

Exact solution of plane Poiseuille flow in a channel filled with a porous medium

Abstract

In this paper, I have presented an analytical solution to the problem of plane Poiseuille flow in a channel filled with a porous medium. Nield et al.,¹ have expressed it in terms of elliptic integrals. I express it in an equivalent, but a more conventional form, in terms of Weierstrass elliptic function. I also present the R code to plot the velocity profile and compute the flux.

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Amey Joshi

Director of Fidelity Investments, Bangalore, India

Correspondence: Amey Joshi, Director of Fidelity Investments, Pinehurst, Embassy Golf Links business park, Bangalore, 56007, Tel +91-8691-2046, Email amey.joshi@outlook.com

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Introduction

The flow of an incompressible, Newtonian fluid through a porous medium of porosity ε and permeability K is described by the equation²

$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \left(\frac{\vec{u}}{\varepsilon} \right) = -\nabla \left(\frac{p\varepsilon}{\rho} \right) + \nu_e \Delta \vec{u} + \vec{F}, \quad (1)$$

where ρ is the density of the fluid, \vec{u} is the velocity, p the pressure, ν_e the effective viscosity, ν the kinematic viscosity, Δ denotes the Laplace operator and the total force is

$$\vec{F} = \vec{F}_b - \frac{\varepsilon \nu}{K} \vec{u} - \frac{\varepsilon F_0}{\sqrt{K}} u \vec{u}. \quad (2)$$

It is a sum of the body force \vec{F}_b and the effect of the porosity of the medium. The geometric function F_0 in equation (2) depends only on the porosity and the nature of the solid particle matrix of the porous medium. For a fully developed and steady flow, the left hand side of equation (1) vanishes. Further, if the flow is driven only by a constant pressure gradient and not an external body force, then equation (1) becomes

$$\nu_e \Delta \vec{u} - \frac{\varepsilon \nu}{K} \vec{u} - \frac{\varepsilon F_0}{\sqrt{K}} u \vec{u} + \varepsilon \vec{G} = 0, \quad (3)$$

where $\rho \vec{G} = -\nabla p$. I will solve equation (3) for a flow between parallel solid plates with a porous medium between them.

The physical setup

Consider a steady, fully-developed flow of a fluid along the x axis between two infinite parallel plates, a distance H apart. Let the plates coincide with planes $y = 0$ and $y = H$. If the channel is long enough that the end effects are negligible along most of the channel's length then we can express the fluid's velocity along the x axis as $u(y)$. Equation (3), in this case, becomes

$$\frac{\nu_e}{\varepsilon} u'' - \frac{\nu}{K} u - \frac{F_0}{\sqrt{K}} u^2 + \varepsilon G = 0, \quad (4)$$

where u'' denotes the second derivative with respect to y . Since the channel is bounded by solid walls, the usual no-slip boundary conditions apply. I will solve (4) subject to the conditions $u(y = 0) = 0$ and $u'(H/2) = 0$, that is, the velocity has an extremum (actually, a

maximum) at the center of the channel. Nield et al.,¹ have given an analytical solution which expresses the coordinate y as an elliptic integral in the velocity u . I will develop an equivalent solution in which u is expressed as a Weierstrass elliptic function of y .

The solution

We convert equation (4) in a non-dimensional form using the relations,

$$u_* = \frac{u}{U} \quad (5)$$

$$y_* = \frac{y}{H} \quad (6)$$

$$G_* = \frac{HG}{U^2}, \quad (7)$$

where U is the velocity at the center of the channel, and the dimensionless quantities

$$Re = \frac{HU}{\nu} \quad (8)$$

$$Da = \frac{K}{H^2} \quad (9)$$

$$J = \frac{\nu_e}{\nu} \quad (10)$$

Re , Da and J are the Reynold's number, the Darcy number and the viscosity ratio of the flow. Dropping the $*$ subscript for sake of notation clarity, we get,

$$u'' - A^2 u^2 - B^2 u + C^2 = 0, \quad (11)$$

where

$$A^2 = \frac{F_0 \varepsilon Re}{J \sqrt{Da}} \quad (12)$$

$$B^2 = \frac{\varepsilon}{J Da} \quad (13)$$

$$C^2 = \frac{\varepsilon^2 Re G}{J} \quad (14)$$

are all positive numbers. Introduce a new variable $u_1(y)$,

$$u_1(y) = Au(y) + \frac{1}{2} \frac{B^2}{A}, \quad (15)$$

so that equation (11) becomes,

$$u_1'' - Au_1^2 + \left(AC^2 + \frac{B^4}{4A} \right) = 0, \tag{16}$$

Multiplying equation (16) by u_1' , the first derivative of $u_1(y)$ with respect to y , and integrating we get,

$$(u_1')^2 - \frac{2}{3}Au_1^3 + 2\left(AC^2 + \frac{B^4}{4A} \right)u_1 + \frac{36g_3}{A^2} = 0, \tag{17}$$

where $36g_3 / A^2$ is a constant of integration. Introduce yet another variable, $u_2(y) = (A/6)u_1(y)$, to transform equation (17) to

$$(u_2')^2 = 4u_2^3 - \frac{1}{3}\left(A^2C^2 + \frac{B^4}{4} \right)u_2 - g_3. \tag{18}$$

This equation is of the form,

$$(u_2')^2 = 4u_2^3 - g_2u_2 - g_3, \tag{19}$$

where

$$g_2 = \frac{1}{3}\left(A^2C^2 + \frac{B^4}{4} \right) \tag{20}$$

Its solution is $u_2(y) = \wp(y + k_1; g_2, g_3)$, where k_1 is another constant of integration. The function \wp is the Weierstrass elliptic function with invariants g_2 and g_3 .³ Therefore, the solution of (11) is

$$u(y) = \frac{6}{A^2} \left[\wp \left(y + k_1 \mid \frac{1}{3} \left(A^2C^2 + \frac{B^4}{4} \right), g_3 \right) - \frac{B^2}{12} \right] \tag{21}$$

The constants g_3 and k_1 are found using the boundary conditions.

Some properties of Weierstrass elliptic function

The Weierstrass elliptic function is defined as the solution of the differential equation,

$$[\wp'(u)]^2 = 4\wp^3(u) - g_2\wp(u) - g_3, \tag{22}$$

where u is a complex variable and g_2 and g_3 are complex constants, called invariants of \wp . Weierstrass elliptic function is a doubly periodic function. We follow Lawden⁴ and call the periods $2\omega_1$ and $2\omega_3$. There is another half period $2\omega_2$, where $\omega_2 = -\omega_1 - \omega_3$. Whittaker et al.,³ denote the fundamental periods by ω_1 and ω_2 and define $\omega_3 = -\omega_1 - \omega_2$. If the constants e_j are defined as

$$e_j = \wp(\omega_j) \tag{23}$$

then it can be shown that [by Lawden⁴ p.160] they are solutions of the cubic equation $4s^3 - g_2s - g_3 = 0$. Further, they are related to the invariants as

$$g_2 = -4(e_1e_2 + e_2e_3 + e_3e_1) \tag{24}$$

$$g_3 = 4e_1e_2e_3 \tag{25}$$

Additionally [Lawden⁴ p.159],

$$e_1 + e_2 + e_3 = 0 \tag{26}$$

The discriminant of the cubic equation $4t^3 - g_2t - g_3 = 0$ is $D = 16(g_2^3 - 27g_3^2)$.

If

- a. $D > 0$, then all roots of $4t^3 - g_2t - g_3 = 0$ are real and distinct.

- b. $D = 0$, then the roots of $4t^3 - g_2t - g_3 = 0$ are real and not distinct.

- c. $D < 0$, then $4t^3 - g_2t - g_3 = 0$ has one real root and two complex roots. The complex roots are conjugates of each other.

Theorem 1: If the invariants of Weierstrass elliptic function \wp are real and if the roots of the cubic $4s^3 - g_2s - g_3 = 0$ are real then ω_1 is real and ω_3 is imaginary.

Proof: The constants e_1, e_2, e_3 are solutions of the cubic $4s^3 - g_2s - g_3 = 0$. If they are all real, we can choose them to be such that $e_1 > e_2 > e_3$. Then (22) can be written as

$$\{(\wp(u) - e_1)(\wp(u) - e_2)(\wp(u) - e_3)\}^{1/2} = -\frac{1}{2}\wp'(u)$$

that is,

$$\frac{d\wp(u)}{\sqrt{4(\wp(u) - e_1)(\wp(u) - e_2)(\wp(u) - e_3)}} = -du$$

equivalently,

$$\frac{dt}{\sqrt{4(t - e_1)(t - e_2)(t - e_3)}} = -du,$$

where $t = \wp(u)$. Integrating this equation so that u goes from 0 to U , and hence t goes from $\wp(0)$ to $\wp(U)$, we have two possibilities because of the double pole of \wp at the origin,

$$\int_{\wp(0)}^{\wp(U)} \frac{dt}{\sqrt{4(t - e_1)(t - e_2)(t - e_3)}} = -U \tag{27}$$

or

$$\int_{-\infty}^{\wp(U)} \frac{dt}{\sqrt{4(t - e_1)(t - e_2)(t - e_3)}} = -U \tag{28}$$

Choose $U = \omega_1$ in equation (27), so that

$$\omega_1 = -\int_{-\infty}^{\wp(\omega_1)} \frac{dt}{\sqrt{4(t - e_1)(t - e_2)(t - e_3)}} = \int_{e_1}^{\infty} \frac{dt}{\sqrt{4(t - e_1)(t - e_2)(t - e_3)}}$$

Since e_1 is the largest root of $4(t - e_1)(t - e_2)(t - e_3) = 4t^3 - g_2t - g_3 = 0$, throughout the path of integration, the integrand stays finite. Further, the integrand is always real and hence ω_1 is real.

Choose $U = -\omega_3$ in equation (28), so that

$$\omega_3 = -\int_{-\infty}^{\wp(\omega_3)} \frac{dt}{\sqrt{4(t - e_1)(t - e_2)(t - e_3)}} = -\int_{-\infty}^{e_3} \frac{dt}{\sqrt{4(t - e_1)(t - e_2)(t - e_3)}}$$

or,

$$\omega_3 = i \int_{-\infty}^{e_3} \frac{dt}{\sqrt{4(e_1 - t)(e_2 - t)(e_3 - t)}}$$

Since e_3 is the smallest root of $4(t - e_1)(t - e_2)(t - e_3) = 4t^3 - g_2t - g_3 = 0$, throughout the path of integration, the integrand stays finite. Further, the integrand is always real and hence ω_3 is imaginary.

Remark: The converse of theorem 1 is also true [Lawden⁴ p.163]. That is, if ω_1 is real and ω_3 is imaginary then e_j , the roots of $4s^3 - g_2s - g_3 = 0$ are all real.

If ω_1 is real and \wp is imaginary then section 6.11 shows that⁴

Theorem 2: \wp takes real values on the the rectangle OXY in

Figure 1, where $O \equiv (0,0)$, $X \equiv (\omega_1,0)$, $A \equiv (\omega_1,\omega_3)$ and $Y \equiv (0,\omega_3)$. Further, if u is taken round $OXAY$, $\wp(u)$ decreases from $+\infty$ at O to e_1 at X , further decreases to e_2 along XA , to e_3 along AY and finally to $-\infty$ as it reaches the origin along YA .

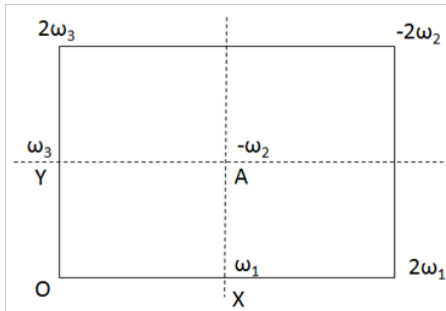


Figure 1 Positive discriminant.

If the invariants g_2 and g_3 of the Weierstrass elliptic function are real but the discriminant of the cubic equation $4t^3 - g_2t - g_3 = 0$ is negative, then referring to Figure 2 or Handbook of Mathematical Functions with Formulas⁵, the function \wp takes value $+\infty$ at O . Its value decreases monotonically upto $-\omega_2$, where it is e_2 and then rises once again to $+\infty$ at $-2\omega_2$. If the function has a singularity at the origin, it will have it at points $-2\omega_2$, $2\omega_3$ and $2\omega_3$ because of its periodicity. The function also takes real values along the segment BD .

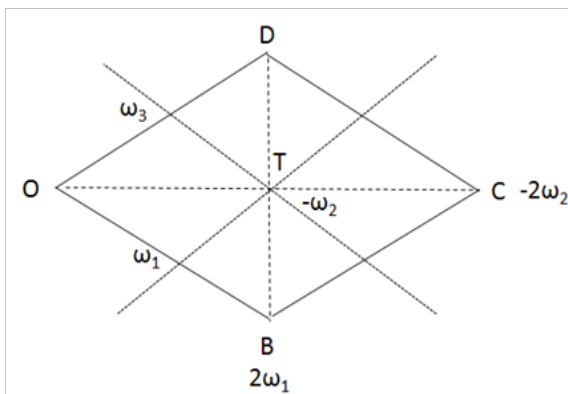


Figure 2 Negative discriminant.

Finding g_3 and k_1

Since the velocity has an extremum at the center of the channel, its derivative with respect to y vanishes there. Therefore, derivatives of u_2 and u_2 also vanish at the center of the channel. If the magnitude of the non-dimensional velocity u^c at the center is $u^c = 1$ then we immediately have $u_1^c = A + B^2 / (2A)$ and

$$u_2^c = \frac{A^2}{6} + \frac{B^2}{12} \tag{29}$$

Since at the center,

$$4[u_2^c]^3 - g_2u_2^c - g_3 = 0,$$

u_2^c is one of the roots of the cubic equation $4s^3 - g_2s - g_3 = 0$. If e_1, e_2, e_3 are the roots, without loss of generality, let

$$u_2^c = e_2 \tag{30}$$

From these equations (24) to (26) we readily get,

$$e_1 = \frac{-e_2 + \sqrt{g_2 - 3e_2^2}}{2} \tag{31}$$

$$e_3 = \frac{-e_2 - \sqrt{g_2 - 3e_2^2}}{2}, \tag{32}$$

Knowing e_1, e_2 and e_3 equation (25) gives,

$$g_3 = 4e_1e_2e_3 \tag{33}$$

We will now use the second boundary condition $u(0) = 0$ to find k_1 . The equation $u(0) = 0$ is equivalent to, $\wp(0 + k_1; g_2, g_3) = B^2 / 12$ or

$$k_1 = \wp^{-1}\left(\frac{B^2}{12} \mid g_2, g_3\right) \tag{34}$$

Computation of the velocity profile

The inverse of Weierstrass elliptic function can be readily evaluated on the wolframalpha website using the Mathematica® function.

Listing 1: Evaluation of \wp^{-1} ,
InverseWeierstrassP[B^2/12, g_2, g_3]

Let us assume that the equation $4s^3 - g_2s - g_3 = 0$ has a positive discriminant. Then we saw in section 3.1 that \wp will take real values on the the rectangle $OXAY$ where $O \equiv (0,0)$, $X \equiv (\omega_1,0)$, $A \equiv (\omega_1,\omega_3)$ and $Y \equiv (0,\omega_3)$. Further, if u is taken round $OXAY$, $\wp(u)$ decreases from $+\infty$ at O to e_1 at X , further decreases to e_2 along XA , to e_3 along AY and finally to $-\infty$ as it reaches the origin along YA . Thus, k_1 will lie on OX is it is greater than e_1 or on XA if it is between e_1 and e_2 or on AB if it is less than e_2 . It cannot be less than e_3 because e_3 is always less than zero while $B^2 / 12 > 0$.

Supposing $e_1 \geq B^2 / 12 \geq e_2$, that is, k_1 lies on XA . Then we have to align the imaginary axis along the y axis and plot values of

$$\frac{6}{A^2} \wp\left(z \mid \frac{1}{3}\left(A^2C^2 + \frac{B^4}{4}\right), \frac{k_0A^2}{36}\right) - \frac{1}{2} \frac{B^2}{A^2}$$

where the complex variable z is $z = \omega_1 + z_0$, where z_0 ranges from 0 to $\pm 2Im(k_1)i$. Since k_1 lies on XA , ω_1 is identical to Rek_1 and hence $z = Re(k_1) + z_0$. The following R code, with the library ‘elliptic’ was used to generate the velocity profile of Figure 3.⁶

Listing 2: Velocity profile of flow in porous medium

The same code can be used if the equation $4s^3 - g_2s - g_3 = 0$ has a negative discriminant and k_1 is complex. In the event k_1 is real, the code should be modified so that the real axis is along the y axis.

When we introduced the non-dimensional variables, we defined $y_* = y / H$. Therefore, y_* should range from 0 to 1. However, we notice in Figure 3 that it does not. We will introduce another transformation that brings the non-dimensional displacement variable between 0 and 1 after proving its validity.

Theorem 3: If we introduce a new non-dimensional displacement variable \tilde{y}_* defined as

$$\tilde{y}_* = \frac{y_*}{\alpha},$$

where α is a positive real number then this transformation keeps the form of equation (11) unchanged.

Listing 2:

```

library(elliptic)
epsilon <- 0.1 # Porosity
Re <- 0.1 # Reynold's number
Da <- 5e-1 # Darcy number
J <- 1 # Viscosity ratio
G <- 1e-4 # Pressure gradient/density

F0 <- 1.75/sqrt(150 * epsilon^3) # Geometric function

# Calculation of A, B, C
A <- sqrt((F0 * epsilon * Re)/(J * sqrt(Da)))
B <- sqrt(epsilon/(J * Da))
C <- sqrt(epsilon^2 * Re * G/J)

g2 <- (A^2*C^2 + B^4/4)/3
u2_center <- A^2/6 + B^4/12

# Choose e2 as u2_center
e2 <- u2_center

e1 <- (-e2 + sqrt(g2 - 3*i*(-1i)*e2^2))/2
e3 <- (-e2 - sqrt(g2 - 3*i*(-1i)*e2^2))/2

g3 <- 4*e1*e2*e3

# Get half periods
Omegas <- half.periods(g = c(g2, g3))

# Recreate e's so that they are consistent with R
es <- e1e2e3(g = c(g2, g3))

# Computed using Mathematica
k1 <- 8.020971603238442 + 3.116271691054093i

# Observe that k1 is equal to Omegas[2].

# Print Omegas. Let omega1 be the real period and omega3 be the imaginary
period.
#
# We want to plot the Weierstrass function along the imaginary # axis. We
know that P(omega1) = e1 while e1 >= P0. The function # P falls below e1
both above and below omega1 if we move along # the imaginary axis. If it
takes a value B^2/12 a distance Im(k1) # above omega1, it will take the same
value a distance Im(k1) below # omega1. Therefore, we plot the function
from omega1 - 1i*Im(k1) # omega1 + 1i * Im(k1).

z <- Omegas[2] + seq(from = 0, to = -2*Im(k1), len = 100)*1i y <- -Im(z)

u <- 6/A^2*( P(z + Im(k1)*1i, g = c(g2, g3)) - B^2/12)

# We choose real part alone. Check that the imaginary part # is very small.
Use the command max(abs(Im(u))) to check # it.

# If the imaginary part is indeed quite small xl <- 'Dimensionless velocity'yl
<- 'y'ml <- 'Velocity profile with porous media'plot(Re(u), y, type = 'l', xlab
= xl, ylab = yl, main = ml)

```

Proof: For sake of clarity, let us write the derivatives in full and re-introduce the '*' subscript for non-dimensional variables. Thus, equation (11) is written as

$$\frac{d^2 u_*}{dy_*^2} - A^2 u_*^2 - B^2 u_* + C^2 = 0 \quad (35)$$

If the non-dimensionalization of y is done with αH , where α is a positive real number then

$$\tilde{y}_* = \frac{y}{\alpha H} = \frac{y_*}{\alpha}$$

If f is a function of y then

$$\frac{df}{dy_*} = \frac{df}{d\tilde{y}_*} \frac{d\tilde{y}_*}{dy_*} = \frac{1}{\alpha} \frac{df}{d\tilde{y}_*}$$

and hence

$$\frac{d^2 u_*}{dy_*^2} = \frac{1}{\alpha^2} \frac{d^2 u_*}{d\tilde{y}_*^2}$$

If we denote the new dimensionless numbers with a tilde on top then from equations (8) to (10),

$$\tilde{Re} = \alpha Re \quad (36)$$

$$\tilde{Da} = \frac{Da}{\alpha^2} \quad (37)$$

$$\tilde{J} = J \quad (38)$$

The new pressure gradient is $\tilde{G}_* = \alpha G_*$ while the new constants \tilde{A} , \tilde{B} and \tilde{C} become

$$\tilde{A}^2 = \frac{F_0 \epsilon \tilde{Re}}{\tilde{J} \sqrt{\tilde{Da}}} = \alpha^2 A^2 \quad (39)$$

$$\tilde{B}^2 = \frac{\epsilon}{\tilde{J} \tilde{Da}} = \alpha^2 B^2 \quad (40)$$

$$\tilde{C}^2 = \frac{\epsilon^2 \tilde{Re} \tilde{G}}{\tilde{J}} = \alpha^2 C^2 \quad (41)$$

Therefore, equation (35) becomes,

$$\frac{1}{\alpha^2} \frac{d^2 u_*}{d\tilde{y}_*^2} - \tilde{A}^2 u_*^2 - \tilde{B}^2 u_* + \tilde{C}^2 = 0$$

or

$$\frac{d^2 u_*}{d\tilde{y}_*^2} - \tilde{A}^2 u_*^2 - \tilde{B}^2 u_* + \tilde{C}^2 = 0$$

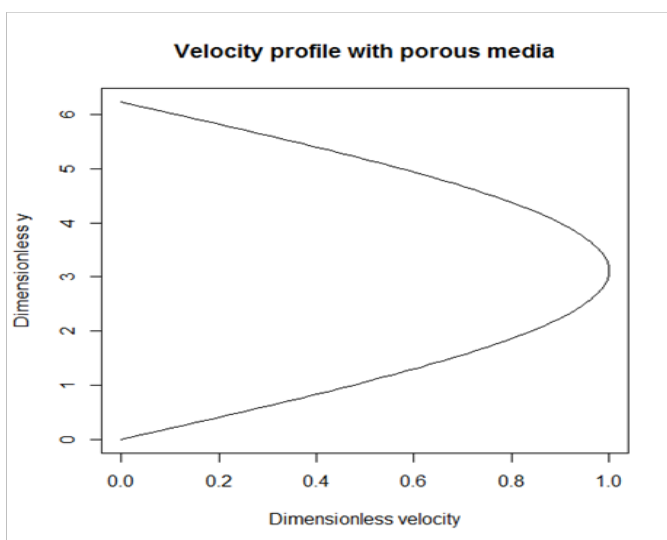


Figure 3 Velocity profile for flow in a porous channel bounded by flat plates.

The vertical axis of the graph in Figure 3 extends from 0 to $2\text{Im}(k_1)$. To shrink it to 0 to 1, we choose $\alpha = 2 * \text{Im}(k_1)$, in this case 6.232543. After changing the constants A , B and C we proceed in similar fashion as before to get the plot in Figure 4.

The code to generate the plot is

Listing 3: Velocity profile of flow in porous medium

```
library(elliptic)

epsilon <- 0.1 # Porosity
Re <- 0.1 # Reynold's number
Da <- 5e-1 # Darcy number
J <- 1 # Viscosity ratio
G <- 1e-4 # Pressure gradient
F0 <- 1.75/sqrt(150 * epsilon^3) # Geometric function

# Re-normalize y using this factor
alpha <- 6.232543

# Calculation of A, B, C
# However, for sake of simplicity, we call the variables A, B and C.
A <- sqrt((F0 * epsilon * Re)/(J * sqrt(Da))) * alpha
B <- sqrt(epsilon/(J * Da)) * alpha
C <- sqrt(epsilon^2 * Re * G/J) * alpha

g2 <- (A^2*C^2 + B^4/4)/3
u2_center <- A^2/6 + B^2/12
# Choose e2 as u2_center
e2 <- u2_center
e1 <- (-e2 + sqrt(g2 - 3*Ii*(-1i)*e2^2))/2
e3 <- (-e2 - sqrt(g2 - 3*Ii*(-1i)*e2^2))/2

g3 <- 4*e1*e2*e3

Omegas <- half.periods(g = c(g2, g3)) print(Omegas)

# Computed using Mathematica
k1 <- 1.286949924601859 + 0.5000000786725393i
P(k1, g = c(g2, g3), give.all.3 = T)

z <- Re(k1) + seq(from = 0, to = -2*Im(k1), len = 100)*1i
y <- -Im(z)
u <- 6/A^2*( P(z + Im(k1)*1i, g = c(g2, g3)) - B^2/12 )

# We choose real part alone. Check that the imaginary part # is very
small. Use the command max(abs(Im(u))) to check # it.
xl <- 'Dimensionless velocity'
yl <- 'Dimensionless y'
ml <- 'Velocity profile with porous media' plot(Re(u), y, type = 'l',
xl = xl, ylab = yl, main = ml)
```

To compare the flow profiles with and without porous medium we choose the pressure gradient in the latter case so that the flow has the same peak velocity as in the former case. If we run the code in listing 4 after the one in listing 3, we can have the two profiles in the same plot as shown in Figure 5.

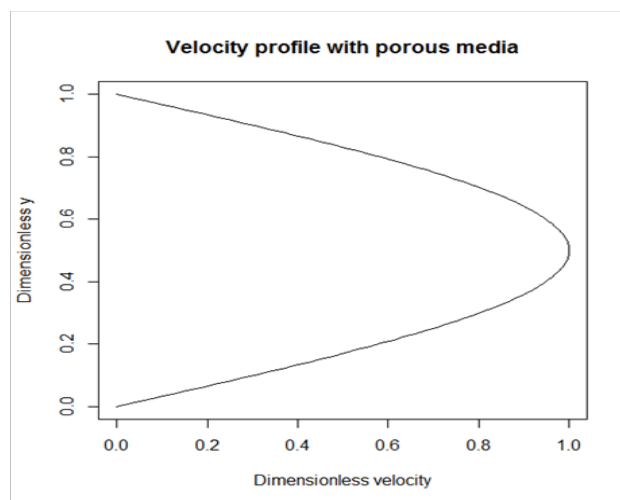


Figure 4 Rescaled velocity profile for flow in a channel bounded by flat plates.

Listing 4: Comparison of velocity profiles

```
# To compare flows with and without porous media,
# adjusting the pressure gradient in the second
# case so that they have the same maximum velocity.
umax <- max(abs(Re(u)))
G.eq <- 2*mu*umax/(Im(k1)^2)
w.eq <- G.eq/mu * y * (Im(k1) - y/2)
xl <- 'Dimensionless velocity'
yl <- 'Dimensionless y'
ml <- 'Comparing flows with and without porous medium.'
plot(Re(u), y, type = 'l', col = 'blue', xl = xl, ylab = yl, main = ml)

lines(w.eq, y, col = 'red')
legend.list <- c('With porous medium', 'Without porous medium')
legend(x = 'topright', legend.list, lty = c(1, 1),
lwd=c(2.5,2.5),col=c('blue','red'))
```

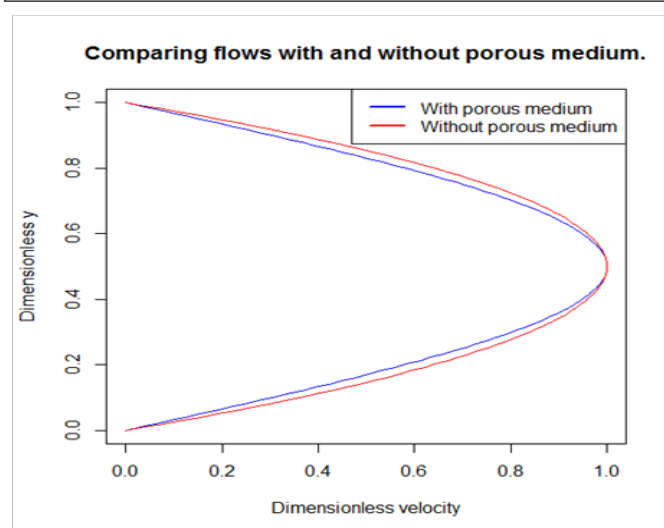


Figure 5 Comparison of velocity profiles.

Calculation of flux

The flux of volume of fluid past a section of the channel is probably more important in practice than the velocity profile. In this section, I will calculate the flux in the channel. If s denotes the transverse dimension of the channel then the flux Q is

$$Q = \int_0^h u ds,$$

where h is the width of the channel and u is velocity of the fluid. If the channel does not have porous medium then

$$u = \frac{G}{2\mu} s(h - s)$$

and hence the flux Q_0 is

$$Q_0 = \frac{Gh^3}{12\mu} \tag{42}$$

On the other hand if the channel is stuffed with porous medium then, the flux Q_1 is

$$Q_1 = \int_0^h \frac{6}{A^2} \left[\wp \left(s + k_1 \mid \frac{1}{3} \left(A^2 C^2 + \frac{B^4}{4} \right), g_3 \right) - \frac{B^2}{12} \right] ds \tag{43}$$

In terms of non-dimensional parameters $h=1$ and hence $Q_0 = 0.6665793$. If we adjust pressure gradient so that peak velocity matches with that when porous medium is present, then code listing 5 gives $Q_0 = 0.6665793$. Carrying out the integration in equation (43) and noting that $s = iy$, we get $Q_1 = 0.6321661$. As expected from Figure 5, $Q_1 < Q_0$. Note that listing 5 is meant to be run after running listing (4).

Listing 5: Calculation of flux,

```
# Flux in presence of porous medium
# up denotes the 'velocity function' when porous medium is present
up <- function(x) {6/A^2*( P(x + Im(k1)*1i, g = c(g2, g3)) - B^2/12)}

# We multiply the integral by i because the integral was carried out
along the imaginary axis.
Q1 <- integrate.segments(up, c(z[1], z[100]), close = F)*1i

# Flux in absence of porous medium
Q0 <- G.eq/(12*mu)
```

There is another way to compute the flux of flow in a porous medium. It uses the relation between Weierstrass \wp function and Weierstrass (not Riemann) ζ function.⁴

$$\wp(y) = -\zeta'(y) \tag{44}$$

Therefore, we get an analytical expression for Q_1 ,

$$Q_1 = \frac{6}{A^2} [\zeta(k_1 \mid g_2, g_3) - \zeta(1 + k_1 \mid g_2, g_3)] - \frac{B^2}{2A^2} \tag{45}$$

The library 'elliptic' allows us to evaluate Weierstrass ζ function. The listing 5, if run after listing 5 computes

$$Q_1 = 0.6321664 .^6$$

Listing 6: Another way to compute flux

```
# We calculate the two terms in equation (45) separately.
Q11 <- 6/A^2*(zeta(z[1] + Im(k1)*1i, g = c(g2, g3)) - zeta(z[100] + Im(k1)*1i, g = c(g2, g3)))
Q12 <- B^2/(2*A^2)*(z[100] - z[1])

# We multiply the result by i because the integral was carried out
along the imaginary axis.
Q1.using.zeta <- (Q11 - Q12)*1i
```

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Conflict of interest

Author declares there is no conflict of interest in publishing the article.

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