due to the exclusion of dependent velocities and constraint multipliers from consideration.

Conclusion

The main function of modern automatic devices is carried out due to controlled movement under the action of control formed at the current time by processing measurement information.

Manipulators are one of the main types of automatic devices, since in many branches of technology the most demanded mode of operation is to perform operations to move objects. At the same time, parallel manipulators are increasingly used, since they have a greater rigidity of the actuator,[1] and, consequently, higher accuracy, as well as significantly less weight compared to sequential kinematics manipulators.[2,3] For the effective use of any automatic device, it is necessary to have an adequate nonlinear mathematical model obtained by rigorous methods. Only under this condition can a sufficiently reliable and efficient use of control algorithms based on the results of mathematical control theory be ensured.

However, mathematical modeling of the controlled dynamics of manipulators with parallel kinematics differs significantly from a similar problem for manipulators with sequential kinematics.

An increase in the rigidity of the manipulator mechanism with parallel kinematics is ensured by closing several kinematic chains in its final link. Therefore, nonlinear conditional relationships inevitably arise between the coordinates of the manipulator nodes and the distances between the nodes, as a result of which, to model the dynamics, it is necessary to apply the analytical mechanics of non-free systems. To describe the configuration, it is necessary to enter coordinates in an amount exceeding the number of degrees of freedom of the system. Linearization of the equations of geometric constraints and the use of only the first approximation of constraints does not always lead[24] to the correct results. Even Routh noted that the linearization of the constraint equations in the vicinity of the motion under study when constructing a mathematical model of the dynamics of systems with non-linear geometric constraints is generally unacceptable.[31]

However, it is possible to reasonably distinguish situations when, for systems with geometric constraints, a reasonable simplification of the study is possible by reducing the dimension of the problem by eliminating dependent coordinates from the linearized equations of nonlinear constraints and passing to independent coordinates.[19,20] But the legitimacy of such a transition can be established only by preliminary construction of a complete nonlinear model of the dynamics of the object under study.

Analytical mechanics has a fairly extensive set of methods that simplify, if applied properly, the modeling of the dynamics of non-free systems. Many of these methods cannot yet find wide application in technical practice, since until recently they were the subject of only abstract theoretical considerations. In this paper, the algorithms of the main methods for constructing models of the dynamics of systems with geometric relationships are considered and a comparative analysis of the models obtained by these methods is carried out in terms of their dimension and complexity of consideration in relation to the problems of studying special parallel manipulators.

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Conflicts of interest

Author declares that there is no conflict of interest**.**

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