Review Article

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The dynamic properties in the matrix model of biological system under optimal harvesting

Abstract

The problem of optimal harvest in biological system is investigated. The structure of system is described by matrix model. The properties of stabilization indexes of harvest components are shown. The harvesting regime is stabilized on low level for long time interval of harvesting. These properties are analogous for turnpike theorems in models of economic dynamics.

Keywords: matrix model, biological system, optimal harvesting, turnpike phenomenon

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Introduction

Matrix models are widely used for describing of ecological, economic and social processes. This article has results for biological applications. The matrices are nonnegative usually by describing of abundance dynamics for biological populations or communities.[1] For example the abundance dynamics for a population with age structure may be described by model (1) with matrix A as Leslie matrix L.[2] The results of O Perron and of G Frobenius for nonnegative matrices are widely known.[3,4] A model of abundance dynamics for biological system has form:

$$\begin{cases} x_{t+1}=Ax_t, t=1,2,... \\ x_1=\bar{x}\geq 0, \end{cases} \quad (1)$$

Here A is a nonnegative irreducible matrix of order n[1, 2, 3], parameter t denotes a discrete time, vector $x_t \in R^n_+$ describes an abundance for all groups of the system.

The next result follows for model (1) from Perron-Frobenius Theorem:[3,5] if h is the order of cyclicity of matrix A, then $\lim_{t\to\infty} \frac{1}{h}\sum_{\tau=t}^{t+h-1} \frac{X_{\tau+1}}{r^\tau} = (q\cdot\bar{x})\,p$. The vector p is positive eigenvector for spectral radius $\rho(A)$ of matrix A, the vector q is positive eigenvector for spectral radius $\rho(A^*)=\rho(A)$ of transposed matrix A^*. These vectors are connected by condition $q\cdot p=1$. This result about asymptotic properties in matrix model is very important. We will compare these asymptotical properties of the abundance dynamics of biological systems with the dynamics under optimal harvesting.

We formulate a problem of the optimal harvest in matrix model (1):

$$\begin{cases} \sum_{t=1}^{T} wU_t x_t \xrightarrow[u_t\in[0;1]]{} \sup \\ x_{t+1}=A(I-U_t)x_t \\ x_1=\bar{x} \end{cases} \quad (2)$$

Vector w characterizes the specific income for specific yield. Matrix $U_t = diag(u_t)$ contains the harvest vector on main diagonal, $0 \leq U_t \leq I$. The time is measured in years. The planning time interval is T. All vectors are nonnegative. The abundance dynamics of biological system is researched in this work. The asymptotic properties for the problem of the optimal harvest have other forms than the properties for model (1).

Results

We study the problem (2) for nonnegative irreducible matrix A. The denotation

$y_t = (I-U_t)x_t$ (3) will be used. Then next equations are true:

$$x_{t+1} = Ay_t, U_{t+1}x_{t+1} = x_{t+1} - y_{t+1} = Ay_t - y_{t+1}, U_1x_1 = \bar{x} - y_1$$

Optimization criterion is being transformed to the form:

$w\left((\bar{x}-y_1)+\sum_{t=1}^{T-1}(Ay_t-y_{t+1})\right) \to \sup$. The equations in model (2) are being transformed to form: $0 \leq y_t \leq \bar{x}$ and $0 \leq y_{t+1} \leq Ay_t$ for $t = 1, ..., T-1$.

Lemma 1: The problem (2) is equivalent to a next problem:

$$\begin{cases} \Phi=\sum_{t=1}^{T-1} w[A-I]y_t - wy_T \to \sup \\ 0\leq y_{t+1} \leq A\,y_t, t=1,...,T-1 \\ 0\leq y_t \leq \bar{x} \end{cases} \quad (4)$$

Proof: Every possible solution for problem (2) is transformed to possible solution for problem (4) by formula (3). Inverse formulas have forms:

$$x_1 = \bar{x}, x_{t+1} = Ay_t, u_{tj} = \begin{cases} x_{tj} - y \; for\, x_{tj} > 0 \\ every\ value\ from\,[0,1]\, for\, x_{tj} \end{cases} = 0, t = 1, ... T-1 \quad (5)$$

The optimal solution for problem (2) corresponds to a optimal solution for problem (4).

The following denotation will be used: $\psi = w(A-I)$.

Lemma 2: The optimal solution for the problem (4) has form:

$$\begin{cases} \hat{y}_1 diag(sg\psi)\bar{x} \\ \hat{y}_{t+1} = diag(sg\psi_{t+1})A\hat{y}_t \ \ for\ t=1,2,...,T-2 \\ \hat{y}_T = 0 \end{cases} \quad (6)$$

We use here the next denotations:

$$\begin{cases} \varphi_{T-1} = \psi \\ \varphi_t = \psi + \varphi^*_{t+1} \ \ for\ \ t=1,2,...,T-2 \end{cases} \quad (7)$$

$$a^+ = \max\{a,0\},\ sga = \left(\begin{matrix} 0\ for\ a<0 \\ 1\ for\ a\geq 0 \end{matrix}\right. . \quad (8)$$

Proof: The equation $\hat{y}_T = 0$ follows from no negativity of vector w (symbol "^" corresponds to some optimal solution). The denotation $\Phi_t = \sum_{\tau=1}^{t} \psi \bullet y_\tau$ is being used. The equation $\Phi = \Phi_{T-1}$ is true for optimization functional. We will find the optimal value for variable y with help of induction by decrease of variable t.

Step (T-1). We are getting formulas:

$\Phi = \Phi_{T-1} = \Phi_{T-2} + \psi \cdot y_{T-1} = \Phi_{T-2} + \varphi_{T-1} y_{T-1}$ for $\varphi_{T-1} = \psi$ The support task $\begin{cases} \varphi_{T-1} y_{T-1} \to \sup \\ 0 \leq y_{T-1} \leq A y_{T-2} \end{cases}$ has the solution in form: $\hat{y}_{T-1} = diag(sg\varphi_{T-1}) A y_{T-2}$. The general form for this solution is $\hat{y}_{T-1} = \hat{y}_{T-1}(y_{T-2})$. Next formula $\varphi_{T-1}\hat{y}_{T-1} = \varphi_{T-1}\hat{y}_{T-1}(y_{T-2}) = \varphi^+_{T-1} A y_{T-2}$ is being calculated. The formula $\Phi = \Phi_{T-2} + \varphi^+_{T-1} A y_{T-2}$ is true.

Step t. Let formula $\Phi = \Phi_T + \varphi^+_{t+1} A y_t$ be true. Then

$\Phi = \Phi_T + \psi y_t + \varphi^+_{t+1} A y_t = \Phi_{t-1} + \varphi_t y_t$ is true for $\varphi_t = \psi + \varphi^+_{t+1} A$

The solution of the subproblem $\begin{cases} \varphi_t y_t \to \sup \\ 0 \leq y_t \leq A y_{t-1} \end{cases}$ is forming

$\hat{y}_t = diag(sg\varphi_t) A y_{t-1}$ The formula $\Phi = \Phi_{t-1} + \varphi^+_t A y_{t-1}$ follows from previous formulas.

The lemma 2 is proved by induction by decrease of variable t. Corollary. The optimal solution in problem (2) exists and has form:

$$\begin{cases} \hat{x}_1 = \bar{x} \\ \hat{x}_{t+1} = A(diag(sg\varphi_t))\hat{x}_t, t=1,2,...,T-1 \\ \hat{u}_T = e \\ \hat{u}_t = \overline{sg}\varphi_t \end{cases} .$$

The denotations $\overline{sg}a = \begin{cases} 1, a<0 \\ 0, a\geq 0 \end{cases}$, $e = (1,1,...,1)$ are used. Proof is based on use of formulas (5). □

Theorem. The solution of the problem (2) with use formulas (3) has the time points t_j and sets $I_j \subset \{1,2,...,n\}$ of indexes for j=1,…,k by some $k \leq n$. These points and sets have next properties: $1 = t_1 < t_1 < ... < t_k = T$ and

$$\begin{cases} \hat{y}_{ti} = 0\ for\ i \in I_j \\ \hat{y}_{ti} > 0\ for\ i \notin I_j\ and\ t \in (t_j; t_{j+1}) \end{cases}, \ I_j \subset I_{j+1}.$$

Proof. The property

$\varphi_{T-1} - \varphi_{T-2} = \psi - (\psi + \varphi^+_{T-1} A) = -\varphi^+_{T-1} A \leq 0 \Rightarrow \varphi_{T-1} \leq \varphi_{T-2}$

follows from formulas (7). We have

$\varphi_t - \varphi_{t-1} = (\varphi^+_{t+1} - \varphi^+_t) A$ for $t = 2, 3, ..., T-2$. The inequalities

$$\varphi_{T-1} \leq \varphi_{T-2} \leq \varphi_1 \quad (8)$$

Follow from previous formula.

The results of the theorem follow from the lemma 2 and formula (8). The meaning of this theorem is in the turnpike phenomenon[6] for the solution of problem (2). This phenomenon consists in stabilization of the indexes of the system components which are taken in harvest. This stabilization is being discussed in next part.

Discussion

The variable y_{ti} denote the abundance of component i for time moment t after harvesting. The theorem means that the solution for optimal harvesting problem (2) has the index sets I_j of zero components after harvesting. These sets have the inclusion properties $I_j \subset I_{j+1}$. This situation is analogous for turnpike phenomenon in models of economic dynamics.[6] But the turnpike properties are specific in this case. These turnpike properties are true for indexes of harvesting components: if the component equals zero after harvest in some time point then this component equal zero in any next time points. The inclusion $I_j \subset I_{j+1}$ means that the new Biosystem components are being included in harvest with increase of time. For example this fact means a decrease of average age of individuals in population with age structure.

The first index set I_l denotes the lightest regime of harvesting. The time moments t_j for $j \geq 2$ go to infinity if interval for harvesting went to infinity $T \to \infty$. The lightest regime of harvesting dominates in this case. The long time planning interval is preferable to efficient use of biological resources.

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Conflicts of interest

None.

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