

Shortest paths of Rubik's Snake prime knots up to 5 crossings

Abstract

A Rubik's Snake is a toy that was invented over 40 years ago together with the more famous Rubik's Cube. It can be twisted into many interesting shapes including knots. Four blocks can form a trivial knot. In this paper, we study how many blocks are needed to form a nontrivial knot with up to 5 crossings. The results are classified using the DT (Dowker-Thistlethwaite) code to make sure each design is indeed the knot we claimed it is. A line representation is used to clearly reveal the knot structure of the Rubik's Snake. Exhaustive local searches are performed to verify that no local improvement is possible for the shortest paths we found.

Volume 8 Issue 2 - 2022

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Received: March 02, 2022 | Published: April 01, 2022

Introduction

The Rubik's Snake is a toy that was invented by Prof. Rubik in 1981.¹ It consists of eight isosceles triangular prisms (what we are calling blocks) that, except for the first and last block, are connected to two other blocks at the centers of the square faces. At each connection, one can twist the Rubik's Snake.

The Rubik's Snake has been used as a tool for the study of protein folding² and for the construction of reconfigurable modular robots.³⁻⁵ More applications of robots can be found in^{6,7} Some ideas in the study of Rubik's Snake such as the use of rotation matrix are also used in rigid Origami folding.^{8,9} In previous papers that the first author collaborated with others, strategies have been given for the design of a Rubik's Snake,¹⁰ and some mathematical problems concerning a Rubik's Snake have been studied.¹¹ Rotations that are not 0° (or 360°), 90°, 180°, or 270° are mentioned in¹⁰ but there was not much theoretical work. On the other hand,¹¹ has quite some theoretical work but is only concerned with integer multiple of 90-degree rotations. In¹² Rubik's Snakes with general rotation angles were studied and some theoretical work was presented. In¹³ we proved some theorems about palindromic, periodic, and Möbius Rubik's Snakes.

Knot theory¹⁴ is an interesting research area in mathematics that drew a lot of attention. However, how could a Rubik's Snake form a knot is not reported in the literature until our recent publication¹⁵ in which possible shortest Rubik's Snake trefoil knot paths with 34 blocks and possible shortest tube version (with general rotation angles not limited to integer multiple of 90 degrees) of Rubik's Snake trefoil knot path were presented.

In protein folding,² a non-trivial knot could be formed. The shortest path is a relevant question to ask. For a trivial knot, 4 blocks of a Rubik's Snake can form a closed loop, which has the shortest path. In this paper, we study the shortest path for a Rubik's Snake non-trivial prime knot with up to 5 crossings. Knots are classified according to their isotopy classes. For non-trivial prime knots up to 5 crossings, we have 3_1 (trefoil), 4_1 , 5_1 , and 5_2 . Only the 3_1 (trefoil) knot was studied in our previous paper.¹⁵ The method of exhausting short palindromic sequences in that paper no longer works for more complicated knots due to the cost. Our method is to use certain key patterns combined with local searches. We used the DT (Dowker-Thistlethwaite) codes

to verify that all the results are indeed the knots we claimed they are. This was not done in our previous work¹⁵ because we only studied the trefoil knot in it which is relatively simple. When the crossing number goes up, the isotopy of knots gets more complicated and the DT codes are used to keep track and make the right classification.

The organization is as follows: In Section 2, we summarize our previous work on the simplest nontrivial knot, i.e., the trefoil knot. In Section 3, we consider prime knots with 4 to 5 crossings. We conclude in Section 4.

The trefoil knot

The Rubik's Snake trefoil knot has been studied by us in the past. We just briefly summarize the results. In¹⁵ 36 solutions of possible shortest Rubik's Snake trefoil knots were given. 6 of them are palindromic sequences. What have been verified are the following: there are no Rubik's Snake trefoil knots with less than 34 blocks that have period more than 1 (period 2 and 3 are verified¹⁵ and trefoil knot cannot have period more than 3). There are no Rubik's Snake trefoil knots with less than 34 blocks that are palindromic. From the construction of the 36 solutions, the key pattern appears to be optimal with no room to improve but there is no rigorous proof and it is too costly to verify all cases and make sure there is no solution with less than 34 blocks.

Prime knots with 4 to 5 crossings

For nontrivial knots other than 3_1 , even exhausting period 2 or palindromic Rubik's Snakes would be too expensive. We have to rely on an approach with affordable computational cost. The main idea is to look for a short loop with a hole so that the snake can go through the hole to form a nontrivial knot. For some knots, the hole needs to be larger so that the snake can go through it twice.

Consider the knot 4_1 . The structure of the knot can be seen in Figure 1. The 4 crossings are clearly seen and it is an alternating knot diagram because the under and overpasses alternate. The shortest 4_1 knot we found using a Rubik's Snake has 46 blocks shown in Figure 2 with the following sequence:

[3,0,0,0,3,0,0,1,2,0,1,1,0,0,0,1,0,1,1,3,3,1,1,0,0,1,0,3,1,1,0,1,3,0,0,1,0,3,3,2,3,0,3,3,0,2].

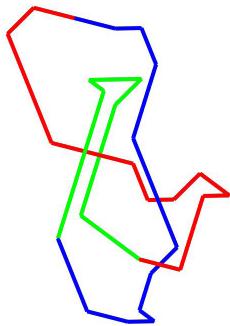


Figure 1 The line representation of A Rubik's Snake 4₁ knot with 46 blocks.

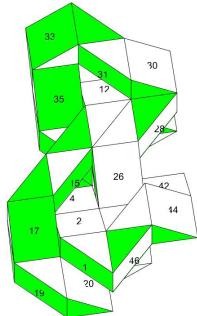


Figure 2 A Rubik's Snake 4₁ knot with 46 blocks.

Here the notation is the same as in¹¹ and¹⁵. Start with a straight ruler of a Rubik's Snake pointing upward. For each joint, “0” means do not rotate, “1” means rotate 90 degrees to the right, “2” means rotate 180 degrees, “3” means rotate 90 degrees to the left. We do not allow collision between any two blocks but they can touch each other. If we connect a line from the center of the entrance square face of a block to the center of the exit square face of a block, in the end, we get the line representation in Figure 1. That is, this figure is not just a random representation of the 4₁ knot. It uses the line segments related to a Rubik's Snake. A DT code can be derived from the figure as [4, 6, 8, 2], which indeed corresponds to 4₁ knot.

Local searches are performed to make sure there is no local substitution to make the number of blocks less without changing the knot's isotopy class. It can be seen from Figure 2 that the structure is indeed compact with no room to improve. Although this is not a rigorous proof that there is no shorter solution, we believe that the result cannot be improved.

In fact, we found at least $4 * (5 + 7 + 19 + 24) = 220$ different solutions of 4₁ knot with 46 blocks.

[30003001201100010A0B00332303302] or
 [30003001201100010A0B00333101101] or
 [30003001201100010A0B10332303302] or
 [30003001201100010A0B10333101101].

Here A has 4 choices [031011], [110130], [113311] and [121121]. For each fixed A, B has 5 or 7 or 19 or 24 solutions based on the sequences above. The length of the subsequence B is 11.

For the first sequence, the 5 solutions for B are [0,1,1,2,1,0,1,0,0,2,3], [1,3,3,2,3,0,3,0,1,3]

[3,0,1,3,0,0,1,0,0,2,3], [3,0,2,1,0,0,3,3,0,2,3], [3,3,3,1,1,0,1,0,0,2,3].

For the second sequence, the 7 solutions for B are [0,1,1,2,1,0,1,0,0,2,3], [1,3,3,2,3,0,3,0,1,3]

[2,0,2,3,0,0,1,1,0,1,3], [2,0,3,1,0,0,3,0,0,1,3]
 [3,0,1,3,0,0,1,0,0,2,3], [3,0,2,1,0,0,3,3,0,2,3]
 [3,3,3,1,1,0,1,0,0,2,3].

For the third sequence, the 19 solutions for B are [0,1,0,3,0,3,1,0,1,1,0], [0,1,0,3,1,1,0,1,3,0,0]

[0,1,0,3,1,1,3,3,1,1,0], [0,1,0,3,1,2,1,1,2,1,0]
 [0,1,1,1,3,0,0,1,3,0,0], [0,1,1,1,3,0,3,3,1,1,0]
 [0,1,1,1,3,1,1,1,2,1,0], [0,1,1,2,1,0,0,3,2,0,0]
 [0,1,1,2,1,3,0,1,2,1,0], [1,3,3,2,3,0,3,0,3,3,1]
 [1,3,3,2,3,0,3,1,1,2,2], [3,0,1,3,0,0,0,3,2,0,0]
 [3,0,1,3,0,3,0,1,2,1,0], [3,0,2,1,0,0,0,1,2,0,1]
 [3,3,3,0,3,0,0,1,3,0,0], [3,3,3,0,3,0,3,3,1,1,0]
 [3,3,3,0,3,1,1,1,2,1,0], [3,3,3,1,1,0,0,3,2,0,0], [3,3,3,1,1,3,0,1,2,1,0].

For the fourth sequence, the 24 solutions for B are [0,1,0,3,0,3,1,0,1,1,0], [0,1,0,3,1,1,0,1,3,0,0]

[0,1,0,3,1,1,3,3,1,1,0], [0,1,0,3,1,2,1,1,2,1,0]
 [0,1,1,1,3,0,0,1,3,0,0], [0,1,1,1,3,0,3,3,1,1,0]
 [0,1,1,1,3,1,1,1,2,1,0], [0,1,1,2,1,0,0,3,2,0,0]
 [0,1,1,2,1,3,0,1,2,1,0], [1,3,3,2,3,0,3,0,3,3,1]
 [1,3,3,2,3,0,3,1,1,2,2], [2,0,2,3,0,0,0,3,1,0,1]
 [2,0,2,3,0,0,1,1,3,3,1], [2,0,2,3,0,0,1,2,1,1,2]
 [2,0,3,1,0,0,3,0,3,3,1], [2,0,3,1,0,0,3,1,1,2]
 [3,0,1,3,0,0,0,3,2,0,0], [3,0,1,3,0,3,0,1,2,1,0]
 [3,0,2,1,0,0,0,1,2,0,1], [3,3,3,0,3,0,0,1,3,0,0]
 [3,3,3,0,3,0,3,3,1,1,0], [3,3,3,0,3,1,1,1,2,1,0]
 [3,3,3,1,1,0,0,3,2,0,0], [3,3,3,1,1,3,0,1,2,1,0].

To better explain our construction, we plot the key pattern [1033230330230003001201100010] of our 4₁ construction in Figure 3. It is one of the 4 sequences involving A and B above (with “AOB” removed and with a rotation). The other 3 sequences are similar. Note that the knot 4₁ has a name “Figure 8 knot” and we indeed let the Rubik's Snake trace a shape like the number “8” if we change our view angle by 90 degrees and focus on the first 22 blocks. Although we do not have a rigorous proof, we believe this pattern will help us to get the shortest path because it does not appear to have room for improvement. After this pattern is determined, the rest is to start with the 29th block and go through the hole on the left surrounded by the 5th, 8th and 14th block then from the back connect to the first block in the end. We could easily find a list of shortest paths from the 29th block to the hole using the computer. The solutions are our list of 4 choices for sequence A. Then a “0” is followed because there is no other choice to avoid collision. After that, it is simply a local search by the computer to determine the sequence B.

Now consider the knot 5₁. The shortest we found has 52 blocks shown in Figure 5:

[0,1,1,1,3,3,0,2,1,1,2,1,0,1,2,1,1,3,0,0,1,0,0,2,3,0,0,0,3,0,1,2,1,0,1,0,1,2,1,0,0,3,0,0,1,3,0,0,0,3,2,0]. The path is based on the key pattern in the following 3₁ with an extra room in a hole to allow crossing another time: [1,1,0,1,2,1,1,3,0,0,1,0,0,2,3,0,0,0,3,0,1,2,1,0,1,0,1,2,1]

,0,3,0,0,1,2,0]. Note that the shortest 3_1 we found has 34 blocks but does not have this extra room. This sequence with 36 blocks should be the shortest with such extra room needed for 5_1 . This can also be understood from Figure 4. If we ignore part of the red and green lines from the lower left and let red connect with green directly, that removes two crossings and the obvious self-crossing and 5_1 is reduced to 3_1 .

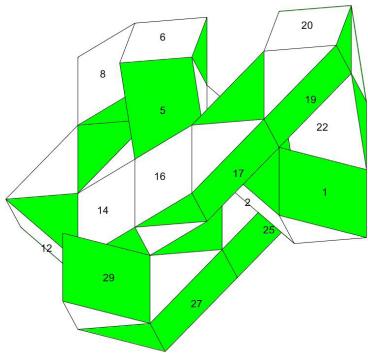


Figure 3 Key pattern of our construction for 4_1 knot.

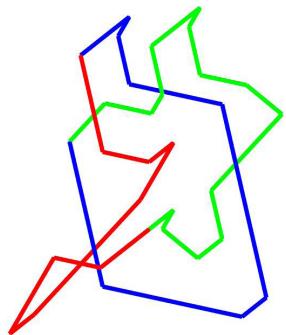


Figure 4 The line representation of A Rubik's Snake 5_1 knot with 52 blocks.

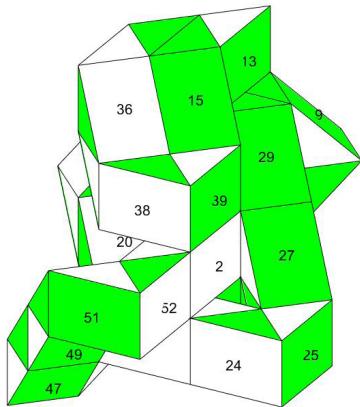


Figure 5 A Rubik's Snake 5_1 knot with 52 blocks.

In Figure 4, we can ignore the obvious removable crossing on the left and the reduced diagram has 5 crossings corresponding to an alternating knot. A DT code can be derived as [6, 8, 10, 2, 4] which indeed corresponds to 5_1 knot.

Now consider the knot 5_2 . The shortest we found has 56 blocks shown in Figure 7:

[0,0,1,3,0,3,0,0,0,1,2,1,0,3,0,0,0,3,2,0,3,3,0,0,1,2,1,2,1,0,1,1,0,2,1,0,0,3,0,0,3,2,0,0,1,0,1,1,0,2,1,0,0,0,0,3].

The path is based on the key pattern in the following 3_1 with an extra room in a hole to allow crossing another time:

[1,2,1,0,1,0,1,2,1,0,3,0,0,0,3,2,0,3,3,0,0,1,2,1,0,1,1,0,2,1,0,0,3,0]. Note that the shortest 3_1 we found has 34 blocks but does not have this extra room. This sequence with 36 blocks should be the shortest with such extra room needed for 5_2 . This can also be understood from Figure 6. If we ignore the red line part and connect directly, it takes one crossing instead of three crossings for the red line part, saving two crossings. 5_2 is reduced to 3_1 .

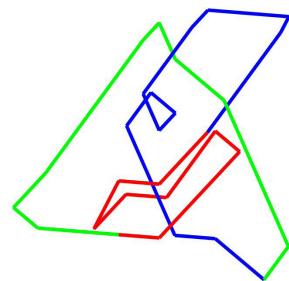


Figure 6 The line representation of A Rubik's Snake 5_2 knot with 56 blocks.

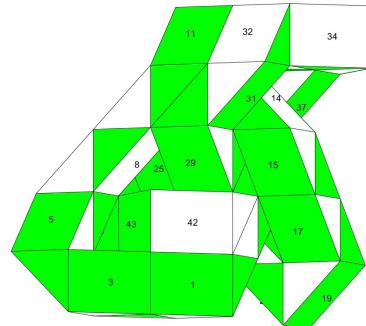


Figure 7 A Rubik's Snake 5_2 knot with 56 blocks.

In Figure 6, we can ignore the obvious removable crossing in the middle and the reduced diagram has 5 crossings corresponding to an alternating knot. A DT code can be derived as [4, 8, 10, 2, 6] which indeed corresponds to 5_2 knot.

The fact that our 5_2 path with 56 blocks is longer than our 5_1 path with 52 blocks can be explained as follows. In Figure 4, to modify a 36-block 3_1 with an extra hole, one only needs to go one more round and go through the extra hole. In Figure 6, to modify a 36-block 3_1 with an extra hole, the additional red path has to push the green path away. That is to say, in addition to the cost of going one more round and going through the extra hole, there is some extra cost for the shortest path due to the structure of the 5_2 knot.

Just like the knots 3_1 and 4_1 , we found many other solutions for 5_1 and 5_2 that tied for the shortest as well. The idea is that we could keep some key patterns and let the computer do a local exhaustive search for less than 12 blocks quickly.

To ensure there is no easy local sequence substitution to improve the results, for the 4_1 , 5_1 , and 5_2 shown in figures, we exhausted all local substitutions by replacing a subsequence of length 10 and did not come up with any solution shorter than our results.

Conclusion

Finding the shortest path for a Rubik's Snake nontrivial knot is challenging. This problem can be viewed as a model for protein

folding. With each joint having 4 choices, the total number of attempts is too many to have an exhaustive search. By using certain key patterns combined with local searches, we found possible shortest 4_1 knot with 46 blocks, 5_1 knot with 52 blocks and 5_2 knot with 56 blocks. All the results are verified by DT codes to make sure they are indeed the knots we claimed they are. We verified that these results cannot be easily improved by substituting a subsequence of length 10 with a shorter subsequence. It would be interesting future work to see if there is a way to verify our results are indeed the shortest.

Acknowledgments

We thank Wen Li for some interesting discussions about the path for 4_1 knot. S. Hou's research is partially supported by the Walter Koss Endowed Professorship. The title of the professorship is made available through the State of Louisiana Board of Regents Support Funds.

Conflicts of interest

The authors declare there are no conflicts of interest.

Funding

None.

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