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Optimal decisions and expected values in two player zero sum games with diagonal game matrixes-explicit functions, general proofs and effects of parameter estimation errors

Abstract

In this paper, the two player zero sum games with diagonal game matrixes, TPZSGD, are analyzed. Many important applications of this particular class of games are found in military decision problems, in customs and immigration strategies and police work. Explicit functions are derived that give the optimal frequences of different decisions and the expected results of relevance to the different decision makers. Arbitrary numbers of decision alternatives are covered. It is proved that the derived optimal decision frequency formulas correspond to the unique optimization results of the two players. It is proved that the optimal solutions, for both players, always lead to a unique completely mixed strategy Nash equilibrium. For each player, the optimal frequency of a particular decision is strictly greater than 0 and strictly less than 1. With comparative statics analyses, the directions of the changes of optimal decision frequences and expected game values as functions of changes in different parameter values, are determined. The signs of the optimal changes of the decision frequences, of the different players, are also determined as functions of risk in different parameter values. Furthermore, the directions of changes of the expected optimal value of the game, are determined as functions of risk in the different parameter values. Finally, some of the derived formulas are used to confirm earlier game theory results presented in the literature. It is demonstrated that the new functions can be applied to solve common military problems.

Keywords: optimal decisions, completely mixed strategy Nash equilibrium, zero sum game theory, stochastic games

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Introduction

What is the optimal strategy of a decision maker, BLUE, such as an individual or organization, when at least one more decision maker, RED, can influence the outcomes? This is a typical question in game theory.

Game theory is a field of research that contains large numbers of studies with different assumptions concerning the number of players, the kinds of decisions that can be taken by the different participants and the degree of information available to the different decision makers at different points in time.

Luce and Raffa[1] give a general description of most of the game theory literature. Some of the highly important and original publications in the field are Nash,[2] von Neumann,[3] and Dresher[4]. Chiang[5] covers two person zero sum games and most other methods and theories of general mathematical economics. Isaacs[6] develops dynamic games with and without stochastic events in continuous and discrete time. In Braun[7] we find a section where differential equations are used to model and describe the development of games of conflict with several examples of real applications.

Lohmander[8] contains a new approach to dynamic games of conflict with two players. It includes a stochastic dynamic programming, SDP, model with a linear programming, LP, or quadratic programming, QP, model as a sub routine. The LP or QP can be used to solve static game problems, such as two person zero sum games, TPZSGs, for each state and stage in the SDP model. The outcomes of the repeated games move the positions in state space (change the states to new states) with different transitions probabilities, in the following periods, within the SDP model. The SDP model solves the complete dynamic and stochastic game over a time horizon with several periods.

During the history of game theory, the TPZSGs have always gained considerable theoretical and practical interest. A detailed treatment is given by Luce and Raffa.[1] Several kinds of TPZSGs with large numbers of military applications are well described by Washburn.[9] This can serve as a good introduction to the analysis in this paper. A Nash equilibrium is the normal outcome of LP solutions to TPZSGs. It is however important to be aware that the Nash equilibrium can not always be expected to be the result in real world games. If the strategies of the players are gradually adjusted based on the observations of the decision frequences of the other players, mixed strategy probability orbits (constrained cycles) may develop. Convergence to the Nash equilibrium can not always be expected. Lohmander[10] has developed a dynamic model and described these possibilities. Herings et al[11] focuses on stationary equilibria in stochastic games. They are interested in model structure, selection and computation. Babu et al[12] give a good historical introduction to the literature on stochastic games. They also develop some new results in the area of equilibrium strategies of dynamic games based on mixed strategy assumptions within static games.

In this paper, a particular class of TPZSGs will be analyzed, namely two player zero sum games with diagonal game matrixes were all diagonal elements are strictly positive. Let us denote them TPZSGDs. This may seem to be a highly particular, constrained and irrelevant class of games. However, this is not true. A large number of obvious and economically very important real world applications of this class of games exist, in particular in military applications, in customs problems and in police work. Lohmander[13] defines, describes and solves four different types of military TPZSGD decision problems with this methodology. These problems include:

a. The selection of roads for transport when enemy forces may prepare attacks along different roads with different expected outcomes,

b. The selection of roads where attacks on enemy transports should be prepared,

c. The positioning of guard squads and

d. The positioning of intelligence, reconaissance and sabotage groups.

Game theory literature usually focuses on very general classes of games, without giving special attention to the TPZSG, and the even more specific TPZSGD, classes.

In this paper, explicit functions of the optimal decision frequences and the expected results of relevance to the different players are derived for situations with arbitrary numbers of decision alternatives.

In the earlier game theory literature, when general classes of games are analyzed, it has usually not been possible to derive explicit functions. Earlier studies are mostly focused on general principles, proofs of the existence of solutions and numerical algorithms to calculate solutions in particular numerically specified situations.

One of the general results derived and proved in this paper is that, for every game in the TPZSGD class, the optimal strategy, for both players, always leads to a unique and completely mixed strategy Nash equilibrium. This means that, for each player, the optimal frequency of every possible decision, is strictly greater than 0 and strictly less than 1.

This result is critical to analytical TPZSGD game theory. It makes it possible to instantly determine the equation system that should be used to calculate the optimal decision frequences. Hence, the optimal decision frequences become possible to analyze with general analytical methods. Explicit functions can be derived for arbitrary numbers of decision options and for all possible elements in the game matrix. In other words, we do not have to handle every particular case with numerical methods.

In the existing literature on game theory, such a proof is not easily found. This problem is usually avoided by intuitive arguments and reasonable assumptions. The book by Washburn[9] is one such example. A similar case is found in Babu et al.[12] They avoid to show that the Nash equilibrium, which they analyze, really is completely mixed. Babu et al[12] simply assume the existence of a particular probability vector. In this paper, the existence of such a probability vector will be proved for a diagonal game matrix where all diagonal elements are strictly positive. It will also be proved that all elements of the probability vector are strictly positive and strictly less than one. Furthermore, explicit functions will be derived for $x, i = 1, 2, ..., n$ and the value of the game.

Thanks to the derived functions, it is also possible to perform explicit sensitivity analyses and to determine the directions of changes of optimal decision frequences and expected results if the direction of change of a particular parameter is known.

In this study, it has been possible to derive explicit results in an area that is highly relevant in real applications: How are the optimal decision frequences of the different players changed if the level of risk of some parameter(s) change(s)? Related results have earlier been derived in stochastic dynamic "one player" problems by Lohmander.[14] First, relevant functions of decisions and expected game values are determined. The first and second derivatives are determined and signed. Then, the Jensen inequality is used to determine the directions of change of the optimal decision frequences and expected game values under the influence of increasing risk in the different parameter values.

Analysis

A TPZSGD will now be analyzed in the most general way. BLUE is the maximizer, who selects the row, i . RED is the minimizer, who selects the column, j . The decision of BLUE is not known by RED before RED takes a decision and the decision of RED is not known by BLUE before BLUE takes the decision. The game matrix, $A(i, j)$, is diagonal. All diagonal elements $c_{ij} = A(i, j)$ are strictly positive and represent the reward that BLUE obtains from RED in case $i = j$. (The reward that BLUE obtains is equal to the loss that RED gets). In case $i \neq j$, the reward is zero. Equations (2.1) and (2.2) define these conditions.

$$c_{ij}\Big|_{i \neq j} = 0, i = 1, ..., n, j = 1, 2, ..., n \qquad (2.1)$$

$$c_{ij}\Big|_{i=j} = g_i > 0 \text{ , } i = 1, ..., n, j = 1, 2, ..., n \qquad (2.2)$$

A concrete example is the following: RED should move an army convoy from one city to another. One road, among the existing n available roads, should be selected. BLUE wants to destroy as many RED trucks as possible. RED sends the convoy via road j and BLUE moves the equipment and troops to road i and prepares an attack there. If $i = j$, BLUE attacks RED and destroys the number of RED trucks found in a diagonal element of the game matrix where $i = j$. $A(i, j) > 0 \; for \; i = j$. If BLUE and RED select different roads, no attack takes place and no trucks are destroyed.

$$A(i, j) > 0 \; for \; i = j \text{ .}$$

Different roads usually have different properties with respect to slope, curvature, protection, options to hide close to the road and so on. As a consequence, the values of the diagonal elements of the game matrix, $A(i, j) > 0 \; for \; i = j$, are usually not the same for different values of i .

The maximization problem of BLUE

The maximization problem of BLUE is defined here. The expected reward, X_0 , is the objective function, which is found in (2.1.1). The number of possible decisions is n and the probability of a particular decision, i , is x_i . The total probability can not exceed 1, which is shown in (2.1.2.). g_i is defined in (2.2). Since RED can select any decision j , x_0 is constrained via (2.1.3). Furthermore, no probability can be negative, which is seen in (2.1.4).

$$\max x_0 \quad (2.1.1)$$

s.t.

$$\sum_{i=1}^{n} x_i \le 1 \quad (2.1.2)$$

$$x_0 \le g_i x_i \,,\, i=1,...,n \quad (2.1.3)$$

$$x_i \ge 0 \,,\, i=1,...,n \quad (2.1.4)$$

Let λ_i denote dual variables. The following Lagrange function is defined:

$$L = x_0 + \lambda_0\left(1-\sum_{i=1}^{n} x_i\right) + \sum_{i=1}^{n} \lambda_i\left(g_i x_i - x_0\right) \quad (2.1.5)$$

The following derivatives will be needed in the proceeding analysis:

$$\frac{dL}{d\lambda_0} = 1-\sum_{i=1}^{n} x_i \ge 0 \quad (2.1.6)$$

$$\frac{dL}{d\lambda_i} = g_i x_i - x_0 \ge 0 \,,\, i=1,...,n \quad (2.1.7)$$

$$\frac{dL}{dx_0} = 1-\sum_{i=1}^{n} \lambda_i \le 0 \quad (2.1.8)$$

$$\frac{dL}{dx_i} = \lambda_i g_i - \lambda_0 \le 0 \,,\, i=1,...,n \quad (2.1.9)$$

Karush Kuhn Tucker conditions in general problems

In general problems, we may have different numbers of decision variables and constraints. Furthermore, the elements $c_{ij}\big|_{i\neq j}$ are not necessarily zero (Table 1).

Table 1 Karush Kuhn Tucker conditions in general maximization problems

$\lambda_i \ge 0\ \forall i$	$\frac{dL}{d\lambda_i} \ge 0\ \forall i$	$\lambda_i \frac{dL}{d\lambda_i} = 0\ \forall i$
$x_j \ge 0\ \forall j$	$\frac{dL}{dx_j} \le 0\ \forall j$	$x_j \frac{dL}{dx_j} = 0\ \forall j$

Particular conditions in problems that satisfy (2.1) and (2.2)

Note that in these problems, $i = j$ in all relevant constraints.

$$\lambda_i \ge 0\ \forall i \quad (2.1.10)$$

$$\frac{dL}{d\lambda_i} \ge 0\ \forall i \quad (2.1.11)$$

$$\lambda_i \frac{dL}{d\lambda_i} = 0\ \forall i \quad (2.1.12)$$

$$x_i \ge 0\ \forall i \quad (2.1.13)$$

$$\frac{dL}{dx_i} \le 0\ \forall i \quad (2.1.14)$$

$$x_i \frac{dL}{dx_i} = 0\ \forall i \quad (2.1.15)$$

Proof 1: Proof that $x_0^* > 0$:

(2.1.2) and (2.1.4) make it feasible to let $x_i > 0 \,,\, i=1,...n$.

(2.2) says that $g_i > 0 \,,\, i=1,2,...,n$.

When $g_i x_i > 0, i=1,...n$, (2.1.3) makes it feasible to let $x_0 > 0$ **.**

(2.1.1) states that we want to maximize x_0 . Let stars indicate optimal values.

Hence, when optimal decisions are taken, $x_0 = x_0^* > 0$ **.**

Proof 2: Proof that $x_i^* > 0 \,,\, i=1,...,n$:

(2.1.7) says that $\frac{dL}{d\lambda_i} = g_i x_i - x_0 \ge 0 \,,\, i=1,...,n$

Proof 1 states that $x_0 > 0$ **.** (2.2) says that $g_i > 0 \,,\, i=1,...,n$.

$x_i \ge \frac{x_0}{g_i} > 0 \,,\, i=1,...,n$.

Hence, $x_i = x_i^* > 0 \,,\, i=0,...,n$.

Proof 3: Proof that $\lambda_i^* \,,\, i=0,...,n$ can be determined from a linear equation system.

$\left(x_i > 0 \,,\, i=0,...,n\right) \wedge$ (2.1.15) $\Rightarrow \left\{ \frac{dL}{dx_0} = 0 \,;\ \frac{dL}{dx_i} = 0 \,,\, i=1,...,n \right\}$
$= \left\{ (2.1.16) \wedge (2.1.17) \right\}$.

$$\frac{dL}{dx_0} = 1-\sum_{i=1}^{n} \lambda_i = 0 \quad (2.1.16)$$

$$\frac{dL}{dx_i} = \lambda_i g_i - \lambda_0 = 0 \,,\, i=1,...,n \quad (2.1.17)$$

Proof 4: Proof that $\lambda_i^* > 0 \,,\, i=0,...,n$.

(2.1.16) $\Rightarrow \exists i\big|_{i>0,\lambda_i>0}$.

Hence, at least for one strictly positive value i, λ_i is strictly greater than zero.

$\left(\exists i\big|_{i>0,\lambda_i>0}\right) \wedge \left(g_i > 0 \,,\, i=1,...,n\right) \wedge$ (2.1.17) $\Rightarrow \lambda_0 > 0$.

$$\lambda_0 > 0 \quad (2.1.18)$$

(2.1.17) $\wedge \left(g_i > 0 \,,\, i=1,...,n\right) \wedge$ (2.1.18) $\Rightarrow \left(\lambda_i > 0 \,,\, i=1,...,n\right)$

$$\lambda_i > 0 \,,\ i=1,...,n \quad (2.1.19)$$

(2.1.18) $\wedge$ (2.1.19) $\Rightarrow \left(\lambda_i > 0 \,,\, i=0,...,n\right)$

$$\lambda_i^* > 0 \,,\, i=0,...,n \quad (2.1.20)$$

Proof 5: Proof that $x_i^* \,,\, i=1,...,n$, can be determined from a linear equation system.

$\left(\lambda_i > 0 \,,\, i=0,...,n\right) \wedge$ (2.1.12) $\Rightarrow$

$\left\{ \frac{dL}{d\lambda_0} = 0 \,;\ \frac{dL}{d\lambda_i} = 0 \,,\, i=1,...,n \right\} = \left\{ (2.1.21) \wedge (2.1.22) \right\}$.

$$\frac{dL}{d\lambda_0} = 1 - \sum_{i=1}^{n} x_i = 0 \quad (2.1.21)$$

$$\frac{dL}{d\lambda_i} = g_i x_i - x_0 = 0 \,,\, i = 1,...,n \quad (2.1.22)$$

Determination of explicit equations that give all values: x_i^*, $i = 0,...,n$:

(2.1.22) $\Rightarrow$ (2.1.23).

$$x_i = \frac{x_0}{g_i} \,,\, i = 1,...,n \quad (2.1.23)$$

(2.1.21) $\Rightarrow$ (2.1.24).

$$\sum_{i=1}^{n} x_i = 1 \quad (2.1.24)$$

$$\sum_{i=1}^{n} \frac{x_0}{g_i} = 1 \quad (2.1.25)$$

$$\sum_{i=1}^{n} \frac{1}{g_i} = \frac{1}{x_0} \quad (2.1.26)$$

$$x_0 = \frac{1}{\sum_{i=1}^{n} \frac{1}{g_i}} \quad (2.1.27)$$

$$x_0^* = \left(\sum_{i=1}^{n} g_i^{-1}\right)^{-1} \quad (2.1.28)$$

$$x_i^* = g_i^{-1}\left(\sum_{q=1}^{n} g_q^{-1}\right)^{-1} \,,\, i = 1,...,n \quad (2.1.29)$$

Determination of explicit equations that give all values: λ_i^*, $i = 0,...,n$:

(2.1.17) $\Rightarrow$ (2.1.30).

$$\lambda_i = \frac{\lambda_0}{g_i} \,,\, i = 1,...,n \quad (2.1.30)$$

(2.1.16) $\Rightarrow$ (2.1.31)

$$\sum_{i=1}^{n} \lambda_i = 1 \quad (2.1.31)$$

$$\sum_{i=1}^{n} \frac{\lambda_0}{g_i} = 1 \quad (2.1.32)$$

$$\sum_{i=1}^{n} \frac{1}{g_i} = \frac{1}{\lambda_0} \quad (2.1.33)$$

$$\lambda_0 = \frac{1}{\sum_{i=1}^{n} \frac{1}{g_i}} \quad (2.1.34)$$

$$\lambda_0^* = \left(\sum_{i=1}^{n} g_i^{-1}\right)^{-1} \quad (2.1.35)$$

$$\lambda_i^* = g_i^{-1}\left(\sum_{q=1}^{n} g_q^{-1}\right)^{-1} \,,\, i = 1,...,n \quad (2.1.36)$$

Observations:

$$x_0^* = \lambda_0^* = \left(\sum_{i=1}^{n} g_i^{-1}\right)^{-1} \quad (2.1.37)$$

$$x_i^* = \lambda_i^* = g_i^{-1}\left(\sum_{q=1}^{n} g_q^{-1}\right)^{-1} \,,\, i = 1,...,n \quad (2.1.38)$$

The minimization problem of RED

We are interested in the solution to $\min y_0$. The objective function is formulated as $\max(-y_0)$. The frequences of the different decisions, i are y_i .

$$\max(-y_0) \quad (2.2.1)$$

s.t.

$$\sum_{i=1}^{n} y_i \geq 1 \quad (2.2.2)$$

$$y_0 \geq g_i y_i \,,\, i = 1,...,n \quad (2.2.3)$$

$$y_i \geq 0 \,,\, i = 1,...,n \quad (2.2.4)$$

Proof that $y_0^* > 0$

(2.2.2) $\Rightarrow$ (2.2.5).

$$\exists i\big|_{1 \leq i \leq n,\, y_i > 0} \quad (2.2.5)$$

$$g_i > 0 \,,\, i = 1,...,n \quad (2.2.6)$$

(2.2.3) $\wedge$ (2.2.5) $\Rightarrow$ (2.2.6) $\Rightarrow$ (2.2.7).

$$y_0^* \geq y_0 > 0 \quad (2.2.7)$$

Let μ_i denote dual variables. The following Lagrange function is defined for RED:

$$L_2 = -y_0 + \mu_0\left(\sum_{i=1}^{n} y_i - 1\right) + \sum_{i=1}^{n} \mu_i\left(y_0 - g_i y_i\right) \quad (2.2.8)$$

These derivatives will be needed in the analysis:

$$\frac{dL_2}{d\mu_0} = \sum_{i=1}^{n} y_i - 1 \geq 0 \quad (2.2.9)$$

$$\frac{dL_2}{d\mu_i} = y_0 - g_i y_i \geq 0 \,,\, i = 1,...,n \quad (2.2.10)$$

$$\frac{dL_2}{dy_0} = -1 + \sum_{i=1}^{n} \mu_i \leq 0 \quad (2.2.11)$$

$$\frac{dL_2}{dy_i} = \mu_0 - \mu_i g_i \leq 0 \,,\, i = 1,...,n \quad (2.2.12)$$

Proof that $y_i^* > 0$, $i = 0,...,n$

According to (2.2.1), we want to maximize $-y_0$, which implies that we minimize y_0 .

(2.2.2) $\Rightarrow$ $\sum_{i=1}^{n} y_i \geq 1$

(2.2.4) $\Rightarrow$ $y_i \geq 0 \,,\, i = 1,...,n$

Let us start from an infeasible point, origo, and move to a feasible point in the way that keeps y_0 as low as possible. Initially, let $(y_1,...,y_n) = (0,...,0)$. According to (2.2.2), this point is not feasible.

(2.2.3) $\Rightarrow$ $\min y_0\big|_{y_i = 0,\, i = 1,...,n} = 0$.

Now, we have to move away from the infeasible point $(y_1,...,y_n)=(0,...,0)$. We have to reach a point that satisfies $\sum_{i=1}^{n} y_i \ge 1$ without increasing y_0 more than necessary. To find a point that satisfies (2.2.2), we have to increase the value of at least one of the $y_i|_{i\in\{1,...,n\}}$. Select one arbitrary index $k|_{1\le k\le n}$. To simplify the exposition, we let $k=1$. According to (2.2.3): If we increase y_1 by dy_1 , $\min y_0$ increases by $g_1 dy_1$, as long as $dy_i = 0$, $i=2,...,n$. Hence, $dy_0 = g_1 dy_1$. Let $z = dy_0 = g_1 dy_1$.

However, when $dy_1 > 0$, we may also partly increase y_i , $i=2,...,n$ without increasing dy_0 above z . This follows from (2.2.3) and (2.2.10). Since we want to satisfy $\sum_{i=1}^{n} y_i \ge 1$, we want to increase y_i , $i=2,...,n$ as much as possible, without increasing dy_0 above z . Hence, we select:

$$g_i dy_i = z = g_1 dy_1 \text{ , } i=2,...,n \tag{2.2.13}$$

$$dy_i = \frac{g_1}{g_i} dy_1 \text{ , } i=2,...,n \tag{2.2.14}$$

$$(dy_1 > 0) \wedge (g_i > 0 \text{ , } i=1,...,n) \Rightarrow dy_i > 0 \text{ , } i=2,...,n \tag{2.2.15}$$

Since we started in origo, we have

$$y_i = dy_i + 0 > 0 \text{ , } i=1,...,n \tag{2.2.16}$$

We already know that $y_0^* \ge y_0 > 0$. Hence,.

$$y_i^* > 0 \text{ , } i=0,...,n \tag{2.2.17}$$

Observation: The following direct method can be used to solve the optimization problem of RED.

First, remember that $y_0^* = dy_0^* + 0 = z$. We may directly determine the optimal values of $y_i^* > 0$, $i=0,...,n$ without using the Lagrange function and KKT conditions, in this way:

$$\sum_{i=1}^{n} y_i = \left((dy_1 + 0) + (dy_2 + 0) ... + (dy_n + 0) \right) = 1 \tag{2.2.18}$$

$$\sum_{i=1}^{n} y_i = (y_1 + y_2 + ... + y_n) = 1 \tag{2.2.19}$$

$$\sum_{i=1}^{n} y_i = \left(\frac{z}{g_1} + \left(\frac{g_1}{g_2}\frac{z}{g_1} \right) + ... + \left(\frac{g_1}{g_n}\frac{z}{g_1} \right) \right) = 1 \tag{2.2.20}$$

$$\sum_{i=1}^{n} y_i = \left(\frac{z}{g_1} + \frac{z}{g_2} + ... + \frac{z}{g_n} \right) = 1 \tag{2.2.21}$$

$$\sum_{i=1}^{n} y_i = \left(\frac{1}{g_1} + \frac{1}{g_2} + ... + \frac{1}{g_n} \right) = \frac{1}{z} \tag{2.2.22}$$

$$\sum_{i=1}^{n} g_i^{-1} = \frac{1}{z} \tag{2.2.23}$$

$$y_0^* = z = \left(\sum_{i=1}^{n} g_i^{-1} \right)^{-1} \tag{2.2.24}$$

$$y_i^* = g_i^{-1} y_0^* = g_i^{-1} \left(\sum_{q=1}^{n} g_q^{-1} \right)^{-1} \text{ , } i=1,...,n \tag{2.2.25}$$

Proof that μ_i^* , $i=0,...,n$ can be solved via a linear equation system and that $\mu_i^* > 0$, $i=0,...,n$.

Since $y_i^* > 0$, $i=0,...,n$, we may determine that $\mu_i^* > 0$, $i=0,...,n$ via a linear equation system.

$$\left(y_i \frac{dL_2}{dy_i} = 0 \text{ , } i=0,...,n \right) \wedge (y_i > 0 \text{ , } i=0,...,n) \Rightarrow \left(\frac{dL_2}{dy_i} = 0 \text{ , } i=0,...,n \right)$$

$$\frac{dL_2}{dy_0} = -1 + \sum_{q=1}^{n} \mu_q = 0 \tag{2.2.26}$$

$$\frac{dL_2}{dy_i} = \mu_0 - \mu_i g_i = 0 \text{ , } i=1,...,n \tag{2.2.27}$$

$$(2.2.26) \Rightarrow \exists i|_{1\le i\le n,\ \mu_i > 0} \tag{2.2.28}$$

$$(g_i > 0 \text{ , } i=1,...,n) \wedge (2.2.27) \wedge (2.2.28) \Rightarrow \mu_0 > 0 \tag{2.2.29}$$

$$(g_i > 0 \text{ , } i=1,...,n) \wedge (2.2.27) \wedge (2.2.29) \Rightarrow (\mu_i > 0 \text{ , } i=1,...,n) \tag{2.2.30}$$

$$(2.2.29) \wedge (2.2.30) \Rightarrow (\mu_i > 0 \text{ , } i=0,...,n) \tag{2.2.31}$$

Proof that y_i^* , $i=0,...,n$ can be solved via a linear equation system and that $y_i^* > 0$, $i=0,...,n$.

Since $\mu_i^* > 0$, $i=0,...,n$, we may determine that $y_i^* > 0$, $i=0,...,n$ via a linear equation system.

$$\left(\mu_i \frac{dL_2}{d\mu_i} = 0 \text{ , } i=0,...,n \right) \wedge (\mu_i > 0 \text{ , } i=0,...,n) \Rightarrow \left(\frac{dL_2}{d\mu_i} = 0 \text{ , } i=0,...,n \right)$$

$$\frac{dL_2}{d\mu_0} = \sum_{q=1}^{n} y_q - 1 = 0 \tag{2.2.32}$$

$$\frac{dL_2}{d\mu_i} = y_0 - g_i y_i = 0 \text{ , } i=1,...,n \tag{2.2.33}$$

$$(2.2.32) \Rightarrow \exists i|_{1\le i\le n,\ y_i > 0} \tag{2.2.34}$$

$$(g_i > 0 \text{ , } i=1,...,n) \wedge (2.2.33) \Rightarrow y_0 > 0 \tag{2.2.35}$$

$$(g_i > 0 \text{ , } i=1,...,n) \wedge (2.2.35) \Rightarrow (y_i > 0 \text{ , } i=1,...,n) \tag{2.2.36}$$

$$(2.2.35) \wedge (2.2.36) \Rightarrow (y_i > 0 \text{ , } i=0,...,n) \tag{2.2.37}$$

Determination of explicit equations that give all values: y_i^* , $i=0,...,n$:

(2.2.33) $\Rightarrow$ (2.2.38).

$$y_i = \frac{y_0}{g_i} \text{ , } i=1,...,n \tag{2.2.38}$$

(2.2.32) $\Rightarrow$ (2.2.39).

$$\sum_{i=1}^{n} y_i = 1 \tag{2.2.39}$$

$$\sum_{i=1}^{n} \frac{y_0}{g_i} = 1 \tag{2.2.40}$$

$$\sum_{i=1}^{n} \frac{1}{g_i} = \frac{1}{y_0} \tag{2.2.41}$$

$$y_0 = \frac{1}{\sum_{i=1}^{n} \frac{1}{g_i}} \quad (2.2.42)$$

$$y_0^* = \left(\sum_{i=1}^{n} g_i^{-1} \right)^{-1} \quad (2.2.43)$$

$$y_i^* = g_i^{-1} \left(\sum_{q=1}^{n} g_q^{-1} \right)^{-1}, \ i = 1,...,n \quad (2.2.44)$$

Determination of explicit equations that give all values: $\mu_i^*,\ i = 0,...,n$:

(2.2.27) $\Rightarrow$ (2.2.45).

$$\mu_i = \frac{\mu_0}{g_i}, \ i = 1,...,n \quad (2.2.45)$$

(2.2.26) $\Rightarrow$ (2.2.46)

$$\sum_{i=1}^{n} \mu_i = 1 \quad (2.2.46)$$

$$\sum_{i=1}^{n} \frac{\mu_0}{g_i} = 1 \quad (2.2.47)$$

$$\sum_{i=1}^{n} \frac{1}{g_i} = \frac{1}{\mu_0} \quad (2.2.48)$$

$$\mu_0 = \frac{1}{\sum_{i=1}^{n} \frac{1}{g_i}} \quad (2.2.49)$$

$$\mu_0^* = \left(\sum_{i=1}^{n} g_i^{-1} \right)^{-1} \quad (2.2.50)$$

$$\mu_i^* = g_i^{-1} \left(\sum_{q=1}^{n} g_q^{-1} \right)^{-1}, \ i = 1,...,n \quad (2.2.51)$$

Observations:

$$y_0^* = \mu_0^* = \left(\sum_{i=1}^{n} g_i^{-1} \right)^{-1} \quad (2.2.52)$$

$$y_i^* = \mu_i^* = g_i^{-1} \left(\sum_{q=1}^{n} g_q^{-1} \right)^{-1}, \ i = 1,...,n \quad (2.2.53)$$

Generalized Observations:

$$x_0^* = \lambda_0^* = y_0^* = \mu_0^* = \left(\sum_{i=1}^{n} g_i^{-1} \right)^{-1} \quad (2.2.54)$$

$$x_i^* = \lambda_i^* = y_i^* = \mu_i^* = g_i^{-1} \left(\sum_{q=1}^{n} g_q^{-1} \right)^{-1}, \ i = 1,...,n \quad (2.2.55)$$

Observations:

$$y_0^* = \mu_0^* = \left(\sum_{i=1}^{n} g_i^{-1} \right)^{-1} \quad (2.2.52)$$

$$y_i^* = \mu_i^* = g_i^{-1} \left(\sum_{q=1}^{n} g_q^{-1} \right)^{-1}, \ i = 1,...,n \quad (2.2.53)$$

Generalized Observations:

$$x_0^* = \lambda_0^* = y_0^* = \mu_0^* = \left(\sum_{i=1}^{n} g_i^{-1} \right)^{-1} \quad (2.2.54)$$

$$x_i^* = \lambda_i^* = y_i^* = \mu_i^* = g_i^{-1} \left(\sum_{q=1}^{n} g_q^{-1} \right)^{-1}, \ i = 1,...,n \quad (2.2.55)$$

Sensitivity analyses

First, the sensitivity analyses will concern these variables: $x_0^* = \lambda_0^* = y_0^* = \mu_0^*$. How do these variables change under the influence of changing elements in the game matrix?

Observation: $x_0^* = \lambda_0^* = y_0^* = \mu_0^* = \left(\sum_{i=1}^{n} g_i^{-1} \right)^{-1}$

Proof that $\frac{dx_0^*}{dg_i} > 0 \wedge \frac{d^2 x_0^*}{dg_i^2} < 0$.

$$x_0^* = \left(\sum_{i=1}^{n} g_i^{-1} \right)^{-1} \quad (2.3.1)$$

$$\frac{dx_0^*}{dg_i} = (-1) \left(\sum_{i=1}^{n} g_i^{-1} \right)^{-2} \left(-g_i^{-2} \right) \quad (2.3.2)$$

$$\frac{dx_0^*}{dg_i} = g_i^{-2} \left(\sum_{i=1}^{n} g_i^{-1} \right)^{-2} > 0 \quad (2.3.3)$$

$$\frac{d^2 x_0^*}{dg_i^2} = -2 g_i^{-3} \left(\sum_{i=1}^{n} g_i^{-1} \right)^{-2} + g_i^{-2} (-2) \left(\sum_{i=1}^{n} g_i^{-1} \right)^{-3} (-1) g_i^{-2} \quad (2.3.4)$$

$$\frac{d^2 x_0^*}{dg_i^2} = -2 g_i^{-3} \left(\sum_{i=1}^{n} g_i^{-1} \right)^{-2} \left(1 - g_i^{-1} \left(\sum_{i=1}^{n} g_i^{-1} \right)^{-1} \right) \quad (2.3.5)$$

$$\frac{d^2 x_0^*}{dg_i^2} = -2 g_i^{-1} \left(x_i^* \right)^2 \left(1 - x_i^* \right) \quad (2.3.6)$$

$$\left(0 < x_i^* < 1 \right) \wedge \left(g_i > 0 \right) \Rightarrow \frac{d^2 x_0^*}{dg_i^2} < 0 \quad (2.3.7)$$

Observation: x_0^* is a strictly increasing and strictly concave function of each g_i. From the Jensen inequality, it follows that increasing risk in g_i will reduce the expected value of x_0^*. Compare Figure 1.

Second, the sensitivity analyses will concern these variables: $x_i^* = \lambda_i^* = y_i^* = \mu_i^*,\ i = 1,...,n$. How do these variables change under the influence of changing elements in the game matrix?

Observation: $x_i^* = \lambda_i^* = y_i^* = \mu_i^* = g_i^{-1} \left(\sum_{q=1}^{n} g_q^{-1} \right)^{-1}, \ i = 1,...,n$

Proof that $\frac{dx_i^*}{dg_i} < 0 \wedge \frac{d^2 x_i^*}{dg_i^2} > 0,\ i \in \{1,...,n\}$.

$$x_i^* = g_i^{-1} \left(\sum_{q=1}^{n} g_q^{-1} \right)^{-1}, \ i = 1,...,n \quad (2.3.8)$$

$$\frac{dx_i^*}{dg_i} = -g_i^{-2} \left(\sum_{q=1}^{n} g_q^{-1} \right)^{-1} + g_i^{-1} (-1) \left(\sum_{q=1}^{n} g_q^{-1} \right)^{-2} \left(-g_q^{-2} \right) \quad (2.3.9)$$

$$\frac{dx_i^*}{dg_i} = g_i^{-2} \left(\sum_{q=1}^{n} g_q^{-1} \right)^{-1} \left(-1 + g_i^{-1} \left(\sum_{q=1}^{n} g_q^{-1} \right)^{-1} \right) \quad (2.3.10)$$

$$\frac{dx_i^*}{dg_i} = g_i^{-1} x_i^* \left(-1 + x_i^* \right) \quad (2.3.11)$$

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$$\left(g_i > 0\right) \wedge \left(0 < x_i^* < 1\right) \Rightarrow \frac{dx_i^*}{dg_i} < 0 \quad (2.3.12)$$

$$\frac{d^2 x_i^*}{dg_i^2} = -g_i^{-2} x_i^* \left(x_i^* - 1\right) + g_i^{-1}\left(g_i^{-1} x_i^* \left(x_i^* - 1\right)\right)\left(x_i^* - 1\right) + g_i^{-1} x_i^* g_i^{-1} x_i^* \left(x_i^* - 1\right) \quad (2.3.13)$$

$$\frac{d^2 x_i^*}{dg_i^2} = -g_i^{-2}\left(x_i^*\left(x_i^* - 1\right) - \left(x_i^*\left(x_i^* - 1\right)\right)\left(x_i^* - 1\right) - x_i^* x_i^* \left(x_i^* - 1\right)\right) \quad (2.3.14)$$

$$\frac{d^2 x_i^*}{dg_i^2} = -g_i^{-2}\left(\left(x_i^*\right)^2 - x_i^* - x_i^*\left(\left(x_i^*\right)^2 - 2x_i^* + 1\right) - \left(x_i^*\right)^2\left(x_i^* - 1\right)\right) \quad (2.3.15)$$

$$\frac{d^2 x_i^*}{dg_i^2} = -g_i^{-2}\left(\left(x_i^*\right)^2 - x_i^* - \left(x_i^*\right)^3 + 2\left(x_i^*\right)^2 - x_i^* - \left(x_i^*\right)^3 + \left(x_i^*\right)^2\right) \quad (2.3.16)$$

$$\frac{d^2 x_i^*}{dg_i^2} = -g_i^{-2}\left(-2\left(x_i^*\right)^3 + 4\left(x_i^*\right)^2 - 2x_i^*\right) \quad (2.3.17)$$

$$\frac{d^2 x_i^*}{dg_i^2} = 2g_i^{-2} x_i^* \left(\left(x_i^*\right)^2 - 2x_i^* + 1\right) \quad (2.3.18)$$

$$\frac{d^2 x_i^*}{dg_i^2} = 2g_i^{-2} x_i^* \left(x_i^* - 1\right)^2 \quad (2.3.19)$$

$$\left(g_i \neq 0\right) \wedge \left(0 < x_i^* < 1\right) \Rightarrow \frac{d^2 x_i^*}{dg_i^2} > 0 \quad (2.3.20)$$

Observation: x_i^* is a strictly decreasing and strictly convex function of g_i. From the Jensen inequality, it follows that increasing risk in g_i will increase the expected value of x_i^*. Compare Figure 2.

Proof that $\frac{dx_k^*}{dg_i} > 0 \wedge \frac{d^2 x_k^*}{dg_i^2} < 0$, $i \in \{1,...,n\}, k \in \{1,...,n\}, i \neq k$.

$$x_k^* = g_k^{-1}\left(\sum_{i=1}^{n} g_i^{-1}\right)^{-1} \quad (2.3.21)$$

$$\frac{dx_k^*}{dg_{i|i\neq k}} = g_k^{-1}(-1)\left(\sum_{i=1}^{n} g_i^{-1}\right)^{-2}\left(-g_i^{-2}\right) \quad (2.3.22)$$

$$\frac{dx_k^*}{dg_{i|i\neq k}} = g_k^{-1} g_i^{-2}\left(\sum_{i=1}^{n} g_i^{-1}\right)^{-2} \quad (2.3.23)$$

$$\left(g_m > 0, m = 1...,n\right) \Rightarrow \frac{dx_k^*}{dg_{i|i\neq k}} > 0 \quad (2.3.24)$$

$$\frac{d^2 x_k^*}{dg_{i|i\neq k}^2} = g_k^{-1}\left(-2g_i^{-3}\left(\sum_{i=1}^{n} g_i^{-1}\right)^{-2} + g_i^{-2}(-2)\left(\sum_{i=1}^{n} g_i^{-1}\right)^{-3}\left(-g_i^{-2}\right)\right) \quad (2.3.25)$$

$$\frac{d^2 x_k^*}{dg_{i|i\neq k}^2} = 2g_k^{-1} g_i^{-3}\left(\sum_{i=1}^{n} g_i^{-1}\right)^{-2}\left(\left(g_i^{-1}\right)\left(\sum_{i=1}^{n} g_i^{-1}\right)^{-1} - 1\right) \quad (2.3.26)$$

$$\frac{d^2 x_k^*}{dg_{i|i\neq k}^2} = 2g_k^{-1} g_i^{-1}\left(x_i^*\right)^2\left(x_i^* - 1\right) \quad (2.3.27)$$

$$\left(g_m > 0, m = 1,...,n\right) \wedge \left(0 < x_i^* < 1\right) \Rightarrow \frac{d^2 x_k^*}{dg_{i|i\neq k}^2} < 0 \quad (2.3.28)$$

Observation: x_k^* is a strictly increasing and strictly concave function of g_i. From the Jensen inequality, it follows that increasing risk in g_i will decrease the expected value of x_k^*. Compare Figure 3.

Numerical illustration

The general definition of the following illustrating game is given in the preceeding section. Let n =2. A very detailed background and interpretation of this particular game, without the new functions and proofs, is given in Lohmander (2019).[14]

$$A = \begin{bmatrix} g_1 & 0 \\ 0 & g_2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \quad (3.1)$$

From (2.2.54) we know that:

$$x_0^* = \lambda_0^* = y_0^* = \mu_0^* = \left(\sum_{i=1}^{n} g_i^{-1}\right)^{-1} \quad (3.2)$$

x_0^*, the expected reward of BLUE, is equal to y_0^***,** the expected loss of RED, in case both optimize the respective strategies. Using the numerical values of the elements in A, we get:

$$x_0^* = \frac{1}{\frac{1}{2} + \frac{1}{3}} = \frac{6}{5} = 1.2 \quad (3.3)$$

Hence, the expected value of the game is 1.2. This value is also shown in Figure 4. and Figure 5. The expected value of the game is a decreasing function of the level of risk of g_1, which is described in connection to, and illustrated in, Figure 1.

From (2.2.55) we know that:

$$x_i^* = \lambda_i^* = y_i^* = \mu_i^* = g_i^{-1}\left(\sum_{q=1}^{n} g_q^{-1}\right)^{-1}, i = 1,...,n \quad (3.4)$$

For BLUE and RED, the optimal probabilities to select different roads are equal. For BLUE, the optimal probability to select road 1 is x_1^*. Via the elements in A, we get:

$$x_1^* = y_1^* = \left(\frac{1}{2}\right) x_0^* = 0.6 \quad (3.5)$$

$$x_2^* = y_2^* = \left(\frac{1}{3}\right) x_0^* = 0.4 \quad (3.6)$$

x_1^* is shown in Figures 6 & 7. In Figure 8, the optimal value is illustrated. The expected value of x_1^* is an increasing function of the level of risk in g_1, which is shown in Figure 2. For BLUE, the optimal probability to select road 2, is x_2^*. In Figure 9, we find this value is 0.4. Figure 3 illustrates that the expected value of x_2^* is a decreasing function of the level of risk in g_1.

The particular results $\left(x_0^*, x_1^*, x_2^*\right)$ discussed in this in this section were also obtained by Lohmander (2019)[14] via the traditional game theory approach of linear programming.

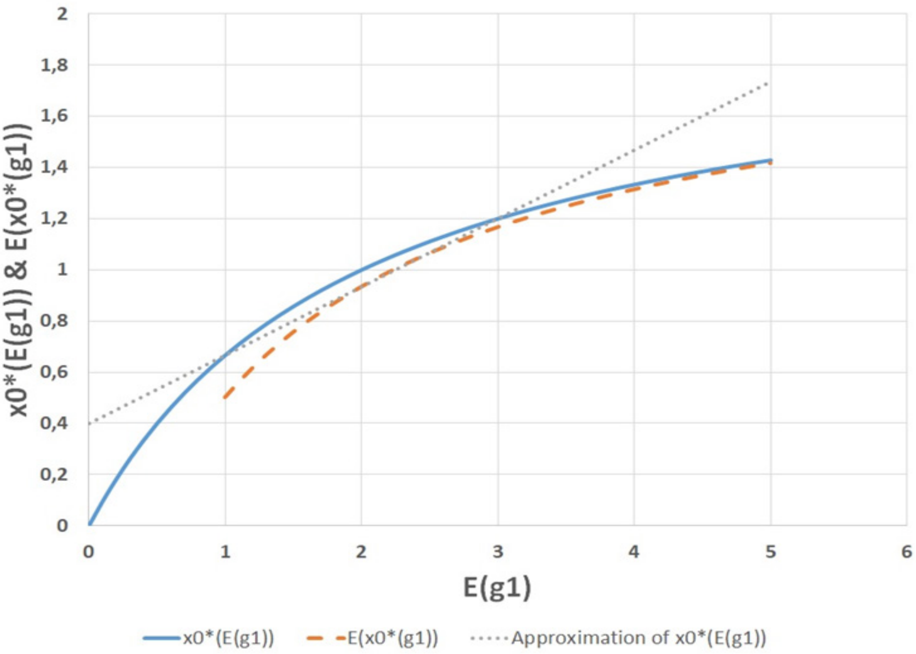

Figure 1 In this graph, the horizontal axes represents $E(g_1)$, the expected value of g_1. Here, g_1 is a stochastic variable. There are two possible outcomes, namely $E(g_1)-1$ and $E(g_1)+1$, with probabilities ½ and ½ respectively. The vertical axes shows $x_0^*(E(g_1))$, the optimal objective function value as a function of the expected value of g_1, and $E\left(x_0^*(g_1)\right)$, the expected value of the optimal objective function value of x_0^* as a function of the value of g_1. The graph also includes a linear approximation of $x_0^*(E(g_1))$ based on the values of $x_0^*(E(g_1))$ for $E(g_1)=1$ and for $E(g_1)=3$. This linear approximation is equal to $E\left(x_0^*(g_1)\right)$ for $E(g_1)=2$. According to the Jensen inequality, $E\left(x_0^*(g_1)\right) < x_0^*(E(g_1))$, when $x_0^*(E(g_1))$ is a strictly concave function and g_1 is a stochastic variable. This graph illustrates that the Jensen inequality is correct. The graph also illustrates the general conclusion that the expected optimal objective function value $E\left(x_0^*(g_1)\right)$ is a strictly decreasing function of the level of risk in g_1.

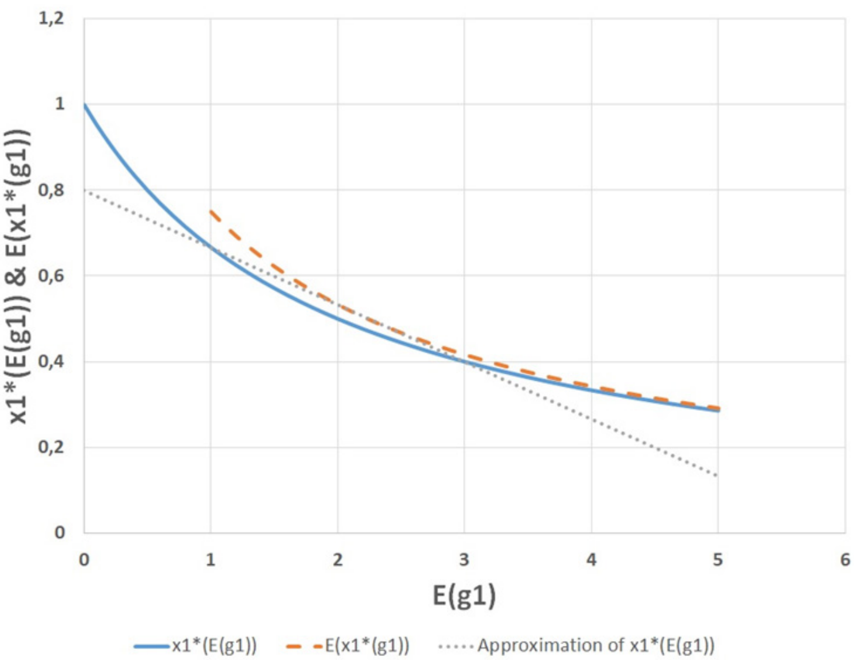

Figure 2 In this graph, the horizontal axes represents $E(g_1)$, the expected value of g_1. Here, g_1 is a stochastic variable. There are two possible outcomes, namely $E(g_1)-1$ and $E(g_1)+1$, with probabilities ½ and ½ respectively. The vertical axes shows the optimal decision frequency $x_1^*(E(g_1))$ as a function of the expected value of g_1, and $E\left(x_1^*(g_1)\right)$, the expected value of the optimal frequency X_1^* as a function of the value of g_1. The graph also includes a linear approximation of $x_1^*(E(g_1))$ based on the values of $x_1^*(E(g_1))$ for $E(g_1)=1$ and for $E(g_1)=3$. This linear approximation is equal to $E\left(x_1^*(g_1)\right)$ for $E(g_1)=2$. According to the Jensen inequality, $E\left(x_1^*(g_1)\right) > x_1^*(E(g_1))$, when $x_1^*(E(g_1))$ is a strictly convex function and g_1 is a stochastic variable. This graph illustrates that the Jensen inequality is correct. The graph also illustrates the general conclusion that the expected optimal decision frequency $E\left(x_1^*(g_1)\right)$ is a strictly increasing function of the level of risk in g_1.

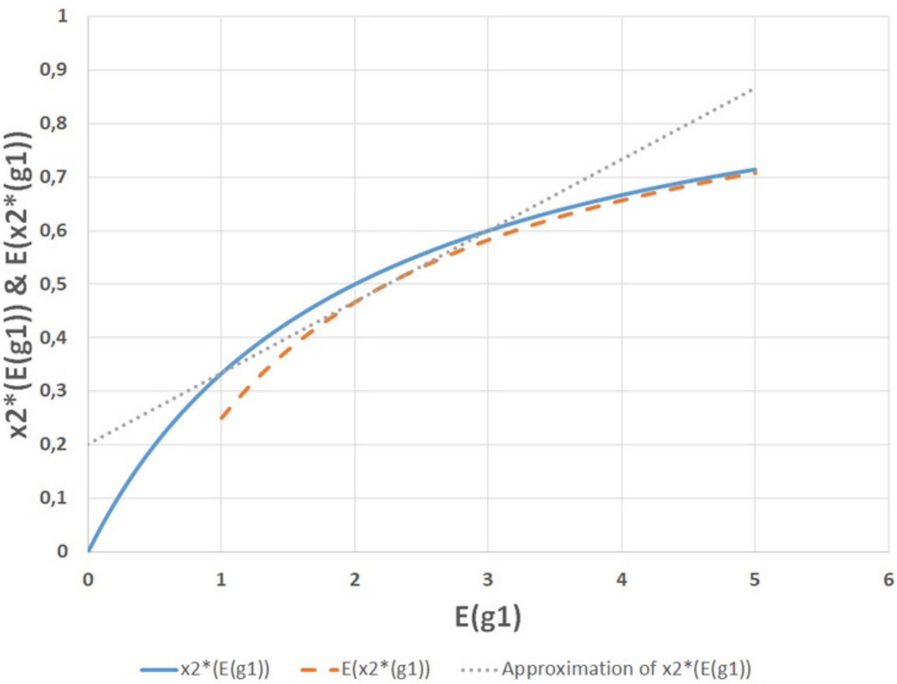

Figure 3 In this graph, the horizontal axes represents $E(g_1)$, the expected value of g_1. Here, g_1 is a stochastic variable. There are two possible outcomes, namely $E(g_1)-1$ and $E(g_1)+1$, with probabilities ½ and ½ respectively. The vertical axes shows the optimal decision frequency $x_2^*\left(E(g_1)\right)$ as a function of the expected value of g_1, and $E\left(x_2^*(g_1)\right)$, the expected value of the optimal frequency x_2^* as a function of the value of g_1. The graph also includes a linear approximation of $x_2^*\left(E(g_1)\right)$ based on the values of $x_2^*(E(g_1))$ for $E(g_1)=1$ and for $E(g_1)=3$. This linear approximation is equal to $E\left(x_2^*(g_1)\right)$ for $E(g_1)=2$. According to the Jensen inequality, $E\left(x_2^*(g_1)\right) < x_2^*\left(E(g_1)\right)$, when $x_2^*(E(g_1))$ is a strictly concave function and g_1 is a stochastic variable. This graph illustrates that the Jensen inequality is correct. The graph also illustrates the general conclusion that the expected optimal decision frequency $E\left(x_2^*(g_1)\right)$ is a strictly decreasing function of the level of risk in g_1.

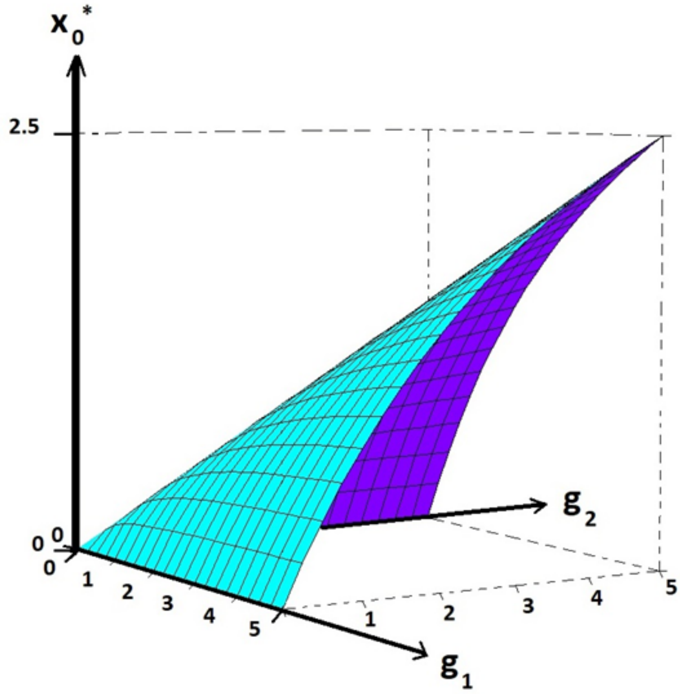

Figure 4 The objective function value x_0^* as a function of the two parameters (g_1, g_2). x_0^* is a strictly increasing function of both parameters.

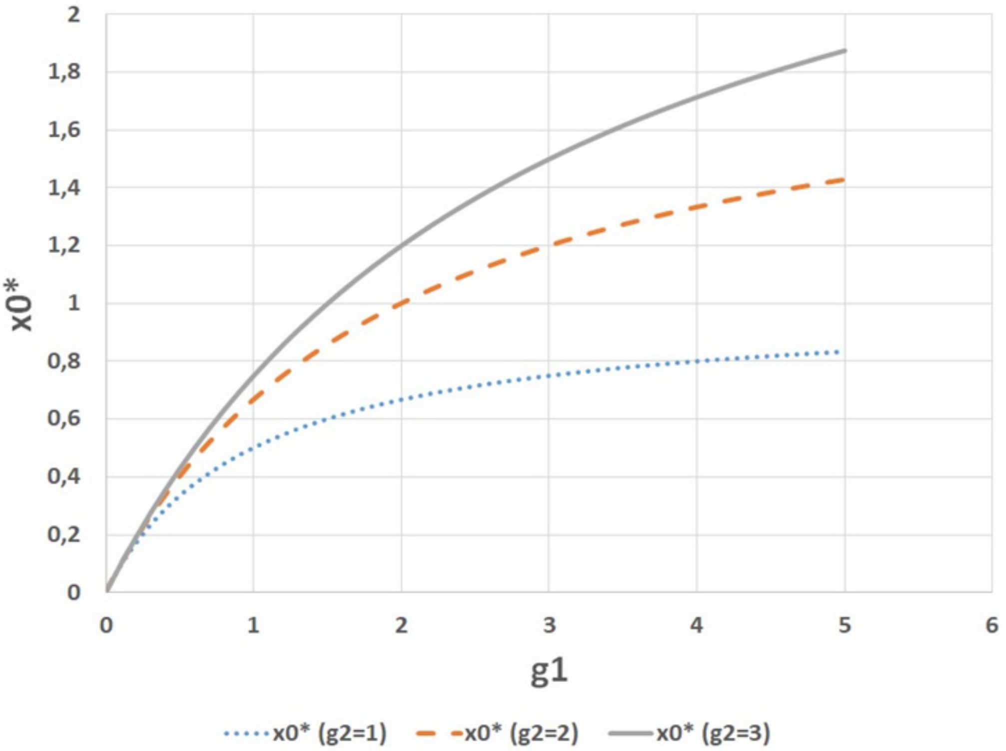

Figure 5 The optimal objective function value x_0^* as a function of the parameter g_1 for alternative values of g_2. x_0^* is a strictly increasing and strictly concave function of g_1. Furthermore, x_0^* is an increasing function of g_2.

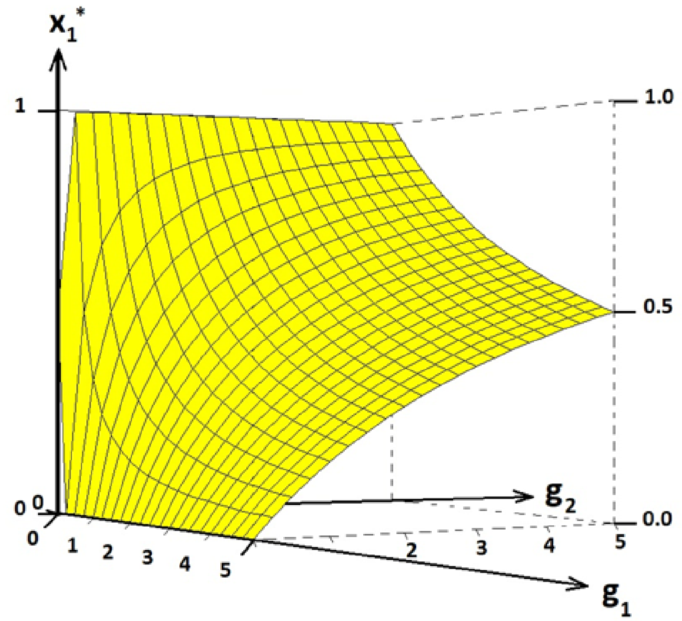

Figure 6 The optimal decision frequency x_1^*, as a function of the two parameters (g_1, g_2). x_1^* is a strictly decreasing and strictly convex function of g_1. x_1^* is a strictly increasing and strictly concave function of g_2.

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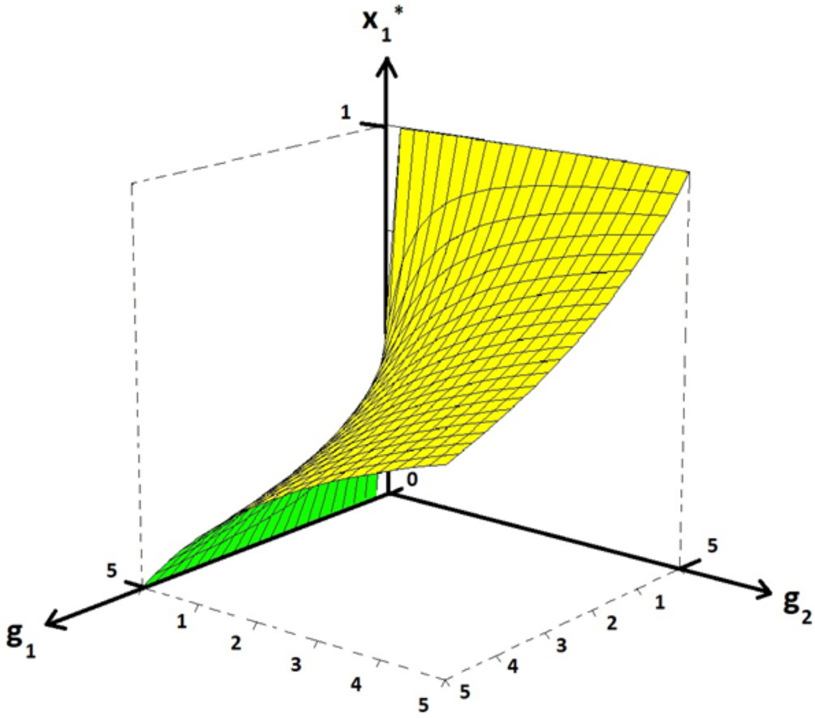

Figure 7 The optimal decision frequency x_1^*, as a function of the two parameters (g_1, g_2). x_1^* is a strictly decreasing and strictly convex function of g_1. x_1^* is a strictly increasing and strictly concave function of g_2. Compare Figure 4., which shows the function from another angle.

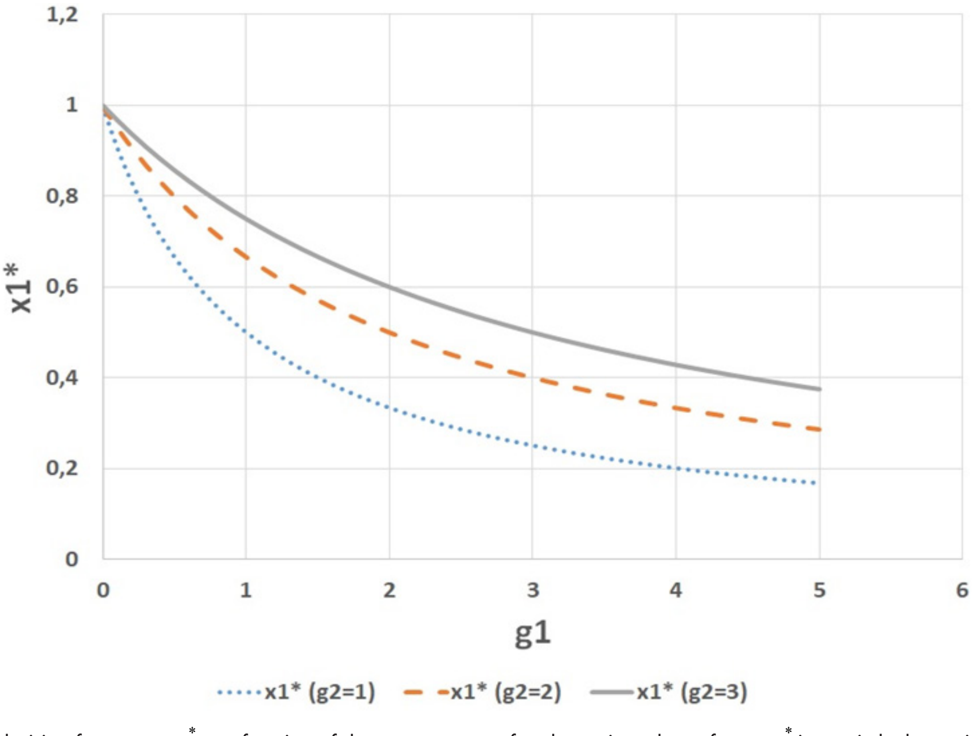

Figure 8 The optimal decision frequency x_1^* as a function of the parameter g_1 for alternative values of g_2. x_1^* is a strictly decreasing and strictly convex function of g_1. Furthermore, x_1^* is an increasing function of g_2.

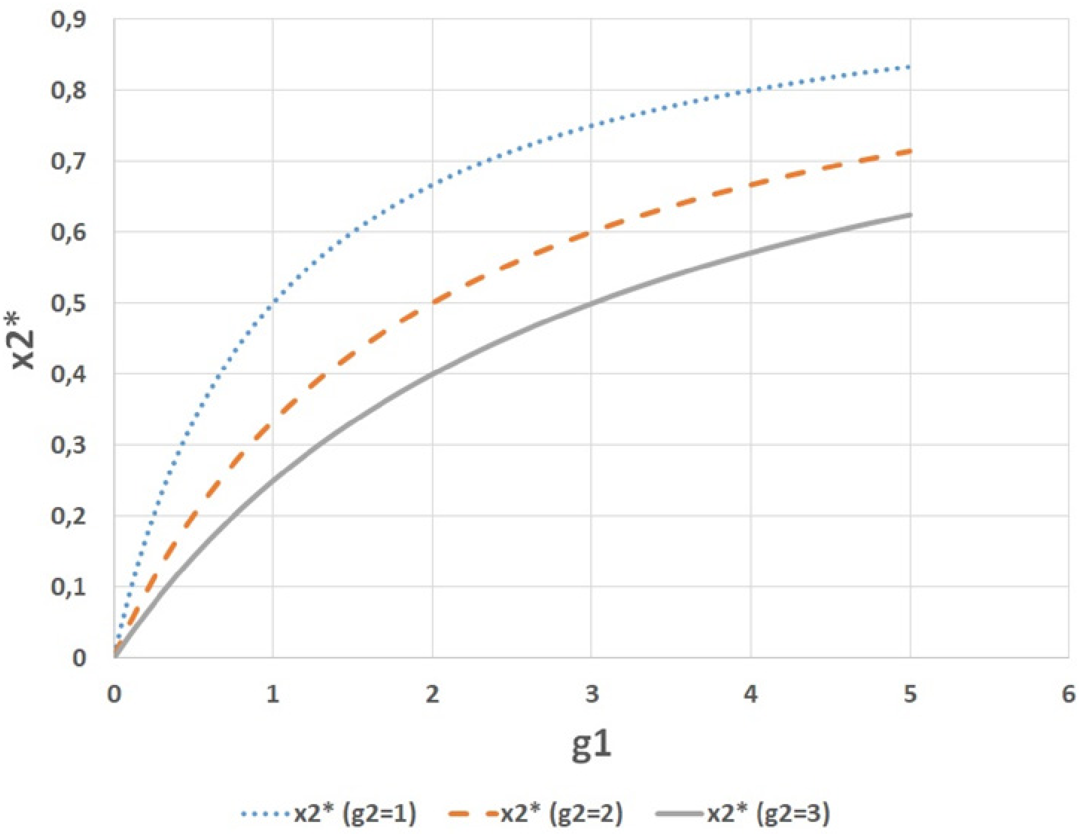

Figure 9 The optimal decision frequency x_2^* as a function of the parameter g_1 for alternative values of g_2. x_2^* is a strictly increasing and strictly concave function of g_1. Furthermore, x_2^* is an decreasing function of g_2.

Conclusion

In this paper, the two player zero sum games with diagonal game matrixes, TPZSGD, are analyzed. Many important applications of this particular class of games are found in military decision problems, in customs and immigration strategies and police work. Explicit functions are derived that give the optimal frequences of different decisions and the expected results of relevance to the different decision makers. Arbitrary numbers of decision alternatives are covered. It is proved that the derived optimal decision frequency formulas correspond to the unique optimization results of the two players. It is proved that the optimal solutions, for both players, always lead to a unique completely mixed strategy Nash equilibrium. For each player, the optimal frequency of a particular decision is strictly greater than 0 and strictly less than 1. With comparative statics analyses, the directions of the changes of optimal decision frequences and expected game values as functions of changes in different parameter values, are determined. Some of the derived formulas are used to confirm earlier game theory results presented in the literature. It is demonstrated that the new functions can be applied to solve a typical military decision problem and that the new functions make it possible to draw clear conclusions concerning issues that were not earlier possible to get via linear programming solutions. With the new approach developed here, it is possible to determine the directions of change of the expected value of the objective function and of the optimal frequences of the different decision alternatives, under the influence of increasing risk in the game matrix elements. Such game matrix elements are in real applications never known with certainty. Hence, this new approach leads to more relevant results than those that can be obtained with earlier methods.

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None.

Conflicts of interest

The author declares that there was no conflicts of interest.

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