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Output control of a single-airscrew helicopter's longitudinal motion spectrum

Abstract

The problem of stabilization law synthesis of a single-airscrew helicopter's longitudinal motion for lack of information about the vertical speed of its motion is analytically solved. The solution is based on the method of output control synthesis of the MIMO-system motion spectrum to be used as the basis of an especially designed multilevel decomposition of the dynamic system model in state space.

Keywords: single-airscrew helicopter, longitudinal motion, linear MIMO-system, stabilization, output feedback, motion spectrum, Control laws synthesis, multilevel decomposition

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Introduction

The physical principle of a single-airscrew helicopter's spatial motion provides the opportunity to categorize it as a complex multidimensional dynamic system (control object). At the same time, it is possible to mark out the following problems in less detail, which can be attributed to the control of helicopters as multidimensional dynamic systems.[1,2]

a. The problem of stabilization, or the problem of stabilizing control law synthesis, i.e. determination of a feedback (of a controller), which provides stability for a disturbed motion. In searching for this kind of problem solution, as a rule, modal control methods or pole control methods are used.[3–6]

b. The closed-loop system decoupling problem, i.e. determination of a controller's coefficients that provide decoupling of the control object's subsystems. The group of methods used is based on the analysis and usage of the reference system operators' kernels.[7,8]

c. The problem of making a set of stabilizing control laws, i.e. obtaining all, or almost all, control laws (algorithms), which provide stability for the closed-loop system, if the solution to the first problem, from those listed above, is not the only one. As a rule, the methods used are based on the solutions to Diophantine equations and the subsequent parameterization (for instance, the Youla-Kuĉera parametrization).[7,9–12]

It is possible to find other, not so commonly used, statements and solutions to the problem. For instance, there is a statement and a solution to the problem when stability and decoupling of the closed-loop controlled system, and also specified placement of the invariant system's zeros, i.e. complex frequencies, on which the closed-loop system "locks" completely or partially, can be provided simultaneously. The strengthened type of decoupling, when it is necessary to provide not only the block-diagonal form of the closed-loop system's operator, but also specific placement of the invariant system's zeros on the complex plane, belongs to it. Many of the practically important solutions to the problems listed above are not accompanied by published methods on how to obtain them. The analytical solutions have to fill this gap. In this paper for the first time

the analytical solution to the problem of a single-airscrew helicopter's stabilization in the vertical plane with the specified placement of poles and the incomplete measurement vector of motion parameters is presented. We use the linearized model of longitudinal motion of a single-airscrew helicopter (SH) having the following form:[13]

$$
\begin{pmatrix} \dot{V}_x \\ \dot{V}_y \\ \dot{\omega}_z \\ \dot{\upsilon} \end{pmatrix} = \begin{pmatrix} a_{V_x}^{V_x} & a_{V_x}^{V_y} & a_{V_x}^{\omega_z} & a_{V_x}^{\upsilon} \\ a_{V_y}^{V_x} & a_{V_y}^{V_y} & a_{V_y}^{\omega_z} & a_{V_y}^{\upsilon} \\ a_{\omega_z}^{V_x} & a_{\omega_z}^{V_y} & a_{\omega_z}^{\omega_z} & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} V_x \\ V_y \\ \omega_z \\ \upsilon \end{pmatrix} + \begin{pmatrix} b_{V_x}^{u_x} & b_{V_x}^{u_{gp}} \\ b_{V_y}^{u_x} & b_{V_y}^{u_{gp}} \\ b_{\omega_z}^{u_x} & b_{\omega_z}^{u_{gp}} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u_x \\ u_{gp} \end{pmatrix}. \quad (1)
$$

Here V_x – deviation from specified value of the longitudinal speed; V_y – deviation from specified value of the vertical speed; υ – deviation from specified value of the pitch angular velocity; υ – deviation from specified value of the angle of pitch; u_x – deviation angle of a main rotor's cone in the longitudinal direction; $a_{V_x}^{V_x}$ – general pitch of a main rotor; $a_{V_x}^{V_x}$, $a_{V_x}^{V_y}$, $a_{V_x}^{\dot{u}_z}$, $a_{V_y}^{V_x}$, $a_{V_y}^{V_x}$, $a_{V_y}^{\delta}$, $a_{V_y}^{\delta}$, $a_{V_y}^{\delta}$, $a_{\dot{u}_z}^{V_x}$, $a_{\dot{u}_z}^{\dot{u}_z}$, $a_{\dot{u}_z}^{\dot{u}_z}$, $b_{V_x}^{u_x}$, $b_{V_y}^{u_x}$, $b_{V_y}^{u_x}$, $b_{V_y}^{u_{gp}}$, $b_{\dot{u}_z}^{u_x}$, $b_{\dot{u}_z}^{u_{gp}}$ – linearization coefficients.[8,14]

We use the following notation

$$
x = \begin{pmatrix} \Delta V_x & \Delta V_y & \Delta \omega_z & \Delta \upsilon \end{pmatrix}^T, u = \begin{pmatrix} \Delta u_x & \Delta u_{gp} \end{pmatrix}^T,
$$

and also

$$
a_{11} = a_{V_x}^{V_x}, \ a_{12} = a_{V_x}^{V_y}, \ a_{13} = a_{V_x}^{\dot{u}_z}, \ a_{14} = a_{V_x}^{\delta},
$$

$$
a_{21} = a_{V_y}^{V_x}, \ a_{22} = a_{V_y}^{V_y}, \ a_{23} = a_{V_y}^{\dot{u}_z}, \ a_{24} = a_{V_y}^{\delta},
$$

$$
a_{31} = a_{\dot{u}_z}^{V_x}, \ a_{32} = a_{\dot{u}_z}^{V_y}, \ a_{33} = a_{\dot{u}_z}^{\dot{u}_z},
$$

$$
b_{21} = b_{V_y}^{u_x}, \ b_{22} = b_{V_y}^{u_{gp}},
$$

$$
b_{21} = b_{V_y}^{u_x}, \ b_{22} = b_{V_y}^{u_{gp}},
$$

$$
b_{31} = b_{\omega_z}^{u_x}, \ b_{32} = b_{\omega_z}^{u_{gp}},
$$

Then instead of expression (1) we will obtain the following linearized model of the SH longitudinal motion:

$$\dot{x} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} x + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \\ 0 & 0 \end{pmatrix} u. \quad (2)$$

Here after it will be considered that information about vertical speed of the SH motion is not available. As a result the system output vector can be expressed as

$$y = \begin{pmatrix} \Delta V_x & \Delta \omega_z & \Delta \upsilon \end{pmatrix}^T .$$

Taking into consideration the assumptions made, the initial mathematical model can be written in the form of a dynamic MIMO-system of the "input – state – output" type:

$$\dot{x} = Ax + Bu, \quad y = Cx , \quad (3)$$

Where $x \in \mathbb{R}^n$, $n = 4$ is the state vector; $u \in \mathbb{R}^r$ is the input vector (control vector), where $r=2$; $y \in \mathbb{R}^m$ is the output vector (measurement vector), where $m=3$. Here $\mathbb{R}$ is the set of real numbers.

In the equations (3) the corresponding matrices can be written as

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{14} & a_{24} \\ a31 & a_{32} & a_{33} & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \\ 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (4)$$

If, as a control law(3), to suggest the expression of the following form:

$$u=Fy=FCx , \quad (5)$$

Where $F \in \mathbb{R}^{r \times m}$ is a matrix of the output controller, then, in accordance with[2] for the system under consideration (3) – (5) a case of the dynamic MIMO-system output control will take place.

Here after we assume that matrix $B \in \mathbb{R}^{4 \times 2}$ (4) has a full rank (rank B=2 in this case), or that its equivalent matrix $B^T B$ is invertible, i.e. $\det\left(B^T B\right) \neq 0$. Let us now consider the matrix spectrum $A \in \mathbb{R}^{4 \times 4}$ (5). It will be understood as a set of matrix A eigen values (poles), $\text{eig}(A) = \left\{ \lambda_i \in \mathbb{C}: \ \det(\lambda_i I_4 - A) = 0, \ i=\overline{1,4} \right\}$. Here I_4 – identity matrix of size 4×4, $\mathbb{C}$ – the set of complex numbers. Let Λ be the given spectrum of the matrix $A+BFC$ of the corresponding close-loop controlled system, i.e.

$$\Lambda = \left\{ \hat{\lambda}_1, \ \hat{\lambda}_2, \ \hat{\lambda}_3, \ \hat{\lambda}_4 \right\} . \quad (6)$$

It is required to determine (i.e. synthesize) explicitly the controller matrix $F \hat{I} \mathbb{R}^{2 \times 3}$, such that the equality $\Lambda = \text{eig}\left(A+BFC\right)$ should be satisfied exactly. The complexity of this problem is a necessity for obtaining a solution in explicit analytical form, since $m \geq r$ matrices in (2) have a piecewise constant form. This paper is dedicated to solving of the problem.

Decomposition of a dynamic system

As a first step of the given problem solution we will consider the multilevel decomposition of the SH model suggested.[15–17] Since in this case the inequality $m \geq r$ (i.e. the number of system's outputs is greater than the number of its inputs) is implemented, then, in general, not taking into consideration specific numerical values for m and r, we consider the multilevel decomposition of system (3) – (5) of the following form: –zero decomposition level

$$\acute{A}_0 = A, \quad B_0 = B, \quad C_0 = C \quad (7)$$

– kth decomposition level ($k=\overline{1, M}$, where $M = \text{ceil}(n/r)$, $\bullet$ –is the operation of rounding the number« * »upwards)

$$A_k = B_{k-1}^{\perp} A_{k-1} B_{k-1}^{\perp+}, \ B_k = B_{k-1}^{\perp} A_{k-1} B_{k-1}, \ C_k = C_{k-1} A_{k-1} B_{k-1}^{\perp+} . \quad (8)$$

Equations (7), (8) for a set of indices $k = \overline{0, M}$ involve the matrices with the following properties:

$$\left(B_k \ \middle| \ B_k^{\perp+} \right)^{-1} = \begin{pmatrix} B_k^+ \\ \hline B_k^{\perp} \end{pmatrix}, \quad B_k^{\perp} B_k = 0, \quad B_k^+ B_k = I_r ,$$

$$\left(\frac{C_k}{C_k^{\perp}} \right)^{-1} = \left(C_k^+ \ \middle| \ C_k^{\perp+} \right), \quad C_k C_k^{\perp} = 0, \quad C_k C_k^+ = I_m ,$$

Where the superscript «⊥» denotes orthogonal annihilators (divisors of zero), and the superscript « + » denotes the Moore-Penrose pseudo inverse matrices.[13,16–19] Also, we consider the recurrence formulae of controllers for the spectrum control on the corresponding decomposition levels, written down in reverse order:

– M-th decomposition level

$$F_M = \left(\Phi_M B_M^+ - B_M^+ {}^{A_M} \right) C_M^+ , \quad (9)$$

– k-th decomposition level ($k=\overline{0, M-1}$)

$$F_k = \left(\Phi_k B_k^- - B_k^- A_k \right) C_k^+, \quad B_k^- = B_k^+ - F_{k+1} C_{k+1} B_k^{\perp} . \quad (10)$$

The multilevel decomposition procedure considered is then implemented.

III. Algorithm for synthesis of the MIMO-system output control

The following statement is true that has been proven[13] and given here with an allowance for replacement of matrix annihilators, which are not possessed of a property of orthogonality, i.e., replacement with matrices, for which, in general, not necessarily the equality $B_k^{\perp} B_k^{\perp+} = I_{n-r}$ is implemented.

Theorem1. Let $m \geq r$, and the following matrices exist and are pair wise completely controllable:

$$G_k^T = B_k^- A_k C_k^{\perp} \left(B_k^- C_k^{\perp} \right)^+, \ H_k^T = \left(B_k^- C_k^{\perp} \right)^{\perp}, \ k=\overline{0, M}, \text{Then,} \quad \text{there exists a}$$

nonempty set of matrices $K \quad i=\overline{0, M}$, such that

$$\Phi_i = G_i + K_i^T H_i = \left(B_i^- A_i C_i^{\perp} \right)\left(B_i^- C_i^{\perp} \right)^+ + K_i^T \left(B_i^- C_i^{\perp} \right)^{\perp} , \quad (11) \text{ and(9), (10)}$$

satisfy the equalities of spectra

$$\text{eig}\left(A_k + B_k F_k C_k \right) = \bigcup_{i=k}^{M} \text{eig}(\Phi_i), \ k=\overline{1, M} ,$$

$$A_0 = A, \quad B_0 = B, \quad C_0 = C ,$$

$$A_k = C_{k-1}^{\wedge} A_{k-1} C_{k-1}^{\wedge+}, \ B_k = B_{k-1} A_{k-1} C_{k-1}^{\wedge+}, \ C_k = C_{k-1} A_{k-1} C_{k-1}^{\wedge+} , \quad k=\overline{1, N}$$

moreover, the following matrices exist and are pair wise completely controllable:

$$G_k = \left(B_k^{\wedge} C_k^+ \right)^+ B_k^{\wedge} A_k C_k^+, \ H_k = \left(B_k^{\wedge} C_k^+ \right)^{\wedge}, \ k = \overline{0, N} .$$

Then, there exists a nonempty set of matrices L_i, $i=\overline{0, N}$,

such that $\Psi_i = G_i + H_i L_i^T = \left(B_i^{\wedge} C_i^+ \right)^+ B_i^{\wedge} A_i C_i^+ + \left(B_i^{\wedge} C_i^+ \right)^{\wedge} L_i^T$ and, for

$$F_N = B_M^+ \left(C_M^+ \Psi_M^- A_M C_M^+ \right) F_k = B_k^+ \left(C_k^+ \Psi_k^- A_k C_k^+ \right),$$

Citation: Zubov NE, Ryabchenko VN, Sorokin IV. Output control of a single-airscrew helicopter's longitudinal motion spectrum. *Int Rob Auto J.* 2018;4(3):157–162. DOI: 10.15406/iratj.2018.04.00114

$$C^-_k = C^+_k - C^{\wedge T}_k \underset{K+1}{B} \underset{K+1}{F}\,,\quad k=\overline{1,\ N\text{-}1}\ \text{it holds that}$$

$$,\ \text{eig}\left(A_k+B_kF_kC_k\right)=\bigcup_{i=k}^{N}\text{eig}(\Psi_i),\ k=\overline{1,\ N}$$

$$\text{eig}\left(A_0+B_0F_0C_0\right)=\bigcup_{i=0}^{N}\text{eig}(\Psi_i)=\Lambda\ .\ \text{It should be noted that}$$

here, as in the algorithms described,[13,16–19] only semi orthogonal and pseudo inverse matrices are used in the transformations, which at least do not reduce the condition number of the equations. Moreover, this approach does not impose restrictions in the form of the differentiation between the algebraic and geometric multiplicities of the elements of the spectrum to be assigned; there are also no restrictions on the size of the problem. This is confirmed by simulation, which shows a high relative accuracy of spectrum control and the practical absence of restrictions on the size of system (3).

Analytical synthesis of a single-airscrew helicopter's longitudinal motion control

In accordance with the problem statement, it is required to find explicitly a formula of controller F in the control law (5) that can be expressed in this case as:

$$\begin{pmatrix}u_x\\u_{gp}\end{pmatrix}=\text{FC}\begin{pmatrix}V_x\\V_y\\\omega_z\\\upsilon\end{pmatrix}=F\begin{pmatrix}V_x\\\omega_z\\\upsilon\end{pmatrix},$$

And provides for the close-loop system «HS + control system» of a specified earlier spectrum(6). We perform for the system (2) with matrices(3), (4) – the multilevel decomposition described in Section 1, which has in this case two $\text{eig}\left(A_0+B_0F_0C_0\right)=\bigcup_{i=0}^{M}\text{eig}(\Phi_i)=\Lambda$. The condition $m\geq r$ in Theorem 1 is not restrictive; it is introduced to indicate that, in the present case F , matrix from (5) is conventionally considered as a matrix of controller (i.e. the number of inputs is less than the number of outputs). For the case $m\leq r$, Theorem 1 has a dual formulation, and matrix F is replaced with the observer matrix L (the number of inputs is greater than the number of outputs).

Theorem 2. Let $m\leq r$, $N=\text{ceil}(n/m)$, and the following decomposition of system (3) hold: decomposition levels (M=1): zero level (7) and first level (8). Therefore, we will have

$$B_0^{\perp}=\begin{pmatrix}l_{11}&l_{11}&1&0\\0&0&0&1\end{pmatrix},\ B_0^{\perp+}=\begin{pmatrix}l_{11}l^{-1}&0\\l_{12}l^{-1}&0\\l^{-1}&0\\0&1\end{pmatrix},\quad(12)$$

$$\text{A}_1=\begin{pmatrix}a^1_{11}&a^1_{12}\\a^1_{21}&0\end{pmatrix},\ \text{B}_1=\begin{pmatrix}b^1_{11}&b^1_{12}\\b^1_{21}&b^1_{22}\end{pmatrix},\ \text{C}_1=\begin{pmatrix}c^1_{11}&c^1_{12}\\c^1_{21}&c^1_{22}\\c^1_{31}&0\end{pmatrix},\quad(13)$$

Where, taking in view the following notation $l^*=b_{11}b_{22}-b_{12}b_{21}$, the matrices' elements in(12), (13) is equal to:

$$l_{11}=\frac{b_{21}b_{32}-b_{22}b_{31}}{l^*},\ l_{12}=\frac{-(b_{11}b_{32}-b_{12}b_{31})}{l^*},\quad l=l^2_{11}+l^2_{12}+1\,,$$

$$a^1_{11}l=a_{33}+a_{13}l_{11}+a_{23}l_{12}+a_{31}l_{11}+a_{32}l_{12}+a_{11}l^2_{11}+$$
$$+a_{22}l^2_{12}+a_{12}l_{11}l_{12}+a_{21}l_{11}l_{12},$$

$$a^1_{12}=a_{13}l_{11}+a_{24}l_{12},\ a^1_{21}=l^{-1},\quad b^1_{21}=b_{31},b^1_{22}=b_{32},$$

$$b^1_{11}=b_{11}(a_{31}+a_{11}l_{11}+a_{21}l_{12})+b_{21}(a_{32}+a_{12}l_{11}+a_{22}l_{12})+$$
$$+b_{31}(a_{33}+a_{13}l_{11}+a_{23}l_{12}),$$

$$b^1_{12}=b_{12}(a_{31}+a_{11}l_{11}+a_{21}l_{12})+b_{22}(a_{32}+a_{12}l_{11}+a_{22}l_{12})+$$
$$+b_{32}(a_{33}+a_{13}l_{11}+a_{23}l_{12}),$$

$$c^1_{11}=(a_{13}+a_{11}l_{11}+a_{12}l_{12})l^{-1},$$
$$c^1_{12}=a_{13},\ c^1_{21}=(a_{33}+a_{31}l_{11}+a_{32}l_{12})l^{-1},\ c^1_{22}=0,\ c^1_{31}=l^{-1}.$$

To check the controllability conditions in Theorem 1, we calculate the matrices:

$$C_0^{\perp T}=(0\ \ 1\ \ 0\ \ 0),\ B_0^+=\begin{pmatrix}b^{0+}_{11}&b^{0+}_{12}&b^{0+}_{14}&0\\b^{0+}_{21}&b^{0+}_{22}&b^{0+}_{23}&0\end{pmatrix},\quad\text{in which, in}$$

presence of the introduced parameter

$$b^{0+}=b^2_{11}b^2_{22}+b^2_{11}b^2_{32}-2b_{11}b_{12}b_{21}b_{22}-2b_{11}b_{12}b_{31}b_{32}+$$
$$+b^2_{12}b^2_{21}+b^2_{12}b^2_{31}+b^2_{21}b^2_{32}-2b_{21}b_{22}b_{31}b_{32}+b^2_{22}b^2_{31},$$

Corresponding elements can be expressed as

$$b^{0+}_{11}=\frac{b_{11}b^2_{22}-b_{12}b_{21}b_{22}+b_{11}b^2_{32}-b_{12}b_{31}b_{32}}{b^{0+}},$$

$$b^{0+}_{12}=\frac{b_{21}b^2_{12}-b_{11}b_{22}b_{12}+b_{21}b^2_{32}-b_{22}b_{31}b_{32}}{b^{0+}},$$

$$b^{0+}_{13}=\frac{b_{31}b^2_{12}-b_{11}b_{32}b_{12}+b_{31}b^2_{22}-b_{21}b_{32}b_{22}}{b^{0+}},$$

$$b^{0+}_{21}=\frac{b_{12}b^2_{21}-b_{11}b_{22}b_{21}+b_{12}b^2_{31}-b_{11}b_{32}b_{31}}{b^{0+}},$$

$$b^{0+}_{22}=\frac{b_{22}b^2_{11}-b_{12}b_{21}b_{11}+b_{22}b^2_{31}-b_{21}b_{32}b_{31}}{b^{0+}},$$

$$b^{0+}_{23}=\frac{b_{32}b^2_{11}-b_{12}b_{31}b_{11}+b_{32}b^2_{21}-b_{22}b_{31}b_{21}}{b^{0+}}.$$

$$b^{0+}_{22}=\frac{b_{22}b^2_{11}-b_{12}b_{21}b_{11}+b_{22}b^2_{31}-b_{21}b_{32}b_{31}}{b^{0+}},$$

$$b^{0+}_{23}=\frac{b_{32}b^2_{11}-b_{12}b_{31}b_{11}+b_{32}b^2_{21}-b_{22}b_{31}b_{21}}{b^{0+}}.$$

Besides, the following equations take place:

$$H_0=\left(B_0^+C_0^{\perp}\right)^{\perp T}=\left(-\frac{b^{0+}_{22}}{b^{0+}_{12}}\ \ 1\right)^T,$$

$$G_0=\left(B_0^-C_0^{\perp}\right)^{+T}\left(B_0^-A_0C_0^{\perp}\right)^T=$$

$$=\begin{pmatrix}\dfrac{b^{0+}_{12}(a_{12}b^{0+}_{11}+a_{22}b^{0+}_{12}+a_{32}b^{0+}_{13})}{(b^{0+}_{12})^2+(b^{0+}_{22})^2}&\dfrac{b^{0+}_{12}(a_{12}b^{0+}_{21}+a_{22}b^{0+}_{22}+a_{32}b^{0+}_{23})}{(b^{0+}_{12})^2+(b^{0+}_{22})^2}\\\dfrac{b^{0+}_{22}(a_{12}b^{0+}_{11}+a_{22}b^{0+}_{12}+a_{32}b^{0+}_{13})}{(b^{0+}_{12})^2+(b^{0+}_{22})^2}&\dfrac{b^{0+}_{22}(a_{12}b^{0+}_{21}+a_{22}b^{0+}_{22}+a_{32}b^{0+}_{23})}{(b^{0+}_{12})^2+(b^{0+}_{22})^2}\end{pmatrix}\quad(14)$$

$$C_1^{\perp T}=0, \quad B_1^+=\begin{pmatrix} b_{11}^{1+} & b_{12}^{1+} \\ b_{21}^{1+} & b_{22}^{1+} \end{pmatrix},$$

$$H_1=\left(B_1^+ C_1^{\perp T}\right)^{\perp T}=I_2, \quad H_1^+=\left(H_1\right)^+=I_2, \qquad (15)$$

$$G_1=\left(B_1^+ C_1^{\perp T}\right)^{+T}\left(B_1^+ A_1 C_1^{\perp T}\right)^T=0,$$

Where, for brevity, the following notation is used:

$$h_1^+=b_{11}^1 b_{22}^1-b_{12}^1 b_{21}^1 \qquad ,$$

$b_{11}^{1+}=b_{22}^1/h_1^+$, $b_{12}^{1+}=-b_{12}^1/h_1^+$, $b_{21}^{1+}=-b_{21}^1/h_1^+$, $b_{22}^{1+}=b_{11}^1/h_1^+$. For the zero and first decomposition levels, we calculate the ranks of the following block matrices $\left(H_1 \mid G_1 H_1\right)$, $\left(H_1 \mid G_1 H_1\right)$ as a result we will obtain: $\text{rank}\left(H_1 \mid G_1 H_1\right)=2$, $\text{rank}\left(H_1 \mid G_1 H_1\right)=2$, this corresponds to the number of "independent" inputs $r=2$. Therefore, each level of decomposition satisfies the controllability condition in Theorem 1. According to the form of controllers(9), (10) we define a matrix whose Eigen values will be assigned to the first decomposition level. Since matrices (15) are invertible, we may choose any matrix $\hat{O}$ that has the prescribed spectrum for the first decomposition level. For simplicity, we set this matrix diagonal $\hat{O}_1=\text{diag}\left(\hat{\lambda}_1, \hat{\lambda}_2\right)$, and calculate the pseudo inverse matrix

$$C_1^+=\begin{pmatrix} c_{11}^{1+} & c_{12}^{1+} & c_{13}^{1+} \\ c_{21}^{1+} & c_{22}^{1+} & c_{23}^{1+} \end{pmatrix}.$$

Here, again for brevity, we introduce the notation

$$c^{1+}=(c_{11}^1 c_{22}^1)^2-2c_{11}^1 c_{12}^1 c_{21}^1 c_{22}^1+(c_{12}^1 c_{21}^1)^2+(c_{12}^1 c_{31}^1)^2+(c_{22}^1 c_{31}^1)^2,$$

$c_{11}^{1+}=c_{22}^1(c_{11}^1 c_{22}^1-c_{12}^1 c_{21}^1)/c^{1+}$, $\quad c_{12}^{1+}=-c_{12}^1(c_{11}^1 c_{22}^1-c_{12}^1 c_{21}^1)/c^{1+}$,

$c_{13}^{1+}=c_{31}^1(c_{12}^1 c_{12}^1+c_{22}^1 c_{22}^1)/c^{1+}$, $\quad c_{21}^{1+}=(c_{12}^1 c_{21}^1 c_{21}^1-c_{11}^1 c_{21}^1 c_{22}^1+c_{12}^1 c_{31}^1 c_{31}^1)/c^{1+}$,

$c_{22}^{1+}=(c_{22}^1 c_{11}^1 c_{11}^1-c_{12}^1 c_{21}^1 c_{11}^1+c_{22}^1 c_{31}^1 c_{31}^1)/c^{1+}$,

$c_{23}^{1+}=-c_{31}^1(c_{11}^1 c_{12}^1-c_{21}^1 c_{22}^1)/c^{1+}$.

Based on equation(9), the first decomposition level yields the following formula for the controller:

$$F_1=B_1^+\left(\Phi_1-A_1\right)C_1^+=\begin{pmatrix} f_{11}^1 & f_{12}^1 & f_{13}^1 \\ f_{21}^1 & f_{22}^1 & f_{23}^1 \end{pmatrix},$$

Where the following notation is used:

$$d_1=a_{12}^1 b_{11}^{1+}-b_{12}^{1+}\hat{\lambda}_1, \quad d_2=a_{11}^1 b_{11}^{1+}+a_{21}^1 b_{12}^{1+}-b_{11}^{1+}\hat{\lambda}_1,$$

$$d_3=a_{12}^1 b_{21}^{1+}-b_{22}^{1+}\hat{\lambda}_2, \quad d_4=a_{11}^1 b_{21}^{1+}+a_{21}^1 b_{22}^{1+}-b_{11}^{1+}\hat{\lambda}_2,$$

and where the components are:

$f_{11}^1=-c_{21}^{1+}d_1-c_{11}^{1+}d_2$, $\quad f_{12}^1=-c_{22}^{1+}d_1-c_{12}^{1+}d_2$, $\quad f_{13}^1=-c_{23}^{1+}d_1-c_{13}^{1+}d_2$,

To calculate matrix B_0^- that is needed for determining the zero level controller, we use the second formula in(10). As a result, we obtain the expression

$$B_0^- = B_0^+ - F_0 C_1 B_0^\perp = \begin{pmatrix} b_{11}^m & b_{12}^m & b_{13}^m & b_{14}^m \\ b_{21}^m & b_{22}^m & b_{23}^m & b_{24}^m \end{pmatrix}.$$

Here we denote linear combinations:

$$b_{11}^m=b_{11}^{0+}-l_{11}(c_{11}^1 f_{11}^1+c_{21}^1 f_{12}^1+c_{31}^1 f_{13}^1),$$

$$b_{12}^m=b_{12}^{0+}-l_{12}(c_{11}^1 f_{11}^1+c_{21}^1 f_{12}^1+c_{31}^1 f_{13}^1),$$

$$b_{13}^m=b_{13}^{0+}-c_{11}^1 f_{11}^1-c_{21}^1 f_{12}^1-c_{31}^1 f_{13}^1,$$

$$b_{14}^m=-c_{12}^1 f_{11}^1-c_{22}^1 f_{12}^1,$$

$$b_{21}^m=b_{21}^{0+}-l_{11}(c_{11}^1 f_{21}^1+c_{21}^1 f_{22}^1+c_{31}^1 f_{23}^1),$$

$$b_{22}^m=b_{22}^{0+}-l_{12}(c_{11}^1 f_{21}^1+c_{21}^1 f_{22}^1+c_{31}^1 f_{23}^1),$$

$$b_{23}^m=b_{23}^{0+}-c_{11}^1 f_{21}^1-c_{21}^1 f_{22}^1-c_{31}^1 f_{23}^1,$$

$$b_{24}^m=-c_{12}^1 f_{21}^1-c_{22}^1 f_{22}^1.$$

According to Theorem 1, we complete the system of the zero decomposition level using (7). For this purpose we use equation (14) and, applying new notation, we obtain

$$G_0 = \begin{pmatrix} a_{11}^a & a_{12}^a \\ a_{21}^a & a_{22}^a \end{pmatrix},$$

Where we denote

$$a_{11}^a=b_{12}^{0+}d_1/d, \quad a_{12}^a=b_{12}^{0+}d_2/d,$$

$$a_{21}^a=b_{22}^{0+}d_1/d, \quad a_{22}^a=b_{22}^{0+}d_2/d,$$

$$d=(b_{12}^{0+})^2+(b_{22}^{0+})^2, \quad d_1=a_{12}b_{11}^m+a_{22}b_{12}^m+a_{32}b_{13}^m,$$

$$d_2=a_{12}b_{21}^m+a_{22}b_{22}^m+a_{32}b_{23}^m.$$

Now, we should determine $\hat{O}$ for the zero decomposition level. For this purpose, we decompose the matrices H_0, G_0 of the zero level into two sublevels and calculate the corresponding matrices. We obtain

$$(H_0)_0=H_0, \ (G_0)_0=G_0, \ (H_0)_0^\perp=\begin{pmatrix} b_{12}^m/b_{22}^m & 1 \end{pmatrix},$$

$$(H_0)_0^{\perp+}=\begin{pmatrix} b_{12}^m b_{22}^m/b^m \\ (b_{12}^m)^2/b^m \end{pmatrix},$$

$$(H_0)_0^+=\begin{pmatrix} -b_{12}^m b_{22}^m/b^m & (b_{12}^m)^2/b^m \end{pmatrix}.$$

Here $b^m=(b_{12}^m)^2+(b_{22}^m)^2$, and, in addition,

$$(G_0)_1 = (H_0)_0 G_0 (H_0)_0^\perp = a_1^{aa} =$$

$$=(b_{22}^m)^2(a_{22}^a+a_{12}^a b_{12}^m/b_{22}^m)/b^m +$$

$$+ b_{12}^m b_{22}^m(a_{21}^a + a_{11}^a b_{12}^m / b_{22}^m) / b^m,$$

$$(H_0)_1=(H_0)_0^\perp G_0 H_0=b_{11}^{bb}.$$

Besides,

$$b_{11}^{bb}=a_{22}^a-(b_{22}^m(a_{21}^a+a_{11}^a b_{12}^m/b_{22}^m)/b^m+a_{12}^a b_{12}^m/b_{22}^m. \quad (16)$$

Using the values of $(H_0)_1=b_{p11}$ (16), we then find the matrix (scalar, in this case) $(H_0)_1^+=b_{p11}$, where $b_{p11}=1/b_{11}^{bb}$.

Let us now assign the Eigen value as a scalar matrix $(\hat{O}_0)_1=\hat{\lambda}_3$ and calculate the matrix of feedback coefficients for the first sublevel of the zero decomposition level. We obtain $k_1=-b_{p11}\left(a_1^{aa}-\hat{\lambda}_3\right)$. Next, we calculate the matrix

$$h_{11}^{0-}=b_{p11}b_{12}^m((a_1^{aa}-\hat{\lambda}_3)/b_{22}^m+b_{12}^m b_{22}^m/b^m,$$

Where

$$h_{11}^{0-}=b_{p11}b_{12}^m((a_1^{aa}-\hat{\lambda}_3)/b_{22}^m+b_{12}^mb_{22}^m/b^m,$$

$h_{12}^{0-}=b_{p11}(a_1^{aa}-\hat{\lambda}_3)+(b_{12}^m)^2/b^m$, and specify again the matrix of Eigen values of the zero sublevel of the zero decomposition level by $\left(\Phi_0\right)_0=\hat{\lambda}$.

Finally, we find matrix K_0 by the rule

$$K_0=\left(\Phi_0\right)_0\left(H_0\right)_0^- -\left(H_0\right)_0^- G_0=\left(K_{11}\quad K_{12}\right),$$

Where in the case under examination

$$K_{11}=a_{11}^a((b_{12}^mb_{22}^m/b^m-b_{p11}b_{12}^m(a_1^{aa}-\hat{\lambda}_3)/b_{22}^m)-a_{21}^a(b_{p11}(a_1^{aa}-\hat{\lambda}_3)$$
$$+(b_{12}^m)^2/b^m)-\hat{\lambda}_4(b_{12}^mb_{22}^m/b^m-b_{p11}b_{12}^m(a_1^{aa}-\hat{\lambda}_3)/b_{22}^m),$$

$$K_{12}=\hat{\lambda}_4(b_{p11}(a_1^{aa}-\hat{\lambda}_3)+(b_{12}^m)^2/b^m)-a_{22}^a(b_{p11}(a_1^{aa}-\hat{\lambda}_3)+$$
$$+(b_{12}^m)^2/b^m)+a_{12}^a((b_{12}^mb_{22}^m/b^m-b_{p11}b_{12}^m(a_1^{aa}-\hat{\lambda}_3)/b_{22}^m).$$

As a result, we obtain using equation(11), the matrix $\hat{O}_0$, whose eigen values $\hat{\lambda}_3$, $\hat{\lambda}_4$, are ensured by the output controller, namely,

$$\Phi_0=\begin{pmatrix} a_{11}^a-\dfrac{b_{22}^m}{b_{12}^m}K_{11} & a_{21}^a+K_{11} \\ a_{12}^a-\dfrac{b_{22}^m}{b_{12}^m}K_{12} & a_{22}^a+K_{12} \end{pmatrix}.$$

Further calculations, which were described, for instance,[13,17–19] finally yield the following formula for the SH output controller vector:

$$F=\begin{pmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \end{pmatrix}. \text{ (17)}$$

Here elements of the following notation:

$$d_{21}=a_{12}^a-\frac{b_{22}^m}{b_{12}^m}K_{12},\ d_{22}=a_{22}^a+K_{12},\ d_{21}=a_{12}^a-\frac{b_{22}^m}{b_{12}^m}K_{12},\ d_{22}=a_{22}^a+K_{12},$$

Can be expressed as

$$f_{11}=b_{11}^md_{11}-b_{12}^ma_{21}-b_{13}^ma_{31}-b_{11}^ma_{11}+b_{21}^md_{12},$$

$$f_{12}=b_{13}^md_{11}-b_{13}^ma_{33}-b_{14}^m+b_{23}^md_{12},$$

$$f_{13}=b_{14}^md_{11}-b_{13}^ma_{34}-b_{11}^ma_{13}-b_{12}^ma_{24}+b_{24}^md_{12},$$

$$f_{21}=b_{11}^md_{21}-b_{22}^ma_{21}-b_{23}^ma_{31}-b_{21}^ma_{11}+b_{21}^md_{22},$$

$$f_{22}=b_{13}^md_{21}-b_{23}^ma_{33}-b_{24}^m+b_{23}^md_{22},$$

$$f_{23}=b_{14}^md_{21}-b_{23}^ma_{34}-b_{22}^ma_{24}-b_{21}^ma_{13}+b_{24}^md_{22}.$$

The synthesized controller (the SH control system laws) ensures exactly the specified spectrum (6) for controlled longitudinal motion of the helicopter. This assertion can be directly checked with the help of appropriate analytical calculations. For this purpose it is sufficient to make use of the package Symbolic Toolbox Matlab; namely, one can use the *eig* instruction to calculate the eigen values of the ***A+BFC*** matrix, where matrices can be expressed as(4), and the ***F*** matrix is determined by (17).

Numerical analysis

Suppose that we want the closed-loop system "SH + control system" to have the following multiple real spectrum

$$\Lambda=\left\{-1.5,\ -1.5,\ -1.5,\ -1.5\right\}\ . \text{ (18)}$$

Let use for the hypothetical SH the ***A***, ***B*** and ***C*** (4) matrices having the following numerical values:

$$A=\begin{pmatrix} -0.0598 & -0.0233 & 0.1095 & -0.1703 \\ -0.0268 & -0.8899 & -1.2091 & -0.0135 \\ 1.5158 & -2.3207 & -2.3464 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$B=\begin{pmatrix} -0.3050 & -0.0228 \\ -1.2817 & 2.0890 \\ 28.178 & 22.7255 \\ 0 & 0 \end{pmatrix}$$

With use of equation (17) for the controller we obtain the numerical value for the matrix *F*. As a result we have

$$F=\begin{pmatrix} 3.6163 & -1.8622 & -3.4996 \\ -2.7936 & 2.2309 & 3.6237 \end{pmatrix}.$$

The computation of Eigen values of the matrix $A+BFC$ yields the set

$\text{eig}(A+BFC)=\{-1.5,\ -1.5151,\ -1.4924\pm0.0131i\}$, Which lies quite close to the specified values (18). Note that to implement the sets (18) using the method[5] is not possible.[20,21]

Conclusion

The problem of a hypothetical single-airscrew helicopter's longitudinal motion stabilization for lack of information about the vertical speed of its motion has been analytically solved. The solution is based on the method of the output signal control synthesis that provides a specified spectrum of the MIMO-system's motion, presented.[13,16–19]

Acknowledgements

Conflict of interest

The author declares there is no conflict of interest.

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