International Journal of Petrochemical Science & Engineering

Research Article

Open Access

Direct applications of homotopy perturbation method for solving nonlinear algebraic and transcendental equations

Abstract

In this work, homotopy perturbation method is directly applied to provide solutions to nonlinear algebraic and transcendental equations. The reliability and efficiency of the method in solving the nonlinear equations are demonstrated through different illustrative numerical examples. The method is shown to be conceptually and computationally simple and straightforward without any ambiguity. Also, the superiority of the direct applications of the approximate analytical method over the other methods shows that it does not require the development of any other iterative scheme that could be used to find the solutions to algebraic and transcendental equations. With the use of the homotopy perturbation method, there is no search for an auxiliary parameter for adjusting and controlling the rate and region of convergence of the solution. Additionally, the approximate analytical method does not entail the determination of Adomian polynomials and finding symbolic or numerical derivatives of any given function. The method does not require finding the correct fixed point and it is free from the problem of choosing an appropriate initial approximation. Therefore, it is hoped that the present work will assist in providing accurate solutions to many practical problems in science and engineering.

Keywords: nonlinear algebraic equations, root-finding method; homotopy perturbation method

Volume 6 Issue 1 - 2023

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Received: March 31, 2023 | **Published:** April 28, 2023

Introduction

The determinations of the roots of nonlinear algebraic equations are important aspects in providing solutions to many practical problems in science and engineering. In fact, providing exact solutions to the nonlinear equations has been among the oldest problems of mathematical methods. However, it is generally difficult to establish a general analytical method that provides exact solutions to nonlinear algebraic equations. Consequently, often time, recourse has been made to numerical methods in finding roots to polynomials, transcendental, and other nonlinear algebraic equations. In such mathematical adventures, bisection method is one of the oldest root-finding numerical methods. Although, the method is simple and its convergence is guaranteed, it is generally slow and works when the root to be estimated is of even-order. Choosing an initial guess or estimate close to root has no advantage in the application of the method and it may lead to carrying out many iterations to converge. Using method of regular falsi which seems to be an improvement over the bisection method, but unfortunately, the method yields an estimate without useable known error bound. Interestingly, the applications of linear fixed-point iteration method to find root of a nonlinear algebraic equations comes with an increased rate of convergence over bisection method and method of regular falsi. However, sometimes, selection of the correct fixed-point or iteration function poses serious challenges in the use of the numerical method. In fact, in fixed-point iteration method, convergence can be slow or non-existence. Newton-Raphson method is taken as the most popular root-finding numerical method with high rate of convergence. Unfortunately, the convergence of the Newton-Raphson method to the required solution is not guaranteed i.e., sometimes, for a given equation and for a given initial guess or estimate, one may not get the required solution. The method converges provided the initial approximation is chosen sufficiently close to the root, otherwise, the procedure may lead to an endless cycle. This shows that the method is very sensitive and grossly dependent on initial guess or starting values. Such an initial guess may be too far from the local root, and it may give a zero derivative and loop indefinitely. In fact, the Newton-Raphson method has poor global convergence properties. It converges slowly near local maxima and local minima, due to oscillation. Furthermore, such slow convergence is witnessed when Newton-Raphson is used for a problem with multiple roots. Moreover, the numerical method encounters problem when the value of the inherent derivative is very small or zero as such can lead to division by zero problem and inflection point issue can occur. Furthermore, root jumping might take place thereby not getting intended solution. Newton-Raphson method requires symbolic derivatives which might be difficult or virtually impossible to get especially for some complicated functions. Secant method has been used to obviate the symbolic derivative and derivative zero problem in Newton-Raphson method as the method does not require the derivatives of the given function. Although, the method is taken as one of the most economical numerical methods that give rapid convergence at a low cost, it requires two initial guesses or estimates for starting and it can produce erratic results when the approximations become close together. In fact, most often in numerical methods, choosing the right initial estimate(s), developing derivative(s) and the finding the correct fixed-point poses serious difficulties.

The limitations of the numerical methods as presented in the preceding sections show that the classical ways of finding analytical solutions (exact or approximate) to the nonlinear problems are still very much important. Although, as stated previously, it is very difficult to develop a general analytical method for solving nonlinear algebraic equations, there have been several submissions such as Cardan's method, Viète's, algebraic geometry, Ferrari's method, Descartes's method, Euler's method, Lagrange resolvent, etc. for the developments of exact solutions to polynomial equations. In fact, the past centuries have witnessed the establishments of various exact analytical solutions to quadratic, cubic and quartic equations.[1–8] However, in the early 19th century, Abel and Galois ingeniously and rigorously demonstrated in their impossibility theorem that there exists no general formula for zeroes of a polynomial equation of degree five or higher.[9] Therefore, the general quintic equation and other higher-order polynomials cannot be solved algebraically in terms of a finite number of additions, subtractions, multiplications, divisions, and root extractions. Such Abel's impossibility theorem puts a period to longstanding search for a 'magic' formula for polynomials of higher orders. However, in recent times, Waston's and Dummit's methods have been used to develop and establish analytical solutions to quintic equations while Buya's method and Hagedorn's method have been applied to solve sextic or hexic equations. Abel-Ruffin theorem and Kulkarni's method were put forward to provide solutions to octic equations while another Kulkarni's method was also developed to establish analytical solutions for nonic equations. De' Moivre theorem can be used to solve polynomial equation of any power but of a reduced form. Nonetheless, all these methods only provide analytical solutions to polynomial equations. At the other hand, transcendental equations and other nonlinear algebraic equations have been solved analytically with the aid of Lambert W-function. However, the method of Lambert W-function is used to solve the nonlinear equations in which the unknown appears both outside and inside an exponential function or a logarithm. Consequently, in the continuous quest of finding roots of nonlinear algebraic equations, there have been unending applications of numerical methods as the viable options, even with all their inherent limitations. However, one major gap in literature is that Abel's impossibility theorem did not state whether polynomial equations can be solved with infinite power series or not. Therefore, in recent years, an infinite power series approach such as Adomian perturbation method (ADM) has been used to find the roots of nonlinear algebraic equations and more, importantly to solve nonlinear differential equations.[10–20] With the aid of the ADM, the approximate solution of the nonlinear equation is considered as an infinite series converging to the accurate solution. However, such power series solution involves determination of Adomian polynomials which increases the computational effort and time. Its slow rate of convergence for problems of wide region or domain is a great shortcoming of the method. Homotopy perturbation method have been applied to develop some iterative methods to solve nonlinear algebraic equations.[21–24] Some other computational schemes have been developed[25–42] to solve nonlinear algebraic equations. A critical review of the developed methods in previous studies point to the fact that they numeric in nature which means that they are based on iterative or numerical procedures and on the idea of successive approximations that start with one or more initial approximations to the required roots. Also, many of the methods in the review works require symbolic derivatives which might be difficult to get in some complicated functions. Motivated by the above limitations and the gaps in the past works and to the best of the author's knowledge, it can be stated that homotopy perturbation method has not been directly applied to solve nonlinear algebraic equations, especially when the given equation does not have a linear term. Therefore, in this study, it is demonstrated that the direct applications of homotopy perturbation method is not only limited to solve nonlinear differential and integral equations but also, it is capable of solving nonlinear algebraic equations. Several numerical examples are given to show the reliability, performance and efficiency of the method in solving nonlinear algebraic equations.

Homotopy perturbation method

Homotopy perturbation method is a total analytical power series method for solving nonlinear equations. It is first proposed by He.[43] The method was also improved by He.[44–47] Its basic principle is stated in the next section.

The basic idea of homotopy perturbation method

In order to establish the basic idea behind homotopy perturbation method, consider a system of nonlinear differential equations given as

$$A(U)-f(r)=0, \quad r\in\Omega, \tag{1}$$

with the boundary conditions

$$B\left(u,\frac{\partial u}{\partial \eta}\right)=0, \qquad r\in\Gamma, \tag{2}$$

where A is a general differential operator, B is a boundary operator, $f(r)$ a known analytical function and Γ is the boundary of the domain Ω

The operator A can be decomposed or divided into two parts, which are L and N, where L is a linear operator, N is a non-linear operator. Eq. (1) can be therefore rewritten as follows

$$L(u)+N(u)-f(r)=0. \tag{3}$$

By the homotopy technique, a homotopy $U(r,p):\Omega\times[0,1]\to R$ can be constructed, which satisfies

$$H(U,p)=(1-p)\left[L(U)-L(U_o)\right]+p\left[A(U)-f(r)\right]=0, \quad p\in[0,1], \tag{4}$$

or

$$H(U,p)=L(U)-L(U_o)+pL(U_o)+p\left[N(U)-f(r)\right]=0. \tag{5}$$

In the above Eqs. (4) and (5), $p\in[0,1]$ is an embedding parameter, u_o is an initial approximation of equation of Eq.(1), which satisfies the boundary conditions.

Also, from Eqs. (4) and Eq. (5), we will have

$$H(U,0)=L(U)-L(U_o)=0, \tag{6}$$

or

$$H(U,1)=A(U)-f(r)=0. \tag{7}$$

The changing process of p from zero to unity is just that of $U(r,p)$ from $u_o(r)$ to $u(r)$. This is referred to deformation in topology. $L(U)-L(U_o)$ and $A(U)-f(r)$ are called homotopic.

Using the embedding parameter p as a small parameter, the solution of Eqs. (4) and Eq. (5) can be assumed to be written as a power series in p as given in Eq. (8)

$$U=U_o+pU_1+p^2U_2+\ldots \tag{8}$$

It should be pointed out that of all the values of p between 0 and 1, $p=1$ produces the best result. Therefore, setting $p=1$, results in the approximation solution of Eq. (9)

$$u=\lim_{p\to 1}U=U_o+U_1+U_2+\ldots \tag{9}$$

Therefore

$$u=U_o+U_1+U_2+\ldots \tag{10}$$

The series Eq. (10) is convergent for most cases.

The basic idea expressed above is a combination of homotopy and perturbation method. Hence, the method is called homotopy perturbation method (HPM), which has eliminated the limitations of the traditional perturbation methods. On the other hand, this technique can have full advantages of the traditional perturbation techniques.

Numerical examples

In order to demonstrate the simplicity, reliability and efficiency of the direct homotopy perturbation method in solving nonlinear algebraic equations, the following polynomial and transcendental equations are clearly solved as presented under this section as presented as follows:

Example 5.1: Find the roots of the following quadratic equation using homotopy perturbation method

$$x^2+4x+3=0. \tag{11}$$

The above equation can be expressed as

$$x+\frac{x^2}{4}+\frac{3}{4}=0 \tag{12}$$

In order to apply homotopy perturbation method, the equation is expressed as

$$(1-p)\left[x-v_0\right]+p\left[x+\frac{3}{4}+\frac{x^2}{4}\right]=0 \tag{13}$$

One can write the above Eq. (13) as

$$x-v_0+p\left[v_0\right]+p\left[\frac{x^2}{4}+\frac{3}{4}\right]=0 \tag{14}$$

Using the embedding parameter p as a small parameter, the solution of Eq. (11) can be assumed to be written as a power series in p as given in Eq. (15)

$$x=x_0+px_1+p^2x_2+p^3x_3+p^4x_4+p^5x_5+p^6x_6+p^7x_7+p^8x_8+p^9x_9+\ldots \tag{15}$$

On substituting Eq. (15) into Eq.(14), we have

$$\begin{gathered}x_0+px_1+p^2x_2+p^3x_3+p^4x_4+p^5x_5+p^6x_6+p^7x_7+p^8x_8+p^9x_9+\ldots-v_0+\\ p[v_0]+\frac{1}{4}p\left[\left(x_0+px_1+p^2x_2+p^3x_3+p^4x_4+p^5x_5+p^6x_6+p^7x_7+p^8x_8+p^9x_9+\ldots\right)^2+\frac{3}{4}\right]=0\end{gathered} \tag{16}$$

Arrange the equation according to the power of the embedding parameter p, we have

$$p^0:\quad x_0-v_0=0,$$

$$p^1:\quad x_1+v_0+\frac{1}{4}x_0^2+\frac{3}{4}=0,$$

$$p^2:\quad x_2+\frac{1}{4}\left(2x_0x_1\right)=0,$$

$$p^3:\quad x_3+\frac{1}{4}\left(x_1^2+2x_0x_2\right)=0,$$

$$p^4:\quad x_4+\frac{1}{4}\left(2x_0x_3+2x_1x_2\right)=0,$$

$$p^5:\quad x_5+\frac{1}{4}\left(x_2^2+2x_1x_3+2x_0x_4\right)=0,$$

$$p^6:\quad x_6+\frac{1}{4}\left(2x_0x_5+2x_1x_4+2x_2x_3\right)=0,$$

$$p^7:\quad x_7+\frac{1}{4}\left(x_3^2+2x_0x_6+2x_1x_5+2x_2x_4\right)=0,$$

$$p^8:\quad x_8+\frac{1}{4}\left(2x_0x_7+2x_1x_6+2x_2x_5+2x_3x_4\right)=0,$$

$$p^9:\quad x_9+\frac{1}{4}\left(x_4^2+2x_0x_8+2x_1x_7+2x_2x_6+2x_3x_5\right)=0,$$

Taking an initial approximation as $v_0=-0.5$

On solving the above equations, we have

$$\begin{gathered}x_0=-0.500000,\ x_1=-0.312500,\ x_2=-0.078125,\ x_3=-0.043945,\ x_4=-0.023193,\\ x_5=-0.014190,\ x_6=-0.008880,\ x_7=-0.005826,\ x_8=-0.003908,\ x_9=-0.002680\end{gathered} \tag{17}$$

Taking an initial approximation as $v_0=-0.75$

$$\begin{gathered}x_0=-0.75,\ x_1=-0.140625,\ x_2=-0.052734,\ x_3=-0.024719,\ x_4=-0.012977,\\ x_5=-0.007300,\ x_6=-0.004302,\ x_7=-0.002621,\ x_8=-0.001634,\ x_9=-0.001043\end{gathered} \tag{18}$$

From the basic principle of HPM,

$$x=x_0+px_1+p^2x_2+p^3x_3+p^4x_4+p^5x_5+p^6x_6+p^7x_7+p^8x_8+p^9x_9+\ldots \tag{19}$$

It should be pointed out that of all the values of p between 0 and 1, $p=1$ produces the best result. Therefore, setting $p=1$, results in the approximation solution of Eq. (9)

$$x=\lim_{p\to 1}x=x_0+x_1+x_2+x_3+x_4+x_5+x_6+x_7+x_8+x_9+\ldots\ldots \tag{20}$$

Which gives

$$x=x_0+x_1+x_2+x_3+x_4+x_5+x_6+x_7+x_8+x_9+\ldots\ldots \tag{21}$$

Therefore, when the initial approximation, $v_0=-0.5$ we have

$$\begin{gathered}x=-0.500000,-0.312500,-0.078125-0.043945-0.023193-0.014190\\ -0.008880,-0.005826-0.003908-0.002680=-0.993252\end{gathered} \tag{22}$$

And, when the initial approximation, $v_0=-0.75$, we have

$$\begin{gathered}x=-0.75-0.140625-0.052734-0.024719-0.012977-0.007300\\ -0.004302-0.002621-0.001634-0.001043+\ldots=-0.997955\end{gathered} \tag{23}$$

The above results show that the closer the initial approximation to the root of the equation, the more accurate is the result of the solution. However, the approach requires an initial estimate. In order to avoid this, one can write a modified homotopy perturbation method so that the scheme can be free from the problem of choosing an appropriate initial approximation.

The equation is expressed as

$$\left[L(u)+c\right]+pN(u)=0. \tag{24}$$

where c is the constant in the nonlinear equation,

$$x+\frac{3}{4}+p\left[\frac{x^2}{4}\right]=0 \tag{25}$$

As done previously, the solution of Eq. (11) can be assumed to be written as a power series in p as

$$x=x_0+px_1+p^2x_2+p^3x_3+p^4x_4+p^5x_5+p^6x_6+p^7x_7+p^8x_8+p^9x_9+\ldots \tag{26}$$

On substituting Eq. (26) into Eq.(25), one has

$$\begin{gathered}x_0+px_1+p^2x_2+p^3x_3+p^4x_4+p^5x_5+p^6x_6+p^7x_7+p^8x_8+p^9x_9+\ldots+\frac{3}{4}\\ +\frac{1}{4}p\left[\left(x_0+px_1+p^2x_2+p^3x_3+p^4x_4+p^5x_5+p^6x_6+p^7x_7+p^8x_8+p^9x_9+\ldots\right)^2\right]=0\end{gathered} \tag{27}$$

Arrange the equation according to the power of the embedding parameter p, we have

$$p^0:\quad x_0+\frac{3}{4}=0,$$

$$p^1: \quad x_1 + \frac{1}{4}x_0^2 = 0,$$

$$p^2: \quad x_2 + \frac{1}{4}\left(2x_0x_1\right) = 0,$$

$$p^3: \quad x_3 + \frac{1}{4}\left(x_1^2 + 2x_0x_2\right) = 0,$$

$$p^4: \quad x_4 + \frac{1}{4}\left(2x_0x_3 + 2x_1x_2\right) = 0,$$

$$p^5: \quad x_5 + \frac{1}{4}\left(x_2^2 + 2x_1x_3 + 2x_0x_4\right) = 0,$$

$$p^6: \quad x_6 + \frac{1}{4}\left(2x_0x_5 + 2x_1x_4 + 2x_2x_3\right) = 0,$$

$$p^7: \quad x_7 + \frac{1}{4}\left(x_3^2 + 2x_0x_6 + 2x_1x_5 + 2x_2x_4\right) = 0,$$

$$p^8: \quad x_8 + \frac{1}{4}\left(2x_0x_7 + 2x_1x_6 + 2x_2x_5 + 2x_3x_4\right) = 0,$$

$$p^9: \quad x_9 + \frac{1}{4}\left(x_4^2 + 2x_0x_8 + 2x_1x_7 + 2x_2x_6 + 2x_3x_5\right) = 0,$$

On solving the above equations, we have

$$\begin{aligned} &x_0 = -0.75,\ x_1 = -0.140625,\ x_2 = -0.052734,\ x_3 = -0.024719,\ x_4 = -0.012977, \\ &x_5 = -0.007300,\ x_6 = -0.004302,\ x_7 = -0.002621,\ x_8 = -0.001634,\ x_9 = -0.001043 \end{aligned} \tag{28}$$

Therefore

$$x = x_0 + x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + \ldots\ldots \tag{29}$$

From Eqs. (28) and (29), we have

$$\begin{aligned} x &= -0.75 - 0.140625 - 0.052734 - 0.024719 - 0.012977 - 0.007300 \\ &\quad - 0.004302 - 0.002621 - 0.001634 - 0.001043 + \ldots = -0.997955 \end{aligned}$$

The exact solutions for the roots of the equation are -1 and -3. The above solution shows that the results of the HPM is approaching a negative root of -1, which is one of the roots of the equation. It should be stated that the rate of convergence can be accelerated using Shank transformation (Table 1).

Table 1 Solution of the equation of example 3.1

Number of Iteration (n)	n-term Solution	Absolute Error
1	-0.75	0.25
2	-0.890625	0.109375
3	-0.943359	0.056641
4	-0.968078	0.031922
5	-0.981055	0.018945
6	-0.988355	0.011645
7	-0.992657	0.007343
8	-0.995278	0.004722
9	-0.996912	0.003088
10	-0.997995	0.002045

Example 5.2: Find the roots of the following cubic equation using homotopy perturbation method

$$x^3 - 5x + 3 = 0\,. \tag{30}$$

The above equation can be expressed as

$$x - \frac{x^3}{5} - \frac{3}{5} = 0 \tag{31}$$

In order to apply homotopy perturbation method, the equation is expressed as

$$x - \frac{3}{5} + p\left[-\frac{x^3}{5}\right] = 0 \tag{32}$$

Which can be written as

$$x - \frac{3}{5} - p\left[\frac{x^3}{5}\right] = 0 \tag{33}$$

According to the procedure of HPM, the solution of Eq. (30) can be assumed to be written as a power series in p as

$$x = x_0 + px_1 + p^2x_2 + p^3x_3 + p^4x_4 + p^5x_5 + p^6x_6 + \ldots \tag{34}$$

After substitution of Eq. (34) into Eq.(33), on arrives at

$$\begin{aligned} &x_0 + px_1 + p^2x_2 + p^3x_3 + p^4x_4 + p^5x_5 + p^6x_6 + \ldots - \frac{3}{5} \\ &-\frac{1}{5}p\left[\left(x_0 + px_1 + p^2x_2 + p^3x_3 + p^4x_4 + p^5x_5 + p^6x_6 + \ldots\right)^3\right] = 0 \end{aligned} \tag{35}$$

Eq. (35) can be arranged according to the power of the embedding parameter p as

$$p^0: \quad x_0 - \frac{3}{5} = 0,$$

$$p^1: \quad x_1 - \frac{1}{5}x_0^3 = 0,$$

$$p^2: \quad x_2 - \frac{1}{5}\left(3x_0^2x_1\right) = 0,$$

$$p^3: \quad x_3 - \frac{1}{5}\left(3x_0^2x_2 + 3x_1^2x_0\right) = 0,$$

$$p^4: \quad x_4 - \frac{1}{5}\left(3x_0^2x_3 + x_1^3 + 6x_0x_1x_2\right) = 0,$$

$$p^5: \quad x_5 - \frac{1}{5}\left(3x_0^2x_4 + 3x_0x_2^2 + 6x_0x_1x_3 + 3x_1^2x_2\right) = 0,$$

$$p^6: \quad x_6 - \frac{1}{5}\left(3x_0^2x_5 + 6x_0x_2x_3 + 6x_0x_1x_4 + 3x_1x_2^2 + 3x_1^2x_3\right) = 0,$$

The solutions of the above equations are

$$\begin{aligned} &x_0 = 0.600000,\ x_1 = 0.0432000,\ x_2 = 0.0093312,\ x_3 = 0.0026874, \\ &x_4 = 0.0008868,\ x_5 = 0.0003169,\ x_6 = 0.0001194 \end{aligned} \tag{36}$$

From the basic principle of HPM,

$$x = 0.600000 + 0.0432000 + 0.0093312 + 0.0026874 + 0.0008868 + 0.0003169 + 0.0001194 + \ldots = 0.6565417$$

The exact solutions of the roots of the given equation are -2.49086362, 0.6566204, 1.83424318. Table 2 shows that the solution through the HPM is approaching the lowest positive root of 0.6566204.

Table 2 Solution of the equation of example 3.2

Number of Iteration (n)	n-term Solution	Absolute Error
1	0.6	0.0566204
2	0.6432	0.0134204
3	0.6525312	0.0040892
4	0.6552186	0.0014018
5	0.6561054	0.000515
6	0.6564223	0.0001981
7	0.6565417	0.0000787

Example 5.3: Determine the root of the following cubic equation with the aid of homotopy perturbation method

$$x^3 - 7x^2 + 14x - 6 = 0\,. \tag{37}$$

The above equation can be expressed as

$$x - \frac{3}{7} + \frac{x^3}{14} - \frac{x^2}{2} = 0 \tag{38}$$

In order to apply homotopy perturbation method, the equation is expressed as

$$x - \frac{3}{7} + p\left[\frac{x^3}{14} - \frac{x^2}{2}\right] = 0 \tag{39}$$

Following the HPM procedure, the solution of Eq. (37) can be assumed to be written as a power series in p as

$$x = x_0 + px_1 + p^2x_2 + p^3x_3 + p^4x_4 + p^5x_5 + p^6x_6 + \ldots \tag{40}$$

When Eq. (40) is substituted into Eq. (39), we have

$$x_0 + px_1 + p^2x_2 + p^3x_3 + p^4x_4 + p^5x_5 + p^6x_6 + \ldots - \frac{3}{7} + p\left[\begin{array}{c}\frac{\left(x_0 + px_1 + p^2x_2 + p^3x_3 + p^4x_4 + p^5x_5 + p^6x_6 + \ldots\right)^3}{14} \\ -\frac{\left(x_0 + px_1 + p^2x_2 + p^3x_3 + p^4x_4 + p^5x_5 + p^6x_6 + \ldots\right)^2}{2}\end{array}\right] = 0 \tag{41}$$

On arranging the Eq. (41) according to the power of the embedding parameter p, gives

$$p^0: \quad x_0 - \frac{3}{7} = 0,$$

$$p^1: \quad x_1 + \frac{1}{14}x_0^3 - \frac{1}{2}x_0^2 = 0,$$

$$p^2: \quad x_2 + \frac{1}{14}\left(3x_0^2x_1\right) - \frac{1}{2}\left(2x_0x_1\right) = 0,$$

$$p^3: \quad x_3 + \frac{1}{14}\left(3x_0^2x_2 + 3x_1^2x_0\right) - \frac{1}{2}\left(x_1^2 + 2x_0x_2\right) = 0,$$

$$p^4: \quad x_4 + \frac{1}{14}\left(3x_0^2x_3 + x_1^3 + 6x_0x_1x_2\right) - \frac{1}{2}\left(2x_0x_3 + 2x_1x_2\right) = 0,$$

$$p^5: \quad x_5 + \frac{1}{14}\left(3x_0^2x_4 + 3x_0x_2^2 + 6x_0x_1x_3 + 3x_1^2x_2\right) - \frac{1}{2}\left(x_2^2 + 2x_1x_3 + 2x_0x_4\right) = 0,$$

$$p^6: \quad x_6 + \frac{1}{14}\left(3x_0^2x_5 + 6x_0x_2x_3 + 6x_0x_1x_4 + 3x_1x_2^2 + 3x_1^2x_3\right) - \frac{1}{2}\left(2x_0x_5 + 2x_1x_4 + 2x_2x_3\right) = 0,$$

The solutions of the above equations are

$$\begin{array}{l} x_0 = 0.4285714,\ x_1 = 0.0862141,\ x_2 = 0.0335563,\ x_3 = 0.0160944, \\ x_4 = 0.0080918,\ x_5 = 0.0046883,\ x_6 = 0.0029661 \end{array} \tag{42}$$

Therefore, we have

$$x = 0.4285714 + 0.0862141 + 0.0335563 + 0.0160944 + 0.0080918 + 0.0046883 + 0.0029661 + \ldots = 0.5801824$$

The exact solutions of the roots of the given equation are 0.5857864, 3.000000 and 3.4142135. The solution of the equation using HPM is approaching the lowest positive root of 0.5857864 as the absolute error is approaching zero (Table 3).

Table 3 Solution of the equation of example 3.3

Number of Iteration (n)	n-term Solution	Absolute Error
1	0.4285714	0.157215
2	0.5147855	0.0710009
3	0.5483418	0.0374446
4	0.5644362	0.0213502
5	0.572528	0.0132584
6	0.5772163	0.0085701
7	0.5801824	0.005604

Example 5.4: Solve the following cubic equation using homotopy perturbation method

$$x^3 + 4x^2 + 8x + 8 = 0\,. \tag{43}$$

The above equation can be expressed as

$$x + 1 + \frac{x^3}{8} + \frac{x^2}{2} = 0 \tag{44}$$

As before, in order to apply homotopy perturbation method, the Eq. (44) can be expressed as

$$x + 1 + p\left[\frac{x^3}{8} + \frac{x^2}{2}\right] = 0 \tag{45}$$

In a similar way, one can say that the solution of Eq. (43) can be expressed as

$$x = x_0 + px_1 + p^2x_2 + p^3x_3 + p^4x_4 + p^5x_5 + p^6x_6 + \ldots \tag{46}$$

On substituting Eq. (456) into Eq.(45), we have

$$x_0 + px_1 + p^2x_2 + p^3x_3 + p^4x_4 + p^5x_5 + p^6x_6 + \ldots + 1 + p\left[\begin{array}{c}\frac{\left(x_0 + px_1 + p^2x_2 + p^3x_3 + p^4x_4 + p^5x_5 + p^6x_6 + \ldots\right)^3}{8} \\ +\frac{\left(x_0 + px_1 + p^2x_2 + p^3x_3 + p^4x_4 + p^5x_5 + p^6x_6 + \ldots\right)^2}{2}\end{array}\right] = 0 \tag{47}$$

Arrange the equation according to the power of the embedding parameter p, produces

$$p^0: \quad x_0 + 1 = 0,$$

$$p^1: \quad x_1 + \frac{1}{8}x_0^3 + \frac{1}{2}x_0^2 = 0,$$

$$p^2: \quad x_2 + \frac{1}{8}\left(3x_0^2x_1\right) + \frac{1}{2}\left(2x_0x_1\right) = 0,$$

$$p^3: \quad x_3 + \frac{1}{8}\left(3x_0^2x_2 + 3x_1^2x_0\right) + \frac{1}{2}\left(x_1^2 + 2x_0x_2\right) = 0,$$

$$p^4: \quad x_4 + \frac{1}{8}\left(3x_0^2x_3 + x_1^3 + 6x_0x_1x_2\right) + \frac{1}{2}\left(2x_0x_3 + 2x_1x_2\right) = 0,$$

$$p^5: \quad x_5 + \frac{1}{8}\left(3x_0^2x_4 + 3x_0x_2^2 + 6x_0x_1x_3 + 3x_1^2x_2\right) + \frac{1}{2}\left(x_2^2 + 2x_1x_3 + 2x_0x_4\right) = 0,$$

$$p^6: \; x_6 + \frac{1}{8}\left(3x_0^2 x_5 + 6x_0 x_2 x_3 + 6x_0 x_1 x_4 + 3x_1 x_2^2 + 3x_1^2 x_3\right) + \frac{1}{2}\left(2x_0 x_5 + 2x_1 x_4 + 2x_2 x_3\right) = 0,$$

When the above equations are solved, we have

$$x_0 = -1.0000000, \; x_1 = -0.37500000, \; x_2 = -0.23437500, \; x_3 = -0.1640625, \; x_4 = -0.1179199, \; x_5 = -0.0835876, \tag{48}$$

Therefore, we have

$$x = (-1.0000000) + (-0.37500000) + (-0.23437500) + (-0.1640625) + (-0.1179199) + (-0.0835876) + ... = -1.974945$$

The exact solutions of the roots of the given equation are -2.00000, -1+ $\sqrt{3i}$ and -1- $\sqrt{3i}$. The result of the HPM is approaching the real root of -2.0000 as the absolute error is reducing to 0. The rate of convergence can be accelerated using Shank transformation (Table 4).

Table 4 Solution of the equation of example 3.4

Number of Iteration (n)	n-term Solution	Absolute Error
1	-1	1
2	-1.375	0.625
3	-1.609375	0.390625
4	-1.7734375	0.2265625
5	-1.8913574	0.1086426
6	-1.974945	0.025055

Example 5.5: Solve the following quintic equation using homotopy perturbation method

$$x^5 - 3x^4 + 2x^3 + 5x^2 - 6x - 4 = 0 . \tag{49}$$

Eq. (4) can be rearranged as

$$x + \frac{2}{3} - \frac{x^5}{6} + \frac{x^4}{2} - \frac{x^3}{3} - \frac{5x^2}{6} = 0 \tag{50}$$

The homotopy perturbation method is applied to write the Eq. (50) as

$$x + \frac{2}{3} + p\left[-\frac{x^5}{6} + \frac{x^4}{2} - \frac{x^3}{3} - \frac{5x^2}{6}\right] = 0 \tag{51}$$

Following the usual procedures of HPM, the solution of Eq. (51) is written as a power series in *p* as

$$x = x_0 + px_1 + p^2 x_2 + p^3 x_3 + p^4 x_4 + p^5 x_5 + p^6 x_6 + ... \tag{52}$$

The substitution of Eq. (52) into Eq.(51) produces

$$x_0 + px_1 + p^2 x_2 + p^3 x_3 + p^4 x_4 + p^5 x_5 + p^6 x_6 + ... + \frac{2}{3}$$
$$+p\left[\begin{array}{l} -\dfrac{\left(x_0 + px_1 + p^2 x_2 + p^3 x_3 + p^4 x_4 + p^5 x_5 + p^6 x_6 + ...\right)^5}{6} \\ +\dfrac{\left(x_0 + px_1 + p^2 x_2 + p^3 x_3 + p^4 x_4 + p^5 x_5 + p^6 x_6 + ...\right)^4}{2} \\ -\dfrac{\left(x_0 + px_1 + p^2 x_2 + p^3 x_3 + p^4 x_4 + p^5 x_5 + p^6 x_6 + ...\right)^3}{3} \\ -\dfrac{5\left(x_0 + px_1 + p^2 x_2 + p^3 x_3 + p^4 x_4 + p^5 x_5 + p^6 x_6 + ...\right)^2}{6} \end{array}\right] = 0 \tag{53}$$

On arranging Eq.(53) according to the power of the embedding parameter *p*, we have

$$p^0: \; x_0 + \frac{2}{3} = 0,$$

$$p^1: \; x_1 - \frac{1}{6}x_0^5 + \frac{1}{2}x_0^4 - \frac{1}{3}x_0^3 - \frac{5}{6}x_0^2 = 0,$$

$$p^2: \; x_2 - \frac{1}{6}\left(5x_0^4 x_1\right) + \frac{1}{2}\left(4x_0^3 x_1\right) - \frac{1}{3}\left(3x_0^2 x_1\right) - \frac{5}{6}\left(2x_0 x_1\right) = 0,$$

$$p^3: \; x_3 - \frac{1}{6}\left(10x_0^3 x_1^2 + 5x_0^4 x_2\right) + \frac{1}{2}\left(4x_0^3 x_2 + 6x_0^2 x_1^2\right) - \frac{1}{3}\left(3x_0^2 x_2 + 3x_1^2 x_0\right) - \frac{5}{6}\left(x_1^2 + 2x_0 x_2\right) = 0,$$

$$p^4: \; x_4 - \frac{1}{6}\left(10x_0^2 x_1^3 + 20x_0^3 x_1 x_2 + 5x_0^4 x_3\right) + \frac{1}{2}\left(4x_0^3 x_3 + 12x_0^2 x_1 x_2 + 4x_0 x_1^3\right)$$
$$-\frac{1}{3}\left(3x_0^2 x_3 + x_1^3 + 6x_0 x_1 x_2\right) - \frac{5}{6}\left(2x_0 x_3 + 2x_1 x_2\right) = 0,$$

$$p^5: \; x_5 - \frac{1}{6}\left(5x_0 x_1^4 + 30x_0^2 x_1^2 x_2 + 10x_0^3 x_2^2 + 20x_0^3 x_1 x_3 + 5x_0^4 x_4\right) + \frac{1}{2}\left(4x_0^3 x_4 + 12x_0^2 x_1 x_3 + 6x_0^2 x_2^2 + 12x_0 x_1^2 x_2 + x_1^4\right)$$
$$-\frac{1}{3}\left(3x_0^2 x_4 + 3x_0 x_2^2 + 6x_0 x_1 x_3 + 3x_1^2 x_2\right) - \frac{5}{6}\left(x_2^2 + 2x_1 x_3 + 2x_0 x_4\right) = 0,$$

The solutions of the above equations give

$$x_0 = -0.66666667, \; x_1 = 0.15089163, \; x_2 = 0.01366097, \; x_3 = -0.03656980, \tag{54}$$

Therefore, we have

$$x = (-0.66666667) + (0.15089163) + (0.01366097) + (-0.03656980) + ... = -0.53868386$$

The exact solutions of the roots of the given equation are -0.528886049, -1.09890396, 1.76518196. It can be seen that the above solution shows that the scheme is approaching the real root of -0.528886049 as the absolute error is approaching 0 (Table 5).

Table 5 Solution of the equation of example 3.5

Number of Iteration (n)	n-term Solution	Absolute Error
1	-0.66666667	0.137780621
2	-0.51577504	0.013111009
3	-0.50211407	0.026771979
4	-0.53868387	0.009797821

Example 5.6: Solve the following nonlinear algebraic simultaneous equations using homotopy perturbation method

$$x^2 - 10x + y^2 + 8 = 0 . \tag{55}$$

$$xy^2 + x - 10y + 8 = 0 \tag{56}$$

The above equation can be expressed as

$$x - \frac{4}{5} - \frac{x^2}{10} - \frac{y^2}{10} = 0 \tag{57}$$

$$y - \frac{4}{5} - \frac{x}{10} - \frac{xy^2}{10} = 0 \tag{58}$$

According to the definitions of HPM, Eqs. (57) and (58) can be written as

$$x - \frac{4}{5} + p\left[-\frac{x^2}{10} - \frac{y^2}{10}\right] = 0 \tag{59}$$

$$y - \frac{4}{5} + p\left[-\frac{x}{10} - \frac{xy^2}{10}\right] = 0 \tag{60}$$

The solutions of Eqs. (59) and (60) can be assumed to be written

$$x = x_0 + px_1 + p^2 x_2 + p^3 x_3 + p^4 x_4 + p^5 x_5 + p^6 x_6 + ... \tag{61}$$

$$y = y_0 + py_1 + p^2 y_2 + p^3 y_3 + p^4 y_4 + p^5 y_5 + p^6 y_6 + ... \quad (62)$$

On substituting Eqs. (61) and (62) into Eq.(59) and (60), we have

$$x_0 + px_1 + p^2x_2 + p^3x_3 + p^4x_4 + p^5x_5 + p^6x_6 + ... - \frac{4}{5} + p\left[\frac{-\left(x_0 + px_1 + p^2x_2 + p^3x_3 + p^4x_4 + p^5x_5 + p^6x_6 + ...\right)^2}{10} - \frac{\left(y_0 + py_1 + p^2y_2 + p^3y_3 + p^4y_4 + p^5y_5 + p^6y_6 + ...\right)^2}{10} \right] = 0 \quad (63)$$

$$y_0 + py_1 + p^2y_2 + p^3y_3 + p^4y_4 + p^5y_5 + p^6y_6 + ... - \frac{4}{5} + p\left[\frac{-\left(x_0 + px_1 + p^2x_2 + p^3x_3 + p^4x_4 + p^5x_5 + p^6x_6 + ...\right)}{10} - \frac{\left\{ \begin{array}{l} \left(x_0 + px_1 + p^2x_2 + p^3x_3 + p^4x_4 + p^5x_5 + p^6x_6 + ...\right) \\ \left(y_0 + py_1 + p^2y_2 + p^3y_3 + p^4y_4 + p^5y_5 + p^6y_6 + ...\right)^2 \end{array} \right\}}{10} \right] = 0 \quad (64)$$

When the Eqs. (63) and (64) are arranged according to the power of the embedding parameter p, one arrives at

$$p^0: \quad x_0 - \frac{4}{5} = 0, \quad y_0 - \frac{4}{5} = 0$$

$$p^1: \quad x_1 - \frac{1}{10}x_0^2 - \frac{1}{10}y_0^2 = 0, \quad y_1 - \frac{1}{10}x_0 - \frac{1}{10}xy_0^2 = 0,$$

$$p^2: \quad x_2 - \frac{1}{10}(2x_0x_1) - \frac{1}{10}(2y_0y_1) = 0, \quad y_2 - \frac{1}{10}x_1 - \frac{1}{10}\left(x_1y_0^2\right) = 0$$

$$p^3: \quad x_3 - \frac{1}{10}\left(x_1^2 + 2x_0x_2\right) - \frac{1}{10}\left(y_1^2 + 2y_0y_2\right) = 0, \quad y_3 - \frac{1}{10}x_2 - \frac{1}{10}\left(x_0y_1^2 + 2x_1y_0y_1\right) = 0,$$

On solving the above equations to the sixth power of the embedding parameter p, we have

$$x = 0.997853, \quad y = 0.997562 \quad (65)$$

The exact solutions of the roots of the given equation are $x = 1$ and $y = 1$. The absolute errors in the approximate solutions are 0.002147 and 0.0024380. It can be seen that the above solution shows that the scheme is approaching the roots of the equations as the absolute errors approach zero.

Example 5.7: Find the roots of the following transcendental equation using homotopy perturbation method

$$x - 2 - e^{-x} = 0 . \quad (66)$$

In order to apply homotopy perturbation method, the equation is expressed as

$$x - 2 + p\left[-e^{-x}\right] = 0 \quad (67)$$

The solution of Eq. (67) can be assumed to be written

$$x = x_0 + px_1 + p^2x_2 + p^3x_3 + p^4x_4 + p^5x_5 + p^6x_6 + ... \quad (68)$$

The substitution Eq. (68) into Eq.(67) gives

$$x_0 + px_1 + p^2x_2 + p^3x_3 + p^4x_4 + p^5x_5 + p^6x_6 + ... - 2 - p\left[e^{-\left(x_0 + px_1 + p^2x_2 + p^3x_3 + p^4x_4 + p^5x_5 + p^6x_6 + ...\right)}\right] = 0 \quad (69)$$

A further simplification of the above equation produces

$$x_0 + px_1 + p^2x_2 + p^3x_3 + p^4x_4 + p^5x_5 + p^6x_6 + ... - 2 - p\left[e^{-x_0}e^{-\left(px_1 + p^2x_2 + p^3x_3 + p^4x_4 + p^5x_5 + p^6x_6 + ...\right)}\right] = 0 \quad (70)$$

The expression in the block bracket can be expanded with the aid of Taylor series as

$$x_0 + px_1 + p^2x_2 + p^3x_3 + p^4x_4 + p^5x_5 + p^6x_6 + ... - 2 + p\left[e^{-x_0} \left[\begin{array}{l} 1 - \left[px_1 + p^2x_2 + p^3x_3 + p^4x_4 + p^5x_5 + p^6x_6 + ...\right] \\ + \dfrac{\left[px_1 + p^2x_2 + p^3x_3 + p^4x_4 + p^5x_5 + p^6x_6 + ...\right]}{2!} \\ - \dfrac{\left[px_1 + p^2x_2 + p^3x_3 + p^4x_4 + p^5x_5 + p^6x_6 + ...\right]}{3!} \\ + \dfrac{\left[px_1 + p^2x_2 + p^3x_3 + p^4x_4 + p^5x_5 + p^6x_6 + ...\right]}{4!} \\ - \dfrac{\left[px_1 + p^2x_2 + p^3x_3 + p^4x_4 + p^5x_5 + p^6x_6 + ...\right]}{5!} \end{array} \right] \right] = 0 \quad (71)$$

The arrangement of Eq.(71) according to the power of the embedding parameter p gives

$$p^0: \quad x_0 - \frac{2}{3} = 0,$$

$$p^1: \quad x_1 - e^{-x_0} = 0,$$

$$p^2: \quad x_2 - x_1e^{-x_0} = 0,$$

$$p^3: \quad x_3 - x_2e^{-x_0} + \frac{x_1^2}{2!}e^{-x_0} = 0,$$

$$p^4: \quad x_4 - x_3e^{-x_0} + x_1x_2e^{-x_0} - \frac{x_1^3}{3!}e^{-x_0} = 0,$$

$$p^5: \quad x_5 - x_4e^{-x_0} + \frac{\left(2x_1x_3 + x_2^2\right)}{2!}e^{-x_0} - \frac{3x_1^2x_2}{3!}e^{-x_0} + \frac{x_1^4}{4!}e^{-x_0} = 0,$$

$$p^6: \quad x_6 - x_5e^{-x_0} + \frac{\left(2x_2x_3 + 2x_1x_4\right)}{2!}e^{-x_0} - \frac{\left(3x_1^2x_3 + 3x_1x_2^2\right)}{3!}e^{-x_0} + \frac{4x_1^3x_2}{4!}e^{-x_0} - \frac{x_1^5}{5!}e^{-x_0} = 0,$$

On solving the above equations, we have

$x_0 = 2.00000000000$, $x_1 = 0.13533528323366$, $x_2 = -0.01831563889$, $x_3 = 0.003718128265$, $x_4 = -0.0008945670078$ $x_5 = 0.0002364579676$, $x_6 = -0.0000663574935$, $x_7 = 0.0000194104211$ $x_8 = -0.0000058532581$, $x_9 = 0.0000018066598$,

Therefore, we have

$$\begin{array}{l} x = 2.00000000000 + 0.13533528323366 - 0.01831563889 + 0.003718128265 \\ \quad - 0.0008945670078 + 0.0002364579676 - 0.0000663574935, +0.0000194104211 \\ \quad - 0.0000058532581 + 0.0000018066598 + ... = 2.1200286699020 \end{array} \quad (72)$$

The exact solutions of the roots of the given equation are 2.1200282389876. It can be seen that the above solution shows that the scheme approaches the exact solution (Table 6).

Table 6 Solution of the equation in example 3.6

Number of Iteration (n)	n-term Solution	Absolute Error
1	2	0.1200282
2	2.135335283	0.015307
3	2.117019644	0.026772
4	2.120737773	0.0007095
5	2.119843206	0.000185
6	2.120079664	5.14E-05
7	2.120013306	1.49E-05
8	2.120032717	4.48E-06
9	2.120026863	1.38E-06
10	2.12002867	4.31E-07

Example 5.8: Determine the root of the following transcendental equation using homotopy perturbation method

$$e^x - x^2 - 3x + 2 = 0 . \tag{73}$$

The above equation can be written as

$$x - \frac{2}{3} + \frac{x^2}{3} - \frac{e^x}{3} = 0 . \tag{74}$$

In order to apply homotopy perturbation method, we can write Eq. (74) as

$$x - \frac{2}{3} + p\left[\frac{x^2}{3} - \frac{e^x}{3}\right] = 0 \tag{75}$$

The assumed solution can be written

$$x = x_0 + px_1 + p^2x_2 + p^3x_3 + p^4x_4 + p^5x_5 + p^6x_6 + \dots \tag{76}$$

From Eq. (76), Eq. (75) can be written as

$$x_0 + px_1 + p^2x_2 + p^3x_3 + p^4x_4 + p^5x_5 + p^6x_6 + \dots - \frac{2}{3}$$
$$-p\left[\frac{\left(x_0 + px_1 + p^2x_2 + p^3x_3 + p^4x_4 + p^5x_5 + p^6x_6 + \dots\right)^2}{3} - \frac{e^{\left(x_0 + px_1 + p^2x_2 + p^3x_3 + p^4x_4 + p^5x_5 + p^6x_6 + \dots\right)}}{3}\right] = 0 \tag{77}$$

Eq. (77) can be written as after applying Taylor series to the second function in the block bracket

$$x_0 + px_1 + p^2x_2 + p^3x_3 + p^4x_4 + p^5x_5 + p^6x_6 + \dots - \frac{2}{3}$$
$$+\frac{1}{3}p\left[\begin{array}{l}\left(x_0 + px_1 + p^2x_2 + p^3x_3 + p^4x_4 + p^5x_5 + p^6x_6 + \dots\right)^2 \\ -e^{x_0}\left[\begin{array}{l}1 + \left[px_1 + p^2x_2 + p^3x_3 + p^4x_4 + p^5x_5 + p^6x_6 + \dots\right] \\ + \dfrac{\left[px_1 + p^2x_2 + p^3x_3 + p^4x_4 + p^5x_5 + p^6x_6 + \dots\right]}{2!} \\ + \dfrac{\left[px_1 + p^2x_2 + p^3x_3 + p^4x_4 + p^5x_5 + p^6x_6 + \dots\right]}{3!} \\ + \dfrac{\left[px_1 + p^2x_2 + p^3x_3 + p^4x_4 + p^5x_5 + p^6x_6 + \dots\right]}{4!} \\ + \dfrac{\left[px_1 + p^2x_2 + p^3x_3 + p^4x_4 + p^5x_5 + p^6x_6 + \dots\right]}{5!}\end{array}\right]\end{array}\right] = 0 \tag{78}$$

Arranging Eq. (77) according to the power of the embedding parameter p, we have

$$p^0: \quad x_0 - 2 = 0,$$

$$p^1: \quad x_1 + \frac{1}{3}x_0^2 - \frac{1}{3}e^{x_0} = 0,$$

$$p^2: \quad x_2 + \frac{1}{3}\left(2x_0x_1\right) - \frac{1}{3}x_1e^{x_0} = 0,$$

$$p^3: \quad x_3 + \frac{1}{3}\left(x_1^2 + 2x_0x_2\right) - \frac{1}{3}\left(x_2 + \frac{x_1^2}{2!}\right)e^{x_0} = 0,$$

$$p^4: \quad x_4 + \frac{1}{3}\left(2x_0x_3 + 2x_1x_2\right) - \frac{1}{3}\left(x_3 + x_1x_2 + \frac{x_1^3}{3!}\right)e^{x_0} = 0,$$

$$p^5: \quad x_5 + \frac{1}{3}\left(x_2^2 + 2x_1x_3 + 2x_0x_4\right) - \frac{1}{3}\left(x_4 + \frac{\left(2x_1x_3 + x_2^2\right)}{2!} + \frac{3x_1^2x_2}{3!} + \frac{x_1^4}{4!}\right)e^{x_0} = 0,$$

$$p^6: \quad x_6 + \frac{1}{3}\left(2x_0x_5 + 2x_1x_4 + 2x_2x_3\right) - \frac{1}{3}\left(x_5 + \frac{\left(2x_2x_3 + 2x_1x_4\right)}{2!} + \frac{\left(3x_1^2x_3 + 3x_1x_2^2\right)}{3!} + \frac{4x_1^3x_2}{4!} + \frac{x_1^5}{5!}\right)e^{x_0} = 0,$$

On solving the above equations, we have

$$x_0 = 0.666667,\ x_1 = -0.501097,\ x_2 = 0.102625,\ x_3 = -0.01883,$$
$$x_4 = 0.016575\ \ x_5 = -0.013209,\ x_6 = 0.007343,\ x_7 = -0.004327$$
$$x_8 = 0.003109,\ x_9 = -0.002240,$$

Therefore, we have

$$x = 0.666667 - 0.501097 + 0.102625 - 0.01883 + 0.016575 - 0.013209$$
$$+0.007343 - 0.004327 + 0.003109 - 0.002240 + \dots = 0.256616 \tag{79}$$

The exact solutions of the roots of the given equation are 0.257530. It can be seen that the above solution shows that the scheme approaches the exact solution (Table 7).

Table 7 Solution of the equation in example 3.7

Number of Iteration (n)	n-term Solution	Absolute Error
1	0.666667	0.409137
2	0.165557	0.091973
3	0.268195	0.010665
4	0.249365	0.008165
5	0.26594	0.00841
6	0.252731	4.80E-03
7	0.260074	2.54E-03
8	0.255747	1.78E-03
9	0.258856	1.33E-03
10	0.256616	9.14E-04

It could be seen from the above examples and from the procedures of homotopy perturbation method that a linear term must be in the equation for the HPM to operate. However, it is found that in some equations, there is no presence of a linear term. Under such scenario, the application of HPM will fail except a kind of modification is done to the given equation to include an artificial linear term. Therefore, in the subsequent examples such problems will be handled.

In order to treat such problems, we adopt that the general nonlinear equation is in the form

$$Lu + Ru + Nu = c \tag{80}$$

The linear terms are decomposed into $L + R$, with L taken as the first linear term which is easily and R as the remainder of the linear operator apart from L. where c is the constant in the equation and u is the variable, Nu represents the nonlinear terms.

Example 5.9: Find the roots of the following cubic equation using homotopy perturbation method

$$x^3 + 4x^2 - 5 = 0 . \tag{81}$$

The above equation does not contain a linear term. A linear term with a convenient coefficient can be introduced to make the equation be easily amendable to the form that homotopy perturbation can easily be applied. Therefore, we write

$$x^3 + 4x^2 + ax - ax - 5 = 0 \quad (82)$$

For the choice of "a", the ratio of the constant in the given equation and that of "a" must be within two values where the needed root of the equation lies. Therefore, we have $0 \le \frac{5}{a} \le$. In this example, we choice $a=8$.

$$x^3 + 4x^2 + 8x - 8x - 5 = 0 \quad (83)$$

We can rewrite Eq. (82) as

$$x - \frac{5}{8} + \frac{x^3}{8} + \frac{x^2}{2} - x = 0 \quad (84)$$

In order to apply homotopy perturbation method, the Eq.(84) is expressed as

$$x - \frac{5}{8} + p\left[\frac{x^3}{8} + \frac{x^2}{2} - x\right] = 0 \quad (85)$$

As done in the previous examples, the solution of the given nonlinear algebraic model can be written as

$$x = x_0 + px_1 + p^2x_2 + p^3x_3 + p^4x_4 + p^5x_5 + p^6x_6 + \dots \quad (86)$$

After the substitution of Eq. (86) into Eq. (85), one arrives at

$$x_0 + px_1 + p^2x_2 + p^3x_3 + p^4x_4 + p^5x_5 + p^6x_6 + \dots - \frac{5}{8}$$
$$+p\left[\begin{array}{l}\frac{\left(x_0 + px_1 + p^2x_2 + p^3x_3 + p^4x_4 + p^5x_5 + p^6x_6 + \dots\right)^3}{8} \\ +\frac{\left(x_0 + px_1 + p^2x_2 + p^3x_3 + p^4x_4 + p^5x_5 + p^6x_6 + \dots\right)^2}{2} \\ -\left(x_0 + px_1 + p^2x_2 + p^3x_3 + p^4x_4 + p^5x_5 + p^6x_6 + \dots\right)^2\end{array}\right] = 0 \quad (87)$$

Arranging Eq. (87) according to the power of the embedding parameter p, we have

$$p^0: \quad x_0 - \frac{5}{8} = 0,$$

$$p^1: \quad x_1 + \frac{1}{8}x_0^3 + \frac{1}{2}x_0^2 - x_0 = 0,$$

$$p^2: \quad x_2 + \frac{1}{8}\left(3x_0^2x_1\right) + \frac{1}{2}\left(2x_0x_1\right) - x_1 = 0,$$

$$p^3: \quad x_3 + \frac{1}{8}\left(3x_0^2x_2 + 3x_1^2x_0\right) + \frac{1}{2}\left(x_1^2 + 2x_0x_2\right) - x_2 = 0,$$

$$p^4: \quad x_4 + \frac{1}{8}\left(3x_0^2x_3 + x_1^3 + 6x_0x_1x_2\right) + \frac{1}{2}\left(2x_0x_3 + 2x_1x_2\right) - x_3 = 0,$$

$$p^5: \quad x_5 + \frac{1}{8}\left(3x_0^2x_4 + 3x_0x_2^2 + 6x_0x_1x_3 + 3x_1^2x_2\right) + \frac{1}{2}\left(x_2^2 + 2x_1x_3 + 2x_0x_4\right) - x_4 = 0,$$

$$p^6: \quad x_6 + \frac{1}{8}\left(3x_0^2x_5 + 6x_0x_2x_3 + 6x_0x_1x_4 + 3x_1x_2^2 + 3x_1^2x_3\right) + \frac{1}{2}\left(2x_0x_5 + 2x_1x_4 + 2x_2x_3\right) = 0,$$

On solving the above equations, we have

$$x_0 = 0.6250000,\ x_1 = 0.3991699,\ x_2 = 0.0912166,\ x_3 = -0.0961684, \quad (88)$$
$$x_4 = -0.0834048,\ x_5 = 0.0257617,\ x_6 = 0.0951325$$

Therefore,

$$x = 0.6250000 + 0.3991699 + 0.0912166 - 0.0961684 - 0.0834048 + 0.0257617 + 0.0951325 + \dots = 1.0567075$$

The exact solutions of the roots of the given equation are 1.0000000, -1.381966011 and -3.618033989. It can be seen that the above solution shows that the scheme is approaching the real positive root of 1.0000 from both ends, as the absolute error tends to zero (Table 8).

Table 8 Solution of the equation of example 3.8

Number of Iteration (n)	n-term Solution	Absolute Error
1	0.625	0.375
2	1.0241699	0.0241699
3	1.1153865	0.1153865
4	1.0192181	0.0192181
5	0.9358133	0.0641867
6	0.961575	0.038425
7	1.0567075	0.0567075

Example 5.10: Find the roots of the following transcendental equation using homotopy perturbation method

$$e^x + sinx + cosx = 3. \quad (89)$$

The above equation does not contain a linear term. A convenient linear term with a coefficient can be introduced to make the equation be easily amendable to the form that homotopy perturbation can easily be applied. Therefore, we write

$$e^x + sinx + cosx + ax - ax = 3 \quad (90)$$

For the choice of "a", the ratio of the constant in the given equation and that of "a" must be within two values where the needed root of the equation lies. Therefore, we have $0 \le \frac{3}{a} \le 1$. In this example, we choice $a = 7$.

A convenient linear term is therefore introduced such that we can write,

$$e^x + sinx + cosx + 7x - 7x = 3 \quad (91)$$

Which can be expressed as

$$x - \frac{3}{7} + \frac{1}{7}\left(e^x + sinx + cosx - 7x\right) = 0 \quad (92)$$

In order to apply homotopy perturbation method, the equation is expressed as

$$x - \frac{3}{7} + \frac{1}{7}p\left(e^x + sinx + cosx - 7x\right) = 0 \quad (93)$$

The solution of the given equation can be assumed to be written as a power series in p as

$$x = x_0 + px_1 + p^2x_2 + p^3x_3 + p^4x_4 + \dots \quad (94)$$

On substituting Eq. (94) into Eq.(93), we have

$$x_0 + px_1 + p^2x_2 + p^3x_3 + p^4x_4 + p^5x_5 + p^6x_6 + \dots - \frac{3}{7}$$
$$+\frac{1}{7}p\left[\begin{array}{l} e^{\left(x_0 + px_1 + p^2x_2 + p^3x_3 + p^4x_4 + p^5x_5 + p^6x_6 + \dots\right)} \\ +sin\left(x_0 + px_1 + p^2x_2 + p^3x_3 + p^4x_4 + p^5x_5 + p^6x_6 + \dots\right) \\ +cos\left(x_0 + px_1 + p^2x_2 + p^3x_3 + p^4x_4 + p^5x_5 + p^6x_6 + \dots\right) \\ -7\left(x_0 + px_1 + p^2x_2 + p^3x_3 + p^4x_4 + p^5x_5 + p^6x_6 + \dots\right)\end{array}\right] = 0 \quad (95)$$

$$x_0 + px_1 + p^2x_2 + p^3x_3 + p^4x_4 + \ldots - \frac{3}{7} + \frac{1}{7}p\begin{bmatrix} e^{x_0}e^{\left(px_1+p^2x_2+p^3x_3+p^4x_4+\ldots\right)} \\ +sin\left[x_0 + \left(px_1 + p^2x_2 + p^3x_3 + p^4x_4 + \ldots\right)\right] \\ +cos\left[x_0 + \left(px_1 + p^2x_2 + p^3x_3 + p^4x_4 + \ldots\right)\right] \\ -7\left(x_0 + px_1 + p^2x_2 + p^3x_3 + p^4x_4 + \ldots\right) \end{bmatrix} = 0 \quad (96)$$

$$x_0 + px_1 + p^2x_2 + p^3x_3 + p^4x_4 + \ldots - \frac{3}{7} + \frac{1}{7}p\begin{bmatrix} e^{x_0}e^{\left(px_1+p^2x_2+p^3x_3+p^4x_4+\ldots\right)} \\ +sinx_0cos\left(px_1 + p^2x_2 + p^3x_3 + p^4x_4 + \ldots\right) \\ +cosx_0sin\left(px_1 + p^2x_2 + p^3x_3 + p^4x_4 + \ldots\right) \\ +cosx_0cos\left(px_1 + p^2x_2 + p^3x_3 + p^4x_4 + \ldots\right) \\ -sinx_0sin\left(px_1 + p^2x_2 + p^3x_3 + p^4x_4 + \ldots\right) \\ -7\left(x_0 + px_1 + p^2x_2 + p^3x_3 + p^4x_4 + \ldots\right) \end{bmatrix} = 0 \quad (97)$$

$$x_0 + px_1 + p^2x_2 + p^3x_3 + p^4x_4 + \ldots - \frac{3}{7} + p\begin{bmatrix} e^{x_0}\begin{bmatrix} 1 - \left[px_1 + p^2x_2 + p^3x_3 + p^4x_4 + \ldots\right] + \frac{\left[px_1 + p^2x_2 + p^3x_3 + p^4x_4 + \ldots\right]}{2!} \\ -\frac{\left[px_1 + p^2x_2 + p^3x_3 + p^4x_4 + \ldots\right]}{3!} + \frac{\left[px_1 + p^2x_2 + p^3x_3 + p^4x_4 + \ldots\right]}{4!} \end{bmatrix} \\ +sinx_0\left[1 - \frac{\left[px_1 + p^2x_2 + p^3x_3 + p^4x_4 + \ldots\right]}{2!} + \frac{\left[px_1 + p^2x_2 + p^3x_3 + p^4x_4 + \ldots\right]}{4!} + \ldots\right] \\ +cosx_0\begin{bmatrix} \left[px_1 + p^2x_2 + p^3x_3 + p^4x_4 + \ldots\right] - \frac{\left[px_1 + p^2x_2 + p^3x_3 + p^4x_4 + \ldots\right]}{3!} \\ + \frac{\left[px_1 + p^2x_2 + p^3x_3 + p^4x_4 + \ldots\right]}{5!} + \ldots \end{bmatrix} \\ +cosx_0\left[1 - \frac{\left[px_1 + p^2x_2 + p^3x_3 + p^4x_4 + \ldots\right]}{2!} + \frac{\left[px_1 + p^2x_2 + p^3x_3 + p^4x_4 + \ldots\right]}{4!} + \ldots\right] \\ -sinx_0\begin{bmatrix} \left[px_1 + p^2x_2 + p^3x_3 + p^4x_4 + \ldots\right] - \frac{\left[px_1 + p^2x_2 + p^3x_3 + p^4x_4 + \ldots\right]}{3!} \\ + \frac{\left[px_1 + p^2x_2 + p^3x_3 + p^4x_4 + \ldots\right]}{5!} + \ldots \end{bmatrix} \\ -7\left(x_0 + px_1 + p^2x_2 + p^3x_3 + p^4x_4 + \ldots\right) \end{bmatrix} = 0 \quad (98)$$

Arrange the equation according to the power of the embedding parameter p, we have

$$p^0: \quad x_0 - \frac{3}{7} = 0,$$

$$p^1: \quad x_1 + \frac{1}{7}\left(e^{x_0} + sinx_0 + cosx_0 - 7x_0\right) = 0,$$

$$p^2: \quad x_2 + \frac{1}{7}\left(x_1e^{x_0} + x_1cosx_0 - x_1sinx_0 - 7x_1\right) = 0,$$

$$p^3: \quad x_3 + \frac{1}{7}\left(x_2e^{x_0} + \frac{x_1^2}{2!}e^{x_0} - \frac{1}{2!}x_1^2sinx_0 + x_2cosx_0 - \frac{1}{2!}x_1^2cosx_0 - x_2sinx_0 - 7x_2\right) = 0,$$

$$p^4: \quad x_4 + \frac{1}{7}\begin{pmatrix} x_3e^{x_0} + x_1x_2e^{x_0} + \frac{x_1^3}{3!}e^{x_0} - \frac{1}{3!}x_1^3cosx_0 - x_1x_2sinx_0 + x_3cosx_0 \\ -\frac{1}{3!}x_1^3sinx_0 - x_1x_2cosx_0 - x_3sinx_0 - 7x_3 \end{pmatrix} = 0,$$

$$p^5: \quad x_5 + \frac{1}{7}\begin{pmatrix} x_4e^{x_0} + \frac{\left(2x_1x_3 + x_2^2\right)}{2!}e^{x_0} + \frac{3x_1^2x_2}{3!}e^{x_0} + \frac{x_1^4}{4!}e^{x_0} \\ +\frac{1}{4!}x_1^4sinx_0 - \frac{3x_1^2x_2}{3!}cosx_0 - \frac{x_2^2}{2!}sinx_0 - x_1x_3sinx_0 + x_4cosx_0 \\ +\frac{1}{4!}x_1^4cosx_0 + \frac{3x_1^2x_2}{3!}sinx_0 - \frac{x_2^2}{2!}cosx_0 - x_1x_3cosx_0 - x_4sinx_0 - 7x_4 \end{pmatrix} = 0,$$

$$p^6: \quad x_6 + \frac{1}{7}\begin{pmatrix} x_5e^{x_0} + \frac{\left(2x_2x_3 + 2x_1x_4\right)}{2!}e^{x_0} + \frac{\left(3x_1^2x_3 + 3x_1x_2^2\right)}{3!}e^{x_0} + \frac{4x_1^3x_2}{4!}e^{x_0} + \frac{x_1^5}{5!}e^{x_0} \\ +\frac{1}{5!}x_1^5cosx_0 + \frac{4x_1^3x_2}{4!}sinx_0 - \frac{6x_2x_3}{3!}sinx_0 - \frac{6x_1x_4}{3!}sinx_0 - \frac{x_1x_2^2}{2!}cosx_0 - \frac{x_1^2x_3}{2!}cosx_0 + x_5cosx_0 \\ -\frac{1}{5!}x_1^5sinx_0 + \frac{4x_1^3x_2}{4!}cosx_0 - \frac{6x_2x_3}{3!}cosx_0 - \frac{6x_1x_4}{3!}cosx_0 + \frac{x_1x_2^2}{2!}sinx_0 + \frac{x_1^2x_3}{2!}sinx_0 - x_5sinx_0 - 7x_5 \end{pmatrix} = 0,$$

On solving the above equations, we have

$$x_0 = 0.4285714, \quad x_1 = 0.0199721, \quad x_2 = 0.0141811, \quad x_3 = 0.0100645, \quad x_4 = 0.0071586, \quad (99)$$

Therefore

$$x = 0.4285714 + 0.0199721 + 0.0141811 + 0.0100645 + 0.0071586 + \ldots = 0.4799477$$

The above solution is converging to 0.4972 at the tenth term which is a root of the equation. Therefore, $x = 0.4972$ (Table 9).

Table 9 Solution of the equation of example 3.9

Number of Iteration (n)	n-term Solution	Absolute Error
1	0.4284714	0.0687268
2	0.4485435	0.0486565
3	0.4627246	0.0344754
4	0.4727891	0.0244109
5	0.4799477	0.0172523

Example 5.11: Find the roots of the following transcendental equation using homotopy perturbation method

$$Inx + sinhx + sin^2xcos^2x = 15\,. \quad (100)$$

The above equation does not contain a linear term. A convenient linear term with a coefficient can be introduced to make the equation be easily amendable to the form that homotopy perturbation can easily be applied. Therefore, we write

$$Inx + sinhx + sin^2xcos^2x + ax - ax = 15 \quad (101)$$

For the choice of "*a*", the ratio of the constant in the given equation and that of "*a*" must be within two values where the needed root of the equation lies. Therefore, we have $3 \le \frac{15}{a} \le 4$. In this example, we choice *a = 5*.

$$Inx + sinhx + sin^2xcos^2x + 5x - 5x = 15 \quad (102)$$

The homotopy perturbation of the equation is given as

$$x - 3 + \frac{1}{5}p\left[Inx + sinhx + sin^2xcos^2x - 5x\right] = 0 \quad (103)$$

The assumed solution is expressed as a power series in p as

$$x = x_0 + px_1 + p^2x_2 + p^3x_3 + p^4x_4 + \ldots \quad (104)$$

On substituting Eq. (104) into Eq.(103), we have

$$x_0+px_1+p^2x_2+p^3x_3+p^4x_4+...-3 +p\left[\begin{array}{l} In\left(x_0+px_1+p^2x_2+p^3x_3+p^4x_4+...\right)+sinh\left(x_0+px_1+p^2x_2+p^3x_3+p^4x_4+...\right) \\ +sin^2\left(x_0+px_1+p^2x_2+p^3x_3+p^4x_4+...\right)cos^2\left(x_0+px_1+p^2x_2+p^3x_3+p^4x_4+...\right) \\ -5\left(x_0+px_1+p^2x_2+p^3x_3+p^4x_4+...\right)\end{array}\right]=0 \quad (105)$$

Arranging the above equation according to the power of the embedding parameter p, we have

$$p^0: \quad x_0-3=0,$$

$$p^1: \quad x_1+\frac{1}{5}\left(Inx_0+sinhx_0+sin^2x_0cos^2x_0-5x_0\right)=0,$$

$$p^2: \quad x_2+\frac{1}{5}\left(\frac{x_1}{x_0}+x_1coshx_0+2x_1sinx_0cos^3x_0-2x_1sin^3x_0cosx_0-5x_1\right)=0,,$$

$$p^3: \quad x_3+\frac{1}{5}\left(\begin{array}{l}\frac{x_2}{x_0}-\frac{x_1^2}{2x_0}+x_2coshx_0+\frac{x_1^2}{2!}sinhx_0+x_1cos^4x_0-6x_1^2sin^2x_0cos^2x_0 \\ +2x_2sinx_0cos^3x_0+x_1^2sin^4x_0-2x_2sin^3x_0cosx_0-5x_2\end{array}\right)=0,$$

$$p^4: \quad x_4+\frac{1}{5}\left(\begin{array}{l}\frac{x_3}{x_0}-\frac{x_1x_2}{x_0^2}+\frac{x_1^3}{3x_0^3}+x_3coshx_0+x_1x_2sinhx_0+\frac{x_1^3}{3!}coshx_0 \\ -\frac{16}{3}x_1^3cos^3x_0sinx_0+2x_1x_2cos^4x_0+\frac{16}{3}x_1^3sin^3x_0cosx_0 \\ -12x_1x_2sin^2x_0cos^2x_0+2x_3sinx_0cos^3x_0+2x_1x_2sin^4x_0 \\ -2x_3sin^3x_0cosx_0-5x_3\end{array}\right)=0,$$

$$p^5: \quad x_5-x_4e^{-x_0}+\frac{\left(2x_1x_3+x_2^2\right)}{2!}e^{-x_0}-\frac{3x_1^2x_2}{3!}e^{-x_0}+\frac{x_1^4}{4!}e^{-x_0}=0,$$

$$p^6: \quad x_6-x_5e^{-x_0}+\frac{\left(2x_2x_3+2x_1x_4\right)}{2!}e^{-x_0}-\frac{\left(3x_1^2x_3+3x_1x_2^2\right)}{3!}e^{-x_0}+\frac{4x_1^3x_2}{4!}e^{-x_0}-\frac{x_1^5}{5!}e^{-x_0}=0,$$

On solving the above equations, we have

$$x=x_1+x_2+x_3+x_4+x_5+x_6+...=3.31705 \quad (106)$$

In the solutions of nonlinear algebraic equations, Shanks transformation can be used to covert a slowly converging sequence to its rapidly converging counterpart effectively. The Shanks transformation is an efficient relation that can acceleratee the convergence rate of the series (Table 10).[48] The Shanks transformation *Sh(U$_n$)* of the sequence U_n is defined as,

$$Sh\left(U_n\right)=\frac{U_{n+1}U_{n-1}-U_n^2}{U_{n+1}-2U_n+U_{n-1}} \quad (107)$$

Further speed-up may be achieved by successive implementation of the Shanks transformation, that is $Sh^2\left(U_n\right)=Sh\left(Sh\left(U_n\right)\right)$, $Sh^3\left(U_n\right)=Sh\left(Sh\left(Sh\left(U_n\right)\right)\right)$, $Sh^4\left(U_n\right)=Sh\left(Sh\left(Sh\left(Sh\left(U_n\right)\right)\right)\right)$ etc.

Table 10 Solutions of some other nonlinear equations using homotopy perturbation method

Equations	Approximate analytical solution
$x^3+x-1=0$	$x=0.6823278$
$x^2-(1-x)^5=0$	$x=0.3459542$
$e^{-x}+cosx=0$	$x=1.7461395$
$cosx-x=0$	$x=0.7390851$
$x-0.2sinx=0.8$	$x=0.9643339$
$sin^2x-x^2+1=0$	$x=1.4044916$
$e^{-x}-3x^2=0$	$x=0.9100075$
$x^6-5x^5+3x^4+x^3+2x^2-8x=0.5$	$x=0.6823278$
$Inx+e^x-2x^2+1=0$	$x=0.1224248$

Conclusion

In this work, homotopy perturbation method has been directly applied to solve nonlinear algebraic and transcendental equations. The reliability and efficiency of the method in solving the nonlinear equations have been demonstrated by different number of illustrative examples. The method has been shown to be conceptually and computationally simple and straightforward without any ambiguity. Also, the advantages and the superiority of the approximate analytical method over the other approximate analytical and numerical methods of finding the roots of nonlinear algebraic and transcendental equations have been presented. Additionally, the following points should be noted.

i. The homotopy perturbation method has capability to find the complex root of nonlinear equation. Through a numerical of numerical experiments, it was found that if an initial term x_o is appropriately chosen as a complex number close to the root, the HPM might converge to a complex root. It should be noted that a diverging series (for the equation with real coefficients) may indicate complex roots. For example, the direct application of HPM to solve equation $x^3-3x-5=0$, produces divergence results which indicate that the equation has complex root(s).

ii. It is not unlikely that the direct approach of the HPM to the solution of nonlinear algebraic and transcendental equations can produce diverging results for an equation that has real roots such as $x^5-4x^4-13x^3+46x^2+11x-43=0$. However, with some mathematical manipulations, the direct application of HPM to such problems is possible. This will be presented in the further works on the study.

iii. It can also be said that with the combination of matrix algebra and direct HPM, all the possible roots of a nonlinear equation can be found. This will also be presented in the subsequent paper of this study.

It is hoped that the present work will assist in providing accurate solutions to many practical problems in science and engineering.

Acknowledgments

None

Conflicts of interest

The author declares there are no conflicts of interest.

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