

Shortest paths of trefoil knot designs using Rubik's Snakes

Abstract

A Rubik's Snake is a toy that has been around for more than 40 years. It can be twisted to many interesting shapes. In particular a Rubik's Snake can be twisted to form a knot. The trefoil knot is the simplest non-trivial knot. In this paper we study how many blocks are needed to form a trefoil knot for the original version of the Rubik's Snake with integer multiple of 90 degree rotations for joints. We also study how many blocks are needed to form a trefoil knot using the tube version of the Rubik's Snake.

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Introduction

The Rubik's Snake is a toy¹ that was invented by Prof. Rubik in 1981.^{2,3} It consists of right isosceles triangular prisms. We call them blocks. They are connected to two other blocks at the centers of the square faces except for the first and last block.

The Rubik's Snake is not just a toy. It has been used as a tool for the study of protein folding^{4,5} and for the construction of reconfigurable modular robots.⁶⁻⁸ There are more applications of robots in^{9,10} Some ideas in the study of Rubik's Snake such as the use of rotation matrix is also used in rigid Origami folding.^{11,12} In previous papers that the first author collaborated with others, strategies have been given for the design of a Rubik's Snake,¹³ and some mathematical problems concerning a Rubik's Snake have been studied.¹⁴ Rotations that are not integer multiple of 90 degrees are mentioned in¹³ but not much theoretical work is presented. On the other hand,¹⁴ has quite some theoretical work but is only concerned with integer multiple of 90 degree rotations. In¹⁵ Rubik's Snakes with general rotation angles were studied with theoretical work presented. In¹⁶ theorems about palindromic, periodic and Möbius Rubik's Snakes were proved.

Knot theory¹⁷ is an interesting research area in mathematics. However, how could a Rubik's Snake form a knot is not reported in the literature. For a trivial knot, 4 blocks of a Rubik's Snake can form a closed loop, which has the shortest path. In this paper, we study the shortest path for a Rubik's Snake trefoil knot which is the simplest non-trivial knot. The organization is as follows: in Section 2, we consider the traditional Rubik's Snake and present our findings for the shortest path. In Section 3, we consider the tube version of the Rubik's Snake and general rotation angles introduced in¹⁵ and present our result for the shortest path. We conclude in Section 4.

The traditional Rubik's snake

We adapt the same notation as in.¹⁴ Start with a straight Rubik's Snake. In a degree sequence, "1" means rotating a joint by 90 degrees to the right. "2" means rotating a joint by 180 degrees. "3" means rotating a joint by 270 degrees to the right (or 90 degrees to the left) and "0" means keeping straight. A knot is automatically a closed loop. In this case, there is an artificial rotation added in the end so that we have n numbers in a degree sequence representing n blocks. Collision is not allowed among blocks.

The simplest nontrivial knot is the trefoil knot. Although it is too expensive to exhaust all possibilities of the Rubik's Snake to look for the shortest possible trefoil knot using the Rubik's Snake, it is realistic to exhaust Rubik's Snakes with palindromic property. Palindromic Rubik's Snake is studied extensively.¹⁶

The difference between the study in this paper and our previous work¹⁴ is that we have to identify that the snake path is a trefoil knot. The idea is as follows. First, by connecting the center of two square faces for each block, we have a line representation of the Rubik's Snake. See Figure 1. Second, in a projection diagram, our code can detect which line is above which, and therefore can distinguish a trefoil knot from a trivial knot. We will discuss how to handle more complicated knots in our future work.

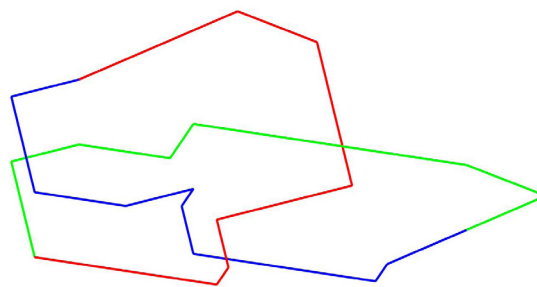


Figure 1 The line representation of a Rubik's snake trefoil (3₁) knot with 34 blocks.

By running our code, we have a complete list of palindromic Rubik's Snake trefoil knots that tied for the shortest with 34 blocks below. Here we do not distinguish left hand and right hand (that is, if 1 and 3 are interchanged in a sequence we do not count as a new sequence) and do not distinguish a rotation of a sequence.

[1, 0, 1, 2, 1, 0, 3, 0, 0, 1, 2, 0, 1, 1, 0, 1, 2,

1, 2, 1, 0, 1, 1, 0, 2, 1, 0, 0, 3, 0, 1, 2, 1, 0]

[1, 0, 1, 2, 1, 0, 3, 0, 0, 1, 3, 0, 3, 0, 3, 1,

1, 1, 3, 0, 3, 3, 0, 3, 1, 0, 0, 3, 0, 1, 2, 1, 0]

[1, 0, 1, 2, 1, 0, 3, 0, 3, 2, 3, 3, 2, 3, 0, 3, 1,

1, 1, 3, 0, 3, 2, 3, 3, 2, 3, 0, 3, 0, 1, 2, 1, 0]
 [1, 0, 1, 2, 1, 0, 3, 0, 3, 3, 0, 2, 3, 0, 0, 1, 2,
 1, 2, 1, 0, 0, 3, 2, 0, 3, 3, 0, 3, 0, 1, 2, 1, 0]
 [1, 0, 1, 2, 1, 0, 3, 0, 3, 3, 0, 3, 1, 0, 0, 3, 1,
 1, 1, 3, 0, 0, 1, 3, 0, 3, 3, 0, 3, 0, 1, 2, 1, 0]
 [1, 0, 1, 2, 1, 0, 3, 0, 3, 3, 1, 1, 3, 3, 0, 3, 1,
 1, 1, 3, 0, 3, 3, 1, 1, 3, 3, 0, 3, 0, 1, 2, 1, 0]

Figure 2 shows the realization of the first sequence. Its line representation is in Figure 1. The key idea is that the computational cost for searching for a palindromic sequence with 34 blocks is only within 4^{17} attempts, not an unrealistic 4^{34} attempts. Similarly searching for period 2 or period 3 solutions are also within reach. We also did an exhaustive search for period 2 and period 3 Rubik's Snake trefoil knots but there is no result with 34 or less blocks.

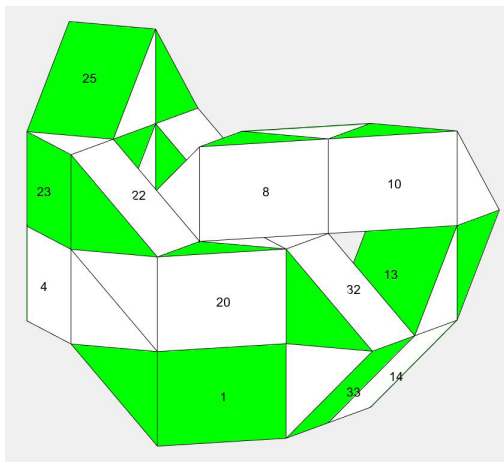


Figure 2 A Rubik's snake trefoil (3₁) knot with 34 blocks.

Based on the above list of sequences, we find that the pattern [0, 3, 0, 1, 2, 1, 0, 1, 0, 1, 2, 1, 0, 3, 0] appears in every single sequence if we change the starting block of the closed loops (i.e., a rotation of the numbers). We plot what this sequence represents in Figure 3. Note that there is a hole which is the key to form a trefoil knot. There is only one way to go through the hole using the Rubik's Snake: a "1" rotation. This also explains why all the 6 sequences have a "1" as their 18th component. Another observation is that the relative location from one end of the hole to one end of other end of this sequence. This motivates us to search for the pattern:

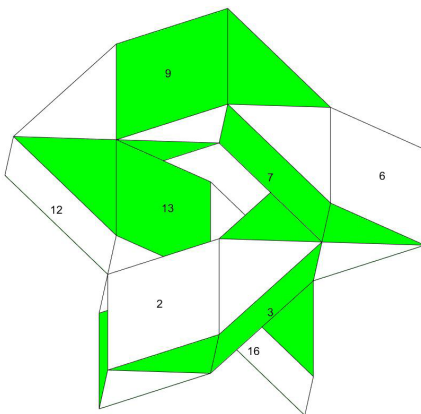


Figure 3 Key local pattern for trefoil (3₁) knot.

$S = [A, 0, 3, 0, 1, 2, 1, 0, 1, 0, 1, 2, 1, 0, 3, 0, \text{reverse}(B), 1]$ where A and B are not necessarily the same but have the same choices. Reverse(B) means the sequence B in reverse order. In the 6 palindromic solutions, A and B are the same with 6 choices:

[1, 3, 0, 0, 1, 3, 0, 3, 3]
 [1, 3, 0, 3, 2, 3, 3, 2, 3]
 [1, 3, 0, 3, 3, 0, 3, 1, 0]
 [1, 3, 0, 3, 3, 1, 1, 3, 3]
 [2, 1, 0, 0, 3, 2, 0, 3, 3]
 [2, 1, 0, 1, 1, 0, 2, 1, 0]

In general, we can therefore generate $6 \times 6 = 36$ solutions allowing A and B chosen from this list but not necessarily the same. We verified that all 36 solutions are valid. Also, we exhausted shorter choices for A and B and verified that there are no shorter solutions. 6 of the 36 solutions are those 6 palindromic solutions found earlier.

A Rubik's Snake Möbius strip is defined¹⁴ and proved to be equivalent to having the sum of degrees in the sequence being $4k + 2$ for integer k. Note that 4 of the 6 choices for A and B have the sum of degrees of the form $4k + 2$ and the other 2 of the 6 choices have the sum of degrees of the form $4k$. The rest of the sequence S has the sum of degrees 16. That implies among the 36 solutions, the number of Möbius strips is $4 \times 2 + 2 \times 4 = 16$.

If there is a shorter trefoil path than 34 blocks, then a local pattern must improve Figure 3. However, this figure appears to be optimal. Therefore, we believe that the shortest path for a trefoil knot has 34 blocks and we have 36 different solutions with 34 blocks (left and right hand are counted as one solution).

Tube version of the Rubik's snake with general rotation angles

As explained¹⁵ there is a unique tube embedded in each Rubik's Snake block with two circles centered at the center of the two square faces and tangent to the rectangular face. Clearly, if these tubes collide, then the traditional Rubik's Snake blocks collide, but not the other way around. It is reasonable to expect the tube version (i.e., a collection of tube blocks) to have shorter path for a trefoil knot.

Note that each joint has infinitely many choices. Even if we restrict to 40 choices for example, 40^n is clearly out of reach for a global search with large n. There is no easy way to search for the optimal solution. Our approach is as follows.

We use the same notation as.¹⁵ For example, "0.75" means rotating the joint by 0.75 times 90 degrees to degrees to the right. "-0.1" means rotating the joint by 0.1 times 90 the left.

First, we define the error to be $\sum_{i=1}^6 \|x_i^{old} - x_i^{new}\|^2$. Here x_i^{old} are the six vertices of the first block of the original version of the Rubik's Snake corresponding to the tube version and x_i^{new} are the six vertices of the artificial block adding to the last block after an artificial rotation, again for the original version of the Rubik's Snake. The norm here is Euclidean norm. Clearly, if a closed loop is formed, this error is supposed to be zero.

Second, it is critical that we have a good initial guess. A good initial guess better already has a trefoil structure without collision. Based on our result that the original Rubik's Snake takes 34 blocks for a trefoil knot, and the fact that the new case is more flexible and should allow

less blocks, we made some attempts with different number of blocks from 20 to 30 and reached the conclusion that 24 is within reach but 23 appears to be hopeless. We admit that we are unable to rigorously prove that 23 is impossible. This would certainly be interesting future research. We believe that 23 blocks is not enough to form a trefoil knot because it is a heavy burden to save an entire block out of 24, and the solution with 24 blocks does not appear to have room for such an improvement.

With trial and error using the above definition of error as a reference we have the following initial guess:

[2, 0, 0.2, 1.9, 0, 0, 3, 0.75, -0.75, 0, -0.8,
2, 0.2, -0.5, 3, 1, 1, -0.3, 2, 1, 0, 0, 1]

The error is 1.32012. This is not small enough. However, because it is reasonably close, we could run a local search focusing on just the first two and last two rotations to make the computational cost within reach. We improved the result to:

[-1.88, -0.09, 0.2, 1.9, 0, 0, 3, 0.75, -0.75, 0, -0.8,
2, 0.2, -0.5, 3, 1, 1, -0.3, 2, 1, 0, 0, 0.20, 1.25]

Now the error is improved to 0.00085. This is accurate enough as could be seen in Figure 4. In real applications with such a small error it is reasonable to accept it as a closed loop. There are 24 identical tube blocks labeled in the figure. Note that this is not the optimal solution for a general rope because our geometry is restricted to have identical tube blocks. The problem we study is not really for a rope with general shape. Also, because one block has significant length and some tubes almost touch each other, it is reasonable to expect that 24 is the optimal solution.

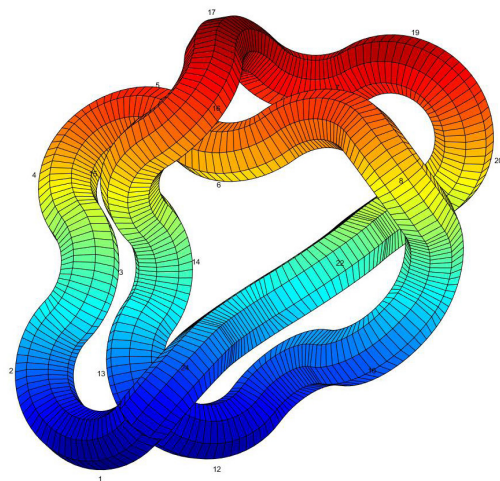


Figure 4 A tube version of Rubik's Snake trefoil (3_1) knot with 24 blocks.

Conclusion

Finding the shortest path for a Rubik's Snake trefoil knot is a challenging problem. With each joint having 4 choices, the total number of attempts is too many to have an exhaustive search. By using the exhaustive search among palindromic Rubik's Snake paths, we found that the shortest trefoil knot has 34 blocks under this constraint. We also extended the results and found 36 different solutions with 34 blocks (left and right hand are counted as the same solution). By using trial and error method combined with a local search, we found a possible shortest tube version of the Rubik's Snake with general rotation angles that has 24 blocks. It would be interesting future research to see if these can be improved at all.

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Conflicts of interest

The author declares that there was no conflict of interest.

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