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The corrected mathematical models for the top motion

Abstract

The mathematical models for the top motions in known publications contain the incorrect expression of the centrifugal torque and do not consider the action of the Coriolis torque generated by the center mass. In reality, the two centrifugal torques, two torques of Coriolis forces, and two changes in the angular momentums formulate the dependency of the angular velocities of the top motions about two axes. The corrected expression of the centrifugal torque and the Coriolis torque generated by the center mass changed the mathematical models for the top motion and its self-stabilization. The new analytical approach for the spinning top motions with the action of all external and inertial torques gives an accurate solution and describes its physics. The derived mathematical models for the spinning top motions and solution represent a good example of the educational process of engineering mechanics.

Keywords: physics of gyroscopic effects, inertial torque, top motions

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Introduction

The top toy and its modifications are the most ancient simple gyroscope that utilizes today and surprised by its properties. Until recent times the motions of the top did not have a correct mathematical model.[1,2] Researchers did not use the fundamental principles of classical mechanics which methods can describe the motions of any objects in.[3-7] The sophisticated motions of the gyroscopes and tops were tried to formulate by several generations of researchers.[8-15] The top motion was described by the inaccurate expression of the centrifugal inertial torque and does not consider the Coriolis torque.[16] This centrifugal torque yielded an inaccurate solution for the motion of the rotating objects about axis of its action, which did not measure because was technically problematic. The gyroscope rotation around another axis with measurement remained without change in the angular velocity. Other researchers confirmed these results and tests. The expression of the centrifugal torque was revised and corrected. The corrected inertial torque changed the axial interrelation of the gyroscope rotations. These new expressions of the inertial torques for horizontal location of the gyroscope are presented in Table 1.[17]

Table 1 Equations of the inertial torques acting on the spinning disc

Type of the torque generated by	Acton	Equation
Centrifugal forces	Resistance Precession	$T_{ct} = \frac{4}{9}\pi^2 J\omega\omega_x$
Coriolis forces	Resistance	$T_{cr} = (8/9)J\omega\omega_x$
Change in angular momentum	Precession	$T_{am} = J\omega\omega_x$
Dependency of angular velocities of spinning disc rotations about axes		$\omega_y = (8\pi^2 + 17)\omega_x$

The new expressions for the top motions on the horizontal surface are derived by the action of the internal and frictional forces and its weight. The spiral motion of the spinning top leg asymptotically drives to the vertical position of its axis. The top preserves vertical spinning until the minimum angular velocity with its following wobbling and side fall. This research work explains the physics of the top motion by the analytical model based on the action of the corrected gyroscopic inertial forces. The resented solution explains in popular form the mechanics of the top motion that are confirmed by practical tests.

Methodology

The spinning top motions is considered when its leg is tilted on the angle γ and rotation in a counter-clockwise direction (Figure 1). The weight of the top, the frictional force of the leg's tip, and inertial torques result in processed motion of the top around its center mass. The top describes a spiral conical surface that drives to its vertical disposition. The action of the frictional force and inertial torques on the top are demonstrated in Figure 1. The analytical approach for the top motion is the same as for the gyroscope with one side support.[16] The analytical models for top motions about axes are as the follows:

$$J_x \frac{d\omega_x}{dt} = T + T_{ct.my} - T_{ctx} - T_{crx} - T_{amy}\eta \quad (1)$$

$$J_y \frac{d\omega_y}{dt} = (T_{ct.x} + T_{amx})\cos\gamma - T_{cry} + T_f \quad (2)$$

$$\omega_y = [4\pi^2 + 8 + (4\pi^2 + 9)\cos\gamma]\omega_x \quad (3)$$

where all parameters are related to the top $J_i = \left(MR^2/4\right) + Ml^2$ is the moment of inertia about axis i [3];

M is mass; $T = Mgl\cos\gamma$ is the torque of the action of the top weight; l is the gravity acceleration, l is the length of the leg; $T_f = Mgfl\cos\gamma$ is the fictional torque of the tip acting in the counter clockwise direction due to rotation of the top around axis oy , where f is the coefficient of the sliding friction, $l\cos\gamma$ is the radius of action of the frictional force, the frictional force reduces the velocity of the top ω that is not considered; $F_{ct} = Ml\cos\gamma\omega_y^2$ is the centrifugal force of the mass rotation around axis oy and acting around axis ox ; ω_y is the precession velocity around axis oy ; γ is the coefficient of the change in the value of the inertial torques; γ is the tilt angle and other expressions are as specified in Table 1.

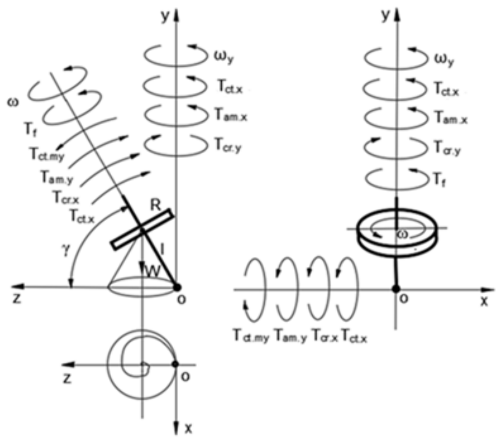

Figure 1 Torques acting on a spinning top.

The precession torque $T_{am.y}$ is changed on the coefficient η because the frictional torque acts [16]:

$$\eta = \frac{(T_{ct.x} + T_{am.x})\cos\gamma + T_f}{(T_{ct.x} + T_{am.x})\cos\gamma} = 1 + \frac{9Mgfl}{(4\pi^2 + 9)J\omega\omega_x} \tag{4}$$

where $T_{ct.x}$ and $T_{am.x}$ are the precession torques acting around axis oy (Table 1).

The solution for Eqs. (1) - (4) with substituting of η, inertial torques (Table 1), $T_{ct'my}$, T_f and T_f into Eq. (1) is the same as represented in publications [16, Chapter 5].

$$J_x\frac{d\omega_x}{dt} = Mgl\cos\gamma + Ml^2\cos^2\gamma\omega_y^2 - \left(\frac{4\pi^2+8}{9}\right)J\omega\omega_x - J\omega\omega_y\left[1 + \frac{9Mgfl}{(4\pi^2+9)J\omega\omega_x}\right] \tag{5}$$

Substituting expression ω_y (Eq. (3)) into Eq. (5) yields:

$$J_x\frac{d\omega_x}{dt} = Mgl\left\{\cos\gamma - \frac{9[4\pi^2+8+(4\pi^2+9)\cos\gamma]f}{4\pi^2+9}\right\} - \left[\frac{4\pi^2+8}{9} + [4\pi^2+8+(4\pi^2+9)\cos\gamma]\right]J\omega\omega_x + Ml^2\cos\gamma\sin\gamma[4\pi^2+8+(4\pi^2+9)\cos\gamma]^2\omega_x^2 \tag{6}$$

Variables are separated and solution of Eq. (6) gives:

$$\frac{J_x d\omega_x}{Ml^2\cos\gamma\sin\gamma[4\pi^2+8+(4\pi^2+9)\cos\gamma]^2\omega_x^2 - \left[\frac{4\pi^2+8}{9} + [4\pi^2+8+(4\pi^2+9)\cos\gamma]\right]J\omega\omega_x + \left\{\cos\gamma - \frac{9[4\pi^2+8+(4\pi^2+9)\cos\gamma]f}{4\pi^2+9}\right\}Mgl} = dt \tag{7}$$

Self-stabilization

A tilted top of the high spinning value will come to vertical disposition because the action of the inertial torques prevails over its weight and centrifugal torque of the center mass. The vertical disposition of the spinning top is defined when the values of the inertial torques are equal to or more than counteracting torques that are expressed by the right side of Eq. (6).

$$Mgl\left\{\cos\gamma - \frac{9[4\pi^2+8+(4\pi^2+9)\cos\gamma]f}{4\pi^2+9}\right\} + Ml^2\cos\gamma\sin\gamma[4\pi^2+8+(4\pi^2+9)\cos\gamma]^2\omega_x^2 = \left[\frac{4\pi^2+8}{9} + [4\pi^2+8+(4\pi^2+9)\cos\gamma]\right]J\omega\omega_x \tag{8}$$

The equilibrium of two groups of torques of Eq. (8) is expressed by the top velocity ω, the velocity of precession ω_x, the tilt angle γ, and the value of the top's leg l. When the value of the inertial torques is bigger than the torques of the left side of Eq. (8), the top will come faster to vertical disposition. In another case, the top will wobble and fall. Analysis of Eq. (8) shows the spinning top does not have the stability of spinning with a long leg, a small radius of its disc, and a low value of the spinning velocity.

Working example

The working example presents the solution of the tilted top motion with the thin disc whose data (Table 2) is the same as considered in publication with simplified solution (Figure 1).[16] The mass of the top leg is neglected and its center mass is disposed on the disc.

Table 2 Technical data of the top

Parameter		**Data**
Angular velocity, ω		1000 rpm
Radius of the disc, R		0,025 m
Length of the leg, l		0,02 m
Radius of the tip		0,001 m
Angle of tilt, γ		75,0°
Mass, M		0,02 kg
Coefficient of friction, f		0,1
Moment of inertia, kgm²	Around axis oz, J = MR²/2	0,625×10⁻⁵
	Around axes ox and oy of the center mass, J = MR²/4	0,3125×10⁻⁵
	Around axes ox and oy, J_x = J_y = MR²/4 + Ml²	1,1125×10⁻⁵

The data of Table 2 is substituted into Eq. (7) and transformation yield:

$$\frac{1,1125\times10^{-5}d\omega_x}{0,02\times0,02^2\cos75^o\sin75^o[4\pi^2+8+(4\pi^2+9)\cos75^o]^2\omega_x^2 - \left[\frac{4\pi^2+8}{9}+4\pi^2+8+(4\pi^2+9)\cos75^o\right]\times0,625\times10^{-5}\times1000\times\frac{2\pi}{60}\omega_x + \left\{\cos75^o - \frac{9\times[4\pi^2+8+(4\pi^2+9)\cos75^o]\times0,1}{4\pi^2+9}\right\}\times0,02\times9,81\times0,02} = dt \tag{9}$$

Solution of Eq. (9) yields:

$$\frac{d\omega_x}{\omega_x^2 - 5,931004\omega_x - 0,465879} = 647,742dt \tag{10}$$

The denominator of Eq. (10) is the quadratic equation which transformation yields:

$$(\omega_x - 6,008540)(\omega_x + 0,077536) \tag{11}$$

Converting of Eq. (11) into integral forms with definite limits yields:

$$\frac{1}{6,086076}\int_0^{\omega_x}\left(\frac{1}{\grave{u}_x - 6,008540} - \frac{1}{\omega_x + 0,077536}\right)d\omega_x = 647,742\int_0^t dt \tag{12}$$

The left side of Eq. (12) is tabulated and presented the integral $\int\frac{dx}{x\pm a} = \ln|a \pm x| + C$, which transformation and solution gives:

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$\ln(\grave{u}_x - 6{,}008540)\Big|_0^{\omega_x} - \ln(\omega_x + 0{,}077536)\Big|_0^{\omega_x} = 3942,207t\Big|_0^t$ that yields the following:

$$\ln\left(\frac{\grave{u}_x - 6{,}008540}{-6{,}008540}\right) - \ln\left(\frac{\omega_x + 0{,}077536}{0{,}077536}\right) = 3942,207t \qquad (13)$$

Transformation of Eq. (13) yields:

$$\omega_x + 0{,}077536 = \frac{-77{,}493551(\grave{u}_x - 6{,}008540)}{e^{3942{,}207t}} \qquad (14)$$

The right side of Eq. (14) is neglected because of a small value of a high order. Equations (14) and (3) yield the values of the angular velocities for the top about two axes:

$$\omega_x = -\ 0{,}077536 \text{ rad/s} \qquad (15)$$

$$\omega_y = [4\pi^2 + 8 + (4\pi^2 + 9)\cos 75^o] \times 0{,}077536 = 4{,}654 \quad \text{rad/s} \qquad (16)$$

where the sign $(-)$for ω_x is the turn of the top in the clockwise direction.

Self-stabilization

The obtained data of Eqs. (15) and (16) and Table 1 are substituted into Eq. (8).

$$0,02 \times 9,81 \times 0,02 \left\{ \cos 75^o - \frac{9[4\pi^2 + 8 + (4\pi^2 + 9)\cos 75^o]0,1}{4\pi^2 + 9} \right\} +$$
$$0,02 \times 0,02^2 \cos 75^o \sin 75^o [4\pi^2 + 8 + (4\pi^2 + 9)\cos 75^o]^2 0{,}077536^2 = \qquad (17)$$
$$\left[\frac{4\pi^2 + 8}{9} + [4\pi^2 + 8 + (4\pi^2 + 9)\cos 75^o] \right] \times 0,625 \times 10^{-5} \times 1000 \times \frac{2\pi}{60} \times 0{,}077536$$

The result of Eq. (17) yields:

$$-0{,}003313 < 0{,}045977 \qquad (18)$$

The value of the inertial torques acting on the top (right component of Eq. (18)) is bigger than the torques generated by the top weight and centrifugal one of its center mass (left component of Eq. (18)). The gyroscopic inertial torques turns the tilted top to vertical that expresses its self-stabilization on the horizontal surface.

Results and discussion

On the spinning top act its weight, frictional force, centrifugal and Coriolis forces of its mass elements and center mass, and the change in the angular momentum. The analytical expression for the tilted top motions on the horizontal surface modified according to the corrected inertial torque that generated by the centrifugal forces of the mass elements of the spinning disc. The motions of the top are interrelated by their angular velocities about axes. The analytical solution of the tilted top motions on the flat surface and a condition for its self-stabilization presented by the corrected components of the acting torques. The physics of the spinning top motion described in previous publication remains the same which a mathematical model was perfected.

Conclusion

The top motions were presented by the analytical solution which components had incorrect expressions of the centrifugal inertial torque and other related dependencies. The recent studies of gyroscopic effects showed the action of the eight inertial torques on spinning objects that are interrelated by the angular velocities about two axes. The corrected mathematical models for the top motion present perfect solution that can be used for popularization and for the course of engineering mechanics. Obtained corrected analytical results of the gyroscopic effects describe the physics of the acting torques on the spinning objects that presented by the mathematical model of the top motions.

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Conflicts of interest

Author declares that there is no conflict of interest.

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