

# Global wellposedness of a free boundary problem for the navier-stokes equations in an exterior domain

## Abstract

In this paper, we prove a unique existence theorem of globally in time strong solutions to free boundary problem for the Navier-Stokes equations in an exterior domain in the case that initial data are small enough. The key step is to prove decay properties of locally in time solutions, which is derived by combination of maximal  $L_p$ - $L_q$  regularity with  $L_p$ - $L_q$  decay estimates for solutions of slightly perturbed Stokes equations with free boundary condition in an exterior domain.

**Keywords:** navier-stokes equations, free boundary problem, global well-posedness, exterior domain, polynomial decay, maximal  $L_p$ - $L_q$  regularity

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## Introduction

This paper deals with global well-posedness of the following free boundary problem for the Navier-Stokes equations. Let  $\Omega$  be an exterior domain in the  $N$  dimensional Euclidean space  $\mathbb{R}^N$  occupied by an incompressible viscous fluid. Let  $\Gamma$  be the boundary of  $\Omega$  that is a  $C^2$  compact hyper surface with the unit outer normal  $\mathbf{n}$ . Let  $\Omega_t$  be the evolution of  $\Omega$  at time  $t$ . Let  $\Gamma_t$  be the boundary of  $\Omega_t$  with the unit outer normal  $\mathbf{n}_t$ . Let  $\rho$  and  $\mu$  be positive numbers denoting the mass density and the viscosity coefficient, respectively. Let  $\mathbf{u} = (u_1, \dots, u_N)^T$  be an  $N$ -vector of functions describing the velocity field, where  ${}^T M$  denotes the transposed  $M$ , and let  $p$  be a scalar function describing the pressure field. We consider the initial boundary value problem for the Navier-Stokes equations in  $\Omega_t$  given by

$$\begin{aligned} \rho(\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}) - \text{Div}(\mu \mathbf{D}(\mathbf{u}) - \mathbf{P}\mathbf{I}) &= 0, & \text{div} \mathbf{u} &= 0 & \text{in } \bigcup_{0 < t < T} \Omega_t \times \{t\}, \\ (\mu \mathbf{D}(\mathbf{u}) - \mathbf{P}\mathbf{I}) \mathbf{n}_t &= 0, & V_{\Gamma_t} &= \mathbf{n}_t \cdot \mathbf{u} & \text{on } \bigcup_{0 < t < T} \Gamma_t \times \{t\}, \\ \mathbf{u}|_{t=0} &= \mathbf{u}_0, & \Omega_t|_{t=0} &= \Omega \end{aligned} \quad (1)$$

Here,  $\mathbf{D}(\mathbf{u}) = \nabla \mathbf{u} + (\nabla \mathbf{u})^T$  denotes the doubled deformation tensor,  $\mathbf{I}$  the  $N \times N$  identity matrix, and  $V_{\Gamma_t}$  the evolution speed of the surface  $\Gamma_t$  in the  $\mathbf{n}_t$  direction. Moreover, for any matrix field  $\mathbf{K}$  with  $(i, j)$  component  $K_{ij}$ , the quantity  $\text{Div} \mathbf{K}$  is an  $N$  vector of functions whose  $i$ th component is  $\sum_{j=1}^N \partial_j K_{ij}$ ,  $\partial_j = \partial / \partial x_j$  and for any  $N$  vector of function  $\mathbf{w} = (w_1, \dots, w_N)$ ,  $\text{div} \mathbf{w} = \sum_{j=1}^N \partial_j w_j$  and the quantity  $\mathbf{w} \cdot \nabla \mathbf{w}$  is an  $N$ -vector of functions whose  $i$ th component is  $\sum_{j=1}^N \partial_j w_i w_j$ . One phase problem for the Navier-Stokes equations formulated in (1) with  $(\mu \mathbf{D}(\mathbf{u}) - \mathbf{P}\mathbf{I}) \mathbf{n}_t = c_\sigma H \mathbf{n}_t$  in place of  $(\mu \mathbf{D}(\mathbf{u}) - \mathbf{P}\mathbf{I}) \mathbf{n}_t = 0$  has been received wide attention for many years, where  $H$  is the doubled mean curvature of  $\Gamma_t$  and  $c_\sigma$  is a non-negative constant describing the coefficient of surface tension. In particular, the

following two cases have been studied by many mathematicians: (1) the motion of an isolated liquid mass and (2) the motion of a viscous incompressible fluid contained in an ocean of infinite extent. In case (1), the initial domain  $\Omega_0$  is bounded and local well posedness in the case that  $c_\sigma > 0$  was proved by Solonnikov<sup>1-4</sup> in the  $L_2$  Sobolev-Slobodetskii space, by Schweizer<sup>5</sup> in the semi group setting, and by Mogilevskii et al.,<sup>6</sup> in the Holder spaces. And, in the case that  $c_\sigma = 0$ , local wellposedness was proved by Solonnikov,<sup>7</sup> Mucha et al.,<sup>8,9</sup> in the  $L_p$  Sobolev-Slobodetskii space, and by Shibata<sup>10,11</sup> in the  $L_p$  in time and  $L_q$  in space setting. Global wellposedness in the case that  $c_\sigma = 0$  for small initial data by Solonnikov<sup>4,7</sup> in the  $L_p$  Sobolev-Slobodetskii space and by Shibata<sup>12</sup> in the  $L_p$  in time and  $L_q$  in space setting. Global wellposedness in the case that  $c_\sigma = 0$  was proved under the assumption that the initial domain  $\Omega_0$  is sufficiently close to a ball and initial data are very small by Solonnikov<sup>13</sup> in the  $L_2$  Sobolev-Slobodetskii space, by Padula et al.,<sup>14</sup> in the Holder spaces, and by Shibata<sup>15</sup> in the  $L_p$  in time and  $L_q$  in space setting.

In case (2), the initial domain  $\Omega_0$  is a perturbed layer like:

$\Omega_0 = \{x \in \mathbb{R}^3 \mid -b < x_3 < \eta(x'), x' = (x_1, x_2) \in \mathbb{R}^2\}$  and local wellposed was proved by Allain,<sup>16</sup> Beale<sup>17</sup> and Tani<sup>18</sup> in the  $L_2$  Sobolev-Slobodetskii space when  $c_\sigma > 0$  and by Abels<sup>19</sup> in the  $L_p$  Sobolev-Slobodetskii space when  $c_\sigma = 0$ .

Global wellposedness for small initial velocity was proved in the  $L_2$  Sobolev-Slobodetskii space by Beale<sup>20</sup> and Tani et al.,<sup>21</sup> in the case that  $c_\sigma > 0$  and by Sylvester<sup>22</sup> in the case that  $c_\sigma = 0$ . The decay rate was studied by Beale et al.,<sup>23</sup> Sylvestre,<sup>24</sup> Hataya<sup>25</sup> & Hataya et al.<sup>26</sup> In the case of the Ocean problem without bottom,  $\Omega_0 = \{x \in \mathbb{R}^3 \mid x_3 < \eta(x'), x' = (x_1, x_2) \in \mathbb{R}^2\}$ .

In this case, global well posedness for small initial data and the decay properties of solutions have been studied by Saito et al.<sup>27-29</sup> Recently, local well-posedness for the one phase problem of the Navie-Stokes equations was proved in the general unbounded domain case by Shibata<sup>12</sup> in the  $c_\sigma = 0$  case and by Shibata<sup>29,30</sup> in the case  $c_\sigma > 0$ .

We remark that two phase problem of the Navier-Stokes equations

has been studied by many mathematicians,<sup>31–46</sup> and references therein. Although many papers dealt with global well-posedness, as far as the author knows, global well-posedness of free boundary problem for the Navier-Stokes equations in an exterior domain has never been treated, and the purpose of this paper is to prove global well-posedness of problem (1) in the  $L_p$  in time and  $L_q$  in space setting. Since only polynomial decays are obtained for solutions of Stokes equations with free boundary conditions in the exterior domain case;<sup>47,48</sup> it is necessary to choose a large exponent  $p$  to guarantee  $L_p$  integrability of solutions, so that the maximal  $L_p$ - $L_q$  regularity for the Stokes equations with free boundary condition proved in Shibata<sup>30,49,50</sup> and also in Pruess et al.,<sup>40</sup> in the different  $p$  and  $q$  case are one of essential tools.

Now we consider the transformation that transforms  $\Omega_t$  to a fixed domain. If  $\Omega$  is a bounded domain, then we have the exponential stability of the corresponding Stokes equations with free boundary conditions in some quotient space, so that we can use the Lagrange transformation to transform  $\Omega_t$  to  $\Omega$ .<sup>7,12</sup> But, is now an exterior domain, so that solutions of the Stokes equations with free boundary conditions decay polynomially as mentioned above. Thus, the Lagrange transformation is not available, because the polynomial decay does not seem to be enough to control the term  $\int_0^t \nabla u ds$ . Another known transformation is the Hanzawa one. But, this transformation requires at least the  $W_q^{3-1/q}$  ( $N < q$ ) regularity of the height function representing  $\Gamma$ , and such regularity is usually derived from surface tension. In our case, surface tension is not taken into account, so that such regularity is unable to be obtained. To overcome such difficulty, our idea is to use the Lagrange transformation only near  $\Gamma$ .

Let  $R$  be a positive number such that  $O = \mathbb{R}^N \setminus \Omega \subset B_{R/2}$ , where  $B_L = \{x \in \mathbb{R}^N \mid |x| < L\}$ , and let  $\kappa$  be a  $C^\infty$  function such that  $\kappa(\xi) = 1$  for  $|\xi| \leq R$  and  $\kappa(\xi) = 0$  for  $|\xi| \geq 2R$ . Let  $\mathbf{v}(\xi, t)$  and  $q(\xi, t)$  be the velocity field and the pressure field in Lagrange coordinates  $\{\xi\}$ . Let us consider a transformation,

$$x = \mathbf{L}(\xi, t) := \xi + \int_0^t \kappa(\xi) \mathbf{v}(\xi, s) ds \quad (2)$$

Let  $\delta$  be a positive number such that the transformation:  $x = \mathbf{L}(\xi, t)$  is bijective from  $\Omega$  on to  $\Omega_t = \{x = \mathbf{L}(\xi, t) \mid \xi \in \Omega\}$  for each  $t \in (0, T)$  provided that

$$\int_0^T \|\nabla(\kappa(\cdot) \mathbf{v}(\cdot, s))\|_{L_\infty(\Omega)} ds \leq \delta \quad (3)$$

Since  $\delta$  will be chosen as a small positive number eventually, we may assume that  $0 < \delta \leq 1$ . Let

$$\ell_{ij}(t) = \delta_{ij} + \int_0^t \frac{\partial}{\partial \xi_j} (\kappa(\xi) v_i(\xi, s)) ds \quad (\mathbf{v} = (v_1, \dots, v_N)), \quad \mathbf{A}(t) (\ell_{ij}(t))^{-1} = (a_{ij}(t)) \quad (4)$$

where  $\delta_{ij}$  are the Kronecker delta symbols, that is  $\delta_{ii} = 1$  and  $\delta_{ij} = 0$  for  $i \neq j$ . Here and hereafter, a function  $a = a(\xi, t)$  is written simply by  $a(t)$  and  $(a_{ij})$  denotes the  $N \times N$  matrix whose  $(i, j)$  component is  $a_{ij}$ , unless confusion may occur. For a while, we

assume that the  $N \times N$  matrix  $(\ell_{ij}(t))$  is invertible.

Let

$$\Omega_t = \{x = \mathbf{L}(\xi, t) \mid \xi \in \Omega\}, \quad \Gamma_t = \{x = \mathbf{L}(\xi, t) \mid \xi \in \Gamma\},$$

$\mathbf{u}(x, t) = \mathbf{v}(\mathbf{L}^{-1}(x, t))$  and  $p(x, t) = q(\mathbf{L}^{-1}(x, t))$  in Eq. (1), and then,  $\mathbf{v}$  and  $q$  satisfy the following

equations:

$$\begin{cases} \partial_t \mathbf{v} - \text{Div}(\mu \mathbf{D}(\mathbf{v}) - q \mathbf{I}) = \mathbf{f}(\mathbf{v}), & \text{in } \Omega \times (0, T), \\ \text{div } \mathbf{v} = g(\mathbf{v}) = \text{div } \mathbf{g}(\mathbf{v}) & \text{in } \Omega \times (0, T), \\ (\mu \mathbf{D}(\mathbf{v}) - q \mathbf{I}) \mathbf{n} = \mathbf{h}(\mathbf{v}) & \text{on } \Gamma \times (0, T), \\ \mathbf{v}|_{t=0} = \mathbf{u}_0 & \text{in } \Omega. \end{cases}$$

Here,  $\mathbf{f}(\mathbf{v})$  is consisting of some linear combinations of nonlinear functions of the forms

$$\begin{aligned} V_0 \left( \int_0^t \nabla(\kappa \mathbf{v}) ds \right) \nabla^2 \mathbf{v}, \quad V_1 \left( \int_0^t \nabla(\kappa \mathbf{v}) ds \right) \partial_t \mathbf{v}, \\ V_2 \left( \int_0^t \nabla(\kappa \mathbf{v}) ds \right) \left( \int_0^t \nabla^2(\kappa \mathbf{v}) ds \right) \nabla \mathbf{v}, \quad (1 - \kappa) (\cdot \mathbf{A})^{-1} \{ \mathbf{v} \cdot (\cdot \mathbf{A} \nabla \mathbf{v}) \}; \end{aligned} \quad (6)$$

and  $g(\mathbf{v})$  and  $\mathbf{h}(\mathbf{v})$  are nonlinear functions of the forms:

$$g(\mathbf{v}) = V_3 \left( \int_0^t \nabla(\kappa \mathbf{v}) ds \right) \nabla \mathbf{v}; \quad \mathbf{g}(\mathbf{v}) = V_4 \left( \int_0^t \nabla(\kappa \mathbf{v}) ds \right) \mathbf{v}; \quad \mathbf{h}(\mathbf{v}) = V_5 \left( \int_0^t \nabla(\kappa \mathbf{v}) ds \right) \nabla \mathbf{v} \quad (7)$$

with some nonlinear functions  $V_i$  such that  $V_i(0) = 0$  except for  $i = 2$ .

The main result of this paper is the following theorem that shows the unique existence theorem of global in time solutions of Eq. (5) and asymptotics as  $t \rightarrow \infty$ .

## Theorem 1

Let  $N \geq 3$  and let  $q_1$  and  $q_2$  be exponents such that  $\max \left( N, \frac{2N}{N-2} \right) < q_2 < \infty$  and  $1/q_1 = 1/q_2 + 1/N$ . Let  $b$  and  $p$  be numbers defined by

$$b = \frac{3N}{2q_2} + \frac{1}{2}, \quad p = \frac{2q_2(1+\sigma)}{q_2 - N} \quad (8)$$

with some very small positive number  $\sigma$ . Then, there exists an  $\varepsilon > 0$  such that if initial data  $\mathbf{u}_0 \in B_{q_2, p}^{2(1-1/p)}(\Omega)^N \cap B_{q_{1/2}, p}^{2(1-1/p)}(\Omega)^N$  satisfies the compatibility condition:

$$\text{div } \mathbf{u}_0 = 0 \text{ in } \Omega, \quad \mathbf{D}(\mathbf{u}_0) \mathbf{n} - \mathbf{D}(\mathbf{u}_0) \mathbf{n} \cdot \mathbf{n} = 0 \text{ on } \Gamma \quad (9)$$

and the smallness condition: with  $I = \|\mathbf{u}_0\|_{B_{q_2, p}^{2(1-1/p)}} + \|\mathbf{u}_0\|_{B_{q_{1/2}, p}^{2(1-1/p)}}$ , then Eq. (5) admits unique solutions  $\mathbf{v}$  and  $q$  with

$$\mathbf{v} \in L_p \left( (0, \infty), H_{q_1}^2(\Omega)^N \right) \cap H_p^1 \left( (0, \infty), L_{q_2}(\Omega)^N \right), \quad q \in L_p \left( (0, \infty), H_{q_1}^1(\Omega) + \dot{H}_{q_1, 0}^1(\Omega) \right),$$

possessing the estimate  $\|v\|_\infty \leq C\varepsilon$  with

$$\begin{aligned} [\mathbf{v}]_T = & \left\{ \int_0^T \left( \langle s \rangle^b \|\mathbf{v}(\cdot, s)\|_{H_{\infty}^1(\Omega)} \right)^p ds \right. \\ & + \int_0^T \left( \langle s \rangle^{\left(b - \frac{N}{2q_1}\right)} \|\mathbf{v}(\cdot, s)\|_{H_{q_1}^1(\Omega)}^p ds + \left( \sup_{0 < s < T} \langle s \rangle^{\frac{N}{2q_1}} \|\mathbf{v}(\cdot, s)\|_{L_{q_1}(\Omega)} \right)^p \right. \\ & \left. \left. + \int_0^T \left( \langle s \rangle^{\left(b - \frac{N}{2q_2}\right)} \|\mathbf{v}(\cdot, s)\|_{H_{q_2}^2(\Omega)} + \|\partial_t \mathbf{v}(\cdot, s)\|_{L_{q_2}(\Omega)} \right)^p ds \right)^{\frac{1}{p}}. \end{aligned}$$

Here,  $\langle s \rangle = (1 + s^2)^{1/2}$  and  $C$  is a constant that is independent of  $\epsilon$ .

### Remark 2

Let  $1/p' = 1 - 1/p$ , that is  $1/p' = 1 - 1/p$ . And then,

$$\frac{1}{p'} = \frac{(1 + 2\sigma)q_2 + N}{2q_2(1 + \sigma)}$$

We choose  $\sigma > 0$  small enough in such a way that the following relations hold,

$$\begin{aligned} 1 < q_1 < 2, \frac{N}{q_1} > b > \frac{1}{p'}, \left( \frac{N}{q_1} - b \right) p > 1, \left( b - \frac{N}{2q_2} \right) p > 1, b \geq \frac{N}{2q_1}, \\ b \geq \frac{N}{q_2}, \left( \frac{1}{2q_2} + \frac{1}{2} \right) p' < 1, \left( b - \frac{N}{2q_2} \right) p' > 1, \frac{N}{q_2} + \frac{2}{p} < 1. \end{aligned}$$

### Remark 3

The exponent  $q_2$  is used to control the nonlinear terms, so that  $q_2$  is chosen as  $N < q_2 < \infty$ .

Let

$$\frac{1}{q_1} = \frac{1}{N} + \frac{1}{q_2}, \frac{1}{q_3} = \frac{1}{q_1} + \frac{1}{q_2}. \quad (12)$$

And the condition:  $q_2 > \frac{2N}{N-2}$  implies that  $q_1 > 2$  and  $q_3 > 1$  which is necessary to prove Theorem 1.

Thus, we assume that

$$\max \left( N, \frac{2N}{N-2} \right) < q_2 < \infty.$$

### Remark 4

We can choose  $\delta > 0$  so small that  $x = L(\xi, t)$  is a diffeomorphism with suitable regularity from  $\Omega$  onto  $\Omega_t$ , so that the original problem (1) is globally well-posed.

Finally, we explain several symbols used in this paper. We use bold small letters to denote  $N$ -vectors of functions and bold capital letters to denote  $N \times N$  matrix of functions. For a scalar function  $h = h(x)$ ,

$\nabla h$  is an  $N$  vector whose  $i^{\text{th}}$  component is  $\partial_i h = \partial h / \partial x_i$ . For  $\mathbf{v}(x) = (v_1(x), \dots, v_N(x))$ ,  $\nabla \mathbf{v}$  is an  $N \times N$  matrix of functions whose  $(i, j)$  component is  $\partial_i v_j$ . Given exponent  $1 < q < \infty$ , let  $q' = q/(q-1)$ .

Let  $L_q(\Omega)$ ,  $H_q^m(\Omega)$  and  $B_{q,p}^s(\Omega)$  be the standard Lebesgue, Sobolev, and Besov spaces on  $\Omega$ , while  $\|\bullet\|_{L_q(\Omega)}$ ,  $\|\bullet\|_{H_q^m(\Omega)}$ ,  $\|\bullet\|_{B_{q,p}^s(\Omega)}$  denote their norms, respectively. For a Banach space  $X$  with norm  $\|\bullet\|_X$ ,

Let  $\{(f_1, \dots, f_d) \mid f_i \in X (i = 1, \dots, d)\}$ , while the norm of  $X_d$  is written simply by  $\|\bullet\|_X$ , which is defined by

$$\|f\|_X = \sum_{j=1}^d \|f_j\|_X \text{ for } f = (f_1, \dots, f_d) \in X_d. \text{ Let } \hat{H}_{q,0}^1(\Omega) = \left\{ \theta \in L_{q,loc}(\Omega) \mid \nabla \theta \in L_q(\Omega)^N, \theta|_{\Gamma} = 0 \right\}$$

$$\hat{H}_q^1(\Omega) = \hat{H}_{q,0}^1(\Omega) + \left\{ p = p_1 + p_2 \mid p_1 \in H_q^1(\Omega), p_2 \in \hat{H}_{q,0}^1(\Omega) \right\}$$

For  $1 \leq p \leq \infty$ ,  $L_p((a, b), X)$  and  $H_p^p((a, b), X)$  denote the standard Lebesgue and Sobolev spaces of

$X$ -valued functions defined on an interval  $(a, b)$ , while  $\|\bullet\|_{H_p^p((a,b),X)}$  and  $\|\bullet\|_{H_m^p((a,b),X)}$  denote their norms, respectively. The letter  $C$  denotes generic constants and  $C_{a,b,c,\dots}$  means that the constant  $C_{a,b,c,\dots}$  depends on  $a, b, c, \dots$ . The values of  $C$  and  $C_{a,b,c}$  may change from line to line.

After Introduction (Sect. 1), the paper is organized as follows: In Sect. 2, we reformulate problem (5) by using the formula:

$$\int_0^t \nabla(\kappa(\xi)\mathbf{v}(\xi, s)) ds = \int_0^t \nabla(\kappa(\xi)\mathbf{v}(\xi, s)) ds - \int_0^t \nabla(\kappa(\xi)\mathbf{v}(\xi, s)) ds.$$

In Sect. 3, we give estimations of the nonlinear terms. In Sect. 4, we explain how to prolong local in time solutions to the infinite time interval  $(0, \infty)$ . Finally, in Sect. 5, we prove Theorem 1 by using maximal  $L_p$ - $L_q$  regularity and  $L_p$ - $L_q$  decay estimate for solutions of the perturbed Stokes equations with free boundary condition in an exterior domain, which was proved in Shibata.<sup>30, 47</sup>

### Another formulation of Eq. (5)

Let  $T > 0$  and let

$$\mathbf{v} \in H_p^1((0, T), L_q(\Omega)^N) \cap L_p((0, T), H_q^2(\Omega)), \quad q \in L_p((0, T), H_q^1(\Omega) + \hat{H}_{q,0}^1(\Omega)) \quad (13)$$

be solutions of Eq. (5) satisfying the condition (3). In what follows, we rewrite Eq. (5) in order that the nonlinear terms have suitable decay properties.

Let  $\mathbf{A}(t) = (a_{ij}(t))$  be the  $N \times N$  matrix given in (4), and let  $\mathbf{n}_t = (n_{t1}, \dots, n_{tN})$  and  $\mathbf{n} = (n_1, \dots, n_N)$ , and then by the transformation (2), we have

$$a_{ji}(t) \frac{\partial}{\partial \xi_j}, \frac{\partial}{\partial x_j} = \sum_{j=1}^N n_{ti} = d(t) \sum_{j=1}^N a_{ji}(t) n_j \quad (14)$$

where  $d(\mathbf{t}) = |T \mathbf{A}(\mathbf{t}) \mathbf{n}|$ .

$$a_{ji}(\mathbf{t}) = \delta_{ij} + \tilde{a}_{ij}(\mathbf{t}), \quad J(\mathbf{t}) = 1 + \tilde{J}(\mathbf{t}), \quad \ell_{ij}(\mathbf{t}) = \delta_{ij} + \tilde{\ell}_{ij}(\mathbf{t}), \quad (15)$$

Where  $\ell_{ij}$  are given in (4) and  $J$  is the Jacobian of the transformation (2), that is  $J = J = \det(\ell_{ij})$ , and then

$$\begin{aligned} \tilde{a}_{ij}(\mathbf{t}) &= b_{ij} \left( \int_0^t \nabla(\kappa(\xi)) \mathbf{v}(\xi, s) ds \right), \quad \tilde{J}(\mathbf{t}) = K \left( \int_0^t \nabla(\kappa(\xi)) \mathbf{v}(\xi, s) ds \right), \\ \tilde{\ell}_{ij}(\mathbf{t}) &= m_{ij} \left( \int_0^t \nabla(\kappa(\xi)) \mathbf{v}(\xi, s) ds \right) := \int_0^t \frac{\partial}{\partial \xi_j} (\kappa(\xi) \mathbf{v}_i(\xi, s)) ds \end{aligned} \quad (16)$$

With some smooth functions  $b_{ij}$  and  $K$  defined on  $\{\mathbf{w} \mid |\mathbf{w}| \leq 1\}$  such that  $b_{ij}(0) = K(0) = 0$ , where  $\mathbf{w}$  is the corresponding variable to  $\int_0^t \nabla(\kappa(\xi)) \mathbf{v}(\xi, s) ds$ . Let  $\mathbf{u}(x, t) = \mathbf{v}(\xi, t)$  and  $p(x, t) = q(\xi, t)$  and then  $\mathbf{u}$  and  $p$  satisfy Eq. (1). By (14),

$$\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} = D_{ij,t}(\mathbf{v}) := D_{ij}(\mathbf{v}) + \tilde{D}_{ij}(\mathbf{t}) \Delta \mathbf{v}$$

With

$$D_{ij}(\mathbf{v}) = \frac{\partial v_i}{\partial \xi_j} + \frac{\partial v_j}{\partial \xi_i}, \quad \tilde{D}_{ij}(\mathbf{t}) \Delta \mathbf{v} = \sum_{K=1}^N \tilde{a}_{kj}(\mathbf{t}) \frac{\partial v_i}{\partial \xi_k} + \tilde{a}_{ki}(\mathbf{t}) \frac{\partial v_j}{\partial \xi_k} \quad (17)$$

We also have an important formula:

$$\operatorname{div} \mathbf{u} = \sum_{j=1}^N \frac{\partial u_j}{\partial x_j} = \sum_{j,k=1}^N J(\mathbf{t}) a_{kj}(\mathbf{t}) \frac{\partial v_j}{\partial \xi_k} = \sum_{j,k=1}^N \frac{\partial}{\partial \xi_k} (J(\mathbf{t}) a_{kj}(\mathbf{t}) v_j) \quad (18)$$

which implies that

$$\sum_{j,k=1}^N \tilde{a}_{kj}(\mathbf{t}) = \tilde{J}(\mathbf{t}) a_{kj}(\mathbf{t}) \frac{\partial v_j}{\partial \xi_k} = \sum_{j,k=1}^N \frac{\partial}{\partial \xi_k} \left\{ (\tilde{a}_{kj}(\mathbf{t}) + \tilde{J}(\mathbf{t}) a_{kj}(\mathbf{t})) v_j \right\} \quad (19)$$

And then, Eq. (5) is written as follows:

$$\begin{cases} \sum_{i=1}^N \ell_{is}(\mathbf{t}) (\partial_t v_i + (1-\kappa) \sum_{j,k=1}^N v_j a_{kj}(\mathbf{t}) \frac{\partial v_i}{\partial \xi_k}) \\ - \mu \sum_{i,j,k=1}^N \ell_{is}(\mathbf{t}) a_{kj}(\mathbf{t}) \frac{\partial}{\partial \xi_k} D_{i,j,t}(\mathbf{v}) - \frac{\partial q}{\partial \xi_s} = 0 & \text{in } \Omega \times (0, T), \\ \sum_{j,k=1}^N J(\mathbf{t}) a_{kj}(\mathbf{t}) \frac{\partial v_j}{\partial \xi_k} = \sum_{j,k=1}^N \frac{\partial}{\partial \xi_k} (J(\mathbf{t}) a_{kj}(\mathbf{t}) v_j) = 0 & \text{in } \Omega \times (0, T) \\ \mu \sum_{i,j,k=1}^N \ell_{is}(\mathbf{t}) a_{kj}(\mathbf{t}) D_{i,j,t}(\mathbf{v}) n_k - q n_s = 0 & \text{on } \Gamma \times (0, T), \\ v|_{t=0} = \mathbf{u}_0 & \text{in } \Omega \end{cases} \quad (20)$$

Where  $s$  runs from 1 through  $N$ . Here, we have used the fact that  $(\ell_{ij}) = \mathbf{A}^{-1}$  which follows from (4).

In order to get some decay properties of the nonlinear terms, we write

$$\int_0^t \nabla(\kappa(\xi)) \mathbf{v}(\xi, s) ds = \int_0^T \nabla(\kappa(\xi)) \mathbf{v}(\xi, s) ds - \int_0^T \nabla(\kappa(\xi)) \mathbf{v}(\xi, s) ds \cdot$$

In (16), by the Taylor formula we write

$$\begin{aligned} a_{ij}(\mathbf{t}) &= a_{ij}(T) + A_{ij}(\mathbf{t}), \quad \ell_{ij}(\mathbf{t}) = \ell_{ij}(T) + L_{ij}(\mathbf{t}), \\ D_{ij,t} &= D_{ij,T}(\mathbf{v}) + D_{ij}(\mathbf{t}) \Delta \mathbf{v} \quad J(\mathbf{t}) = J(T) + J(\mathbf{t}) \end{aligned} \quad (21)$$

With

$$\begin{aligned} A_{ij}(\mathbf{t}) &= \int_0^1 -b'_{ij} \int_0^T \nabla(\kappa(\xi)) \mathbf{v}(\xi, s) ds - \theta \int_0^T \nabla(\kappa(\xi)) \mathbf{v}(\xi, s) ds d\theta \int_0^T \nabla(\kappa(\xi)) \mathbf{v}(\xi, s) ds, \\ L_{ij}(\mathbf{t}) &= - \int_0^T \frac{\partial}{\partial \xi_j} \kappa(\xi) v_i(\xi, s) ds, \quad D_{ij}(\mathbf{t}) \Delta \mathbf{v} = \sum_{k=1}^N (A_{kj}(\mathbf{t}) \frac{\partial v_i}{\partial \xi_k} + A_{ki}(\mathbf{t}) \frac{\partial v_j}{\partial \xi_k}), \\ J(\mathbf{t}) &= - \int_0^1 K' \left( \int_0^T \nabla(\kappa(\xi)) \mathbf{v}(\xi, s) ds - \theta \int_0^T \nabla(\kappa(\xi)) \mathbf{v}(\xi, s) ds \right) d\theta \int_0^T \nabla(\kappa(\xi)) \mathbf{v}(\xi, s) ds \end{aligned}$$

Where  $b'_{ij}$  and  $K'$  are derivatives of  $b_{ij}$  and  $K$  with respect to  $\mathbf{w}$ . By the relation:

$$\sum_{s=1}^N \ell_{is}(T) a_{sm}(T) = \delta_{im}, \quad (22)$$

the first equation in (20) is rewritten as follows:

$$\partial_t v_m - \mu \sum_{j,k=1}^N a_{jk}(T) \frac{\partial}{\partial \xi_k} (\mu D_{mj,T}(\mathbf{v}) - \delta_{mj} q) = f_m(\mathbf{v})$$

With

$$\begin{aligned} f_m(\mathbf{v}) &= - \sum_{s=1}^N a_{sm}(T) \left\{ \sum_{i=1}^N L_{is}(\mathbf{t}) \partial_i v_i + \sum_{i,j,k=1}^N (1-\kappa) \ell_{is}(\mathbf{t}) a_{kj}(\mathbf{t}) v_i \frac{\partial v_j}{\partial \xi_k} \right\} \\ &+ \mu \sum_{s=1}^N a_{sm}(T) \left\{ \sum_{i,j,k=1}^N \ell_{is}(T) a_{kj}(T) \frac{\partial}{\partial \xi_k} (D_{ij}(\mathbf{t}) \nabla \mathbf{v}) + \sum_{i,j,k=1}^N \ell_{is}(T) A_{kj}(\mathbf{t}) \frac{\partial}{\partial \xi_k} D_{ij,t}(\mathbf{v}) \right\} \\ &+ \sum_{i,j,k=1}^N L_{is}(\mathbf{t}) a_{kj}(\mathbf{t}) \frac{\partial}{\partial \xi_k} D_{ij,t}(\mathbf{v}) \end{aligned} \quad (23)$$

Next, by (18)

$$\widetilde{\operatorname{div} \mathbf{v}} = g(\mathbf{v}) = \operatorname{div} g(\mathbf{v})$$

With

$$\widetilde{\operatorname{div} \mathbf{v}} = \sum_{j,k=1}^N J(T) a_{kj}(T) \frac{\partial v_i}{\partial \xi_k} = \sum_{j,k=1}^N \frac{\partial}{\partial \xi_k} J(T) a_{kj}(T) v_j$$

$$g(\mathbf{v}) = \sum_{k=1}^N (J(T) A_{kj}(\mathbf{t}) + J(\mathbf{t}) a_{kj}(\mathbf{t})) \frac{\partial v_j}{\partial \xi_k} \quad (24)$$

$$g_k(\mathbf{v}) = \sum_{j=1}^N (J(T) A_{kj}(\mathbf{t}) + J(\mathbf{t}) a_{kj}(\mathbf{t})) v_j, \quad g(\mathbf{v}) = (g_1(\mathbf{v}), \dots, g_N(\mathbf{v}))$$

Finally, we consider the boundary condition. Let  $\tilde{\mathbf{n}}$  be an  $N$ -vector defined on  $\mathbb{R}^N$  such that  $\tilde{\mathbf{n}} = \mathbf{n}$  on  $\Gamma$  and  $\|\tilde{\mathbf{n}}\|_{H_\infty^2(\mathbb{R}^N)} \leq C$ . In what follows,  $\tilde{\mathbf{n}}$  is simply written by  $\mathbf{n} = (n_1, \dots, n_N)$ . By (14) and (22)

$$\sum_{j,k=1}^N a_{jk}(T) (\mu D_{mj,T}(\mathbf{v}) - \delta_{mj} q) n_k = h_m(\mathbf{v})$$

with

$$h_m(v) = -\mu \sum_{j,k=1}^N \left( a_{kj}(T) D_{mj}(t) \nabla v + A_{kj}(t) D_{mj,t}(v) \right) n_k - \mu \sum_{i,j,k,s=1}^N a_{sm}(T) L_{is}(t) a_{kj}(t) D_{ij,t}(v) n_k \quad (25)$$

By (18),

$$\mu \sum_{j,k=1}^N a_{kj}(T) \frac{\partial}{\partial \xi_k} (\mu D_{mj,T}(v) - \delta_{mj} q) = J(T)^{-1} \sum_{j,k=1}^N \frac{\partial}{\partial \xi_k} \left\{ J(T) a_{jk}(T) (D_{mj,T}(v) - \delta_{mj} q) \right\}$$

And

$$\sum_{j,k=1}^N a_{kj}(T) D_{mj,T}(v) n_k - \left( \sum_{s=1}^N a_{sm}(T) n_s \right) q = \sum_{k=1}^N \left[ \sum_{j=1}^N \left\{ J(T) a_{kj}(T) (D_{mj,T}(v) - \delta_{mj} q) \right\} \right] n_k$$

Thus, letting

$$S_{mk}(v, q) = \sum_{j=1}^N J(T) a_{kj}(T) (\mu D_{mj,T}(v) - \delta_{mj} q), \quad \tilde{S}(v, q) = (S_{ij}(v, q)),$$

$$f(v) = (f_1(v), \dots, f_N(v)), \quad h(v) = (h_1(v), \dots, h_N(v)),$$

and using (18), we see that  $v$  and  $q$  satisfy the following equations:

$$\begin{cases} \partial_t v - J(T)^{-1} \operatorname{Div} \tilde{S}(v, q) = f(v) & \text{in } \Omega \times (0, T), \\ \operatorname{div} v = g(v) = \operatorname{div} g(v) & \text{in } \Omega \times (0, T), \\ \tilde{S}(v, q) n = h(v) & \text{on } \Gamma \times (0, T), \\ v|_{t=0} = u_0 & \text{in } \Omega. \end{cases} \quad (26)$$

## Estimates for the nonlinear terms

Let  $f(v)$ ,  $g(v)$ , and  $h(v)$  are functions defined in Sect. 2. In this section, we estimate these functions. In what follows we write

$$\| \langle t \rangle^\alpha \mathbf{w} \|_{L_p((0, T), X)} = \left\{ \int_0^T \langle t \rangle^\alpha \| \mathbf{w}(\cdot, t) \|_X^p dt \right\}^{1/p}, \quad 1 \leq p < \infty,$$

$$\| \langle t \rangle^\alpha \mathbf{w} \|_{L_\infty((0, T), X)} = \operatorname{esssup}_{0 < t < T} \langle t \rangle^\alpha \| \mathbf{w}(\cdot, t) \|_X \quad p = \infty.$$

First, we prove that

$$\| \langle t \rangle^b \mathbf{f} \|_{L_p((0, T), L_{q/2}(\Omega))} + \| \langle t \rangle \mathbf{f} \|_{L_p((0, T), L_{q_2}(\Omega))} \leq C(I + [\mathbf{v}]_T^2). \quad (27)$$

with  $I = \| u_0 \|_{B_{q/2, p}^{2(1-1/p)}(\Omega)} + \| u_0 \|_{B_{q_2, p}^{2(1-1/p)}(\Omega)}$ . Here and in what follows,  $C$  denotes generic constants independent of  $I$ ,  $[\mathbf{v}]_T$ ,  $\delta$ , and  $T$ . The value of  $C$  may change from line to line. Since we choose  $I$  small enough eventually, we may assume that  $0 < I \leq 1$ . Especially, we use the estimates:

$$I^2 \leq I, \quad I [\mathbf{v}]_T \leq \frac{1}{2} (I^2 + [\mathbf{v}]_T^2) \leq I + [\mathbf{v}]_T^2$$

Since

$$\begin{aligned} \int_\alpha^\beta \| \nabla(\kappa \mathbf{v}(\cdot, s)) \|_{L_\infty(\Omega)} ds &\leq C(1 + \alpha)^{-b+\frac{1}{p'}} \left( \int_\alpha^\beta \langle s \rangle^{>b} \| \mathbf{v}(\cdot, s) \|_{H_\infty^1(\Omega)}^p ds \right)^{1/p} \\ \int_\alpha^\beta \| \nabla^2(\kappa \mathbf{v}(\cdot, s)) \|_{L_q(\Omega)} ds &\leq C(1 + \alpha)^{-b+\frac{N}{2q_2}+\frac{1}{p'}} \left( \int_\alpha^\beta \langle s \rangle^{>\frac{b-N}{2q_2}} \| \mathbf{v}(\cdot, s) \|_{H_{q_2}^2(\Omega)}^p ds \right)^{1/p} \end{aligned}$$

for any  $0 \leq \alpha < \beta \leq T$ , where  $q \in [1, q_2]$ , we have

$$\int_\alpha^\beta \| \nabla(\kappa \mathbf{v}(\cdot, s)) \|_{L_\infty(\Omega)} ds \leq C[\mathbf{v}]_T (1 + \alpha)^{-b+\frac{1}{p'}}$$

$$\int_\alpha^\beta \| \nabla^2(\kappa \mathbf{v}(\cdot, s)) \|_{L_q(\Omega)} ds \leq C[\mathbf{v}]_T$$

for any  $0 \leq \alpha < \beta \leq T$ , where  $q \in [1, q_2]$ , because  $b > \frac{N}{2q_2} + \frac{1}{p'}$  as follows from (11). By real interpolation

theorem, we have

$$\begin{aligned} \sup_{t \in (0, T)} \langle t \rangle^{b-\frac{N}{2q_2}} \| \mathbf{v}(\cdot, t) \|_{B_{q_2, p}^{2(1-1/p)}(\Omega)} &\leq C \| u_0 \|_{B_{q_2, p}^{2(1-1/p)}(\Omega)} \\ &+ \| \langle t \rangle^{\frac{N}{2q_2}} \mathbf{v} \|_{L_p((0, T), H_{q_2}^2(\Omega))} + \| \langle t \rangle^{b-\frac{N}{2q_2}} \partial_t \mathbf{v} \|_{L_p((0, T), L_{q_2}(\Omega))} \end{aligned}$$

To prove (29), we introduce an operator  $\mathcal{T}(t)$  acting of  $g \in B_{q_2, p}^{2(1-1/p)} \mathbb{R}^N$  defined by

$$\mathcal{T}(t)g = \mathcal{F}^{-1} [e^{-(|\xi|+1)t} \mathcal{F}[g](\xi)], \quad (30)$$

where  $\mathcal{F}$  and  $\mathcal{F}^{-1}$  denote the Fourier transform in  $\mathbb{R}^N$  and its inverse transform. We have

$$e^{\gamma t} \mathcal{T}(t)g \|_{L_p((0, \infty), H_q^2(\mathbb{R}^N))} + e^{\gamma t} \partial_t \mathcal{T}(t)g \|_{L_p((0, \infty), L_q(\mathbb{R}^N))} \leq C \| g \|_{B_{q_2, p}^{2(1-1/p)}(\mathbb{R}^N)}$$

Given  $f(t)$  defined on  $(0, T)$  with  $f|_{t=0} = 0$ , let

$$[e_T f](t) = \begin{cases} 0 & (t < 0), \\ f(t) & (0 < t < T), \\ f(2T-t) & (T < t < 2T), \\ 0 & (2T < t), \end{cases} \quad (32)$$

and then  $[e_T f](t) = f(t)$  for  $t \in (0, T)$  and

$$\partial_t [e_T f](t) = \begin{cases} 0 & (t < 0), \\ \partial_t f(t) & (0 < t < T), \\ -(\partial_t f)(2T-t) & (T < t < 2T), \\ 0 & (2T < t), \end{cases} \quad (33)$$

Let  $\tilde{u}_0$  be an  $N$ -vector of functions in  $B_{q_2, p}^{2(1-1/p)}(\mathbb{R}^N)^N$  such that  $\tilde{u}_0 = u_0$  in  $\Omega$  and  $\| \tilde{u}_0 \|_{B_{q_2, p}^{2(1-1/p)} \mathbb{R}^N} < C \| u_0 \|_{B_{q_2, p}^{2(1-1/p)}(\Omega)}$ .

$$\text{Let } z = e_T \langle t \rangle^{b-\frac{N}{2q_2}} \mathbf{v} - \mathcal{T}(|t|) \tilde{u}_0 + \mathcal{T}(|t|) \tilde{u}_0 \text{ for } t \in \mathbb{R}.$$

Since  $\langle t \rangle^{b-\frac{N}{2q_2}} \mathbf{v} - \mathcal{T}(|t|) \tilde{u}_0|_{t=0} = 0$  in  $\Omega$  by (31), (32) and (33),

$$\|z\|_{L_p((0,\infty),H_{q_2}^2(\Omega))} + \|\partial_t z\|_{L_p((0,\infty),L_{q_2}(\Omega))} \leq C(\|v\|_{L_p((0,T),H_{q_2}^2(\Omega))} + \|\partial_t v\|_{L_p((0,T),L_{q_2}(\Omega))} + \|u_0\|_{B_{q_2,p}^{2(1-1/p)}(\Omega)})$$

is continuously imbedded into  $BUC((0, \infty), (E_0, E_1)_{1-1/p,p})$ , where  $E_0$  and  $E_1$  are two Banach spaces such that  $E_1$  is a dense subset of  $E_0$ , and  $BUC$  denotes the set of all uniformly bounded continuous functions. Noting that

$$z(t) = \langle t \rangle^{b-\frac{N}{2q_2}} v(t) \text{ for } t \in (0, T), \text{ we have}$$

It is known (Tanabe)<sup>51</sup> that  $L_p((0, \infty), E_1) \cap H_p^1((0, \infty), E_0)$

$$\sup_{0 < t < T} \langle t \rangle^{b-\frac{N}{2q_2}} \|v(t)\|_{B_{q_2,p}^{2(1-1/p)}(\Omega)} \leq \sup_{t \in (0, \infty)} \|z(t)\|_{B_{q_2,p}^{2(1-1/p)}(\Omega)} \leq C\|z\|_{L_p((0,\infty),H_{q_2}^2(\Omega))} + \|\partial_t z\|_{L_p((0,\infty),L_{q_2}(\Omega))}$$

which, combined with (34), furnishes (29).

Since  $\frac{2}{p} + \frac{N}{q_2} < 1$ ,  $B_{q_2,p}^{2(1-1/p)}(\Omega)$  is continuously imbedded into  $H_\infty^1(\Omega)$  so that by (29)

$$\langle t \rangle^{b-\frac{N}{2q_2}} \|v(t)\|_{L_\infty((0,T),H_\infty^1(\Omega))} \leq C(I + [v]_T) \quad (35)$$

Applying (3), (28) and (29) to the formulas in (15) and (16) and using the fact that  $-b + \frac{1}{p'} \frac{N}{2q_2} \leq -\frac{N}{2q_2}$  and  $-b + \frac{N}{2q_2} \leq -\frac{N}{2q_2}$ , which follows from (11), give

$$\begin{aligned} & \|(a_{ij}(t), J(t), \ell_{ij}(t), A_{ij}(t), J(t), L_{ij}(t))\|_{L_\infty(\Omega)} \leq C, \\ & \|(A_{ij}(t), J(t), L_{ij}(t))\|_{L_\infty(\Omega)} \leq C \int_t^T \|\nabla(\kappa v(\cdot, s))\|_{L_\infty(\Omega)} ds \leq C[v]_T + \langle t \rangle^{-b+\frac{1}{p'}} \leq C[v]_T \langle t \rangle^{-\frac{N}{2q_2}}, \\ & \|\nabla(a_{ij}(t), J(t), \ell_{ij}(t), A_{ij}(t), J(t), L_{ij}(t))\|_{L_q(\Omega)} \leq C \int_0^T \|\nabla^2(\kappa v(\cdot, s))\|_{L_q(\Omega)} ds \leq C[v]_T, \\ & \|\partial_t(a_{ij}(t), J(t), \ell_{ij}(t), A_{ij}(t), J(t), L_{ij}(t))\|_{L_\infty(\Omega)} \\ & \leq C \|\nabla(\kappa v(\cdot, t))\|_{L_\infty(\Omega)} \leq C(I + [v]_T) \langle t \rangle^{-b+\frac{N}{2q_2}} \leq C(I + [v]_T) \langle t \rangle^{-\frac{N}{2q_2}} \end{aligned} \quad (36)$$

for any  $t \in (0, T]$ , where  $q \in [1, q_2]$ . Moreover, we have

$$(\tilde{a}_{ij}, \tilde{J}, \tilde{\ell}_{ij}, \tilde{A}_{ij}, \tilde{J}, \tilde{L}_{ij})(x, t) = 0 \text{ for } x \notin B_{2R} \text{ and } t \in [0, T] \quad (37)$$

By (36) and (37),

$$\|(a_{sm}(T)L_{is}(t)\partial_t v_i)\|_{L_q(\Omega)} \leq C[v]_T + \langle t \rangle^{-b+\frac{1}{p'}} \|\partial_t v_i(t)\|_{L_{q_2}(\Omega)}$$

for any  $q \in [1, q_2]$ . Since  $\frac{1}{p'} < b - \frac{N}{2q_2}$  as follows from (11), we have

$$\langle t \rangle^b \|a_{sm}(T)L_{is}(t)\partial_t v_i\|_{L_p((0,T),L_q(\Omega))} \leq C(I + [v]_T^2)$$

for any  $q \in [1, q_2]$

Next, by Holder's inequality,

$$\langle t \rangle^b \|v \cdot \nabla v(\cdot, t)\|_{L_{q_1/2}(\Omega)} \leq \langle t \rangle^{\frac{N}{2q_1}} \|\nabla v(\cdot, t)\|_{L_{q_1}(\Omega)}$$



so that by (36), we have

$$\left\| \langle t \rangle^b a_{sm}(T) \ell_{is} a_{kj} v_j \frac{\partial v_i}{\partial \xi_k} \right\|_{L_p((0,T), L_{q_1/2}(\Omega))} \leq C[\mathbf{v}]_T^2.$$

Since

$$\langle t \rangle^b \|\mathbf{v} \cdot \nabla \mathbf{v}(\cdot, t)\|_{L_{q_2}(\Omega)} \leq \langle t \rangle^{\frac{N}{2q_2}} \|\nabla \mathbf{v}(\cdot, t)\|_{L_\infty(\Omega)} \langle t \rangle^{b - \frac{N}{2q_2}} \|\nabla \mathbf{v}(\cdot, t)\|_{L_{q_2}(\Omega)},$$

by (36)

$$\left\| \langle t \rangle^b a_{sm}(T) \ell_{is} a_{kj} v_j \frac{\partial v_i}{\partial \xi_k} \right\|_{L_p((0,T), L_{q_2}(\Omega))} \leq C(I + [\mathbf{v}]_T)[\mathbf{v}]_T \leq C(I + [\mathbf{v}]_T^2)$$

Since

$$\frac{\partial}{\partial \xi_k} (D_{ij}(t) \nabla \mathbf{v}) = \sum_{m=1}^N (A_{mj}(t) \frac{\partial^2 v_i}{\partial \xi_k \partial \xi_m} + A_{mi}(t) \frac{\partial^2 v_j}{\partial \xi_k \partial \xi_m}) + \sum_{m=1}^N ((\frac{\partial}{\partial \xi_m} A_{mj}(t) \frac{\partial v_i}{\partial \xi_m} + \frac{\partial}{\partial \xi_k} A_{mi}(t) \frac{\partial v_j}{\partial \xi_m}))$$

by (36)

$$\langle t \rangle^b \left\| \frac{\partial}{\partial \xi_k} (D_{ij}(\cdot) \nabla \mathbf{v}) \right\|_{L_q(\Omega)} \leq C[\mathbf{v}]_T \{ \langle t \rangle^{b - \frac{N}{2q_2}} \|\nabla^2 \mathbf{v}(\cdot, t)\|_{L_q(\Omega)} + \langle t \rangle^b \|\nabla \mathbf{v}(\cdot, t)\|_{L_\infty(\Omega)} \} \quad (38)$$

for any  $q \in [1, q_2]$  and therefore

$$\left\| \langle t \rangle^b a_{sm}(T) \ell_{is}(T) a_{kj}(T) \frac{\partial}{\partial \xi_k} (D_{ij}(\cdot) \nabla \mathbf{v}) \right\|_{L_p((0,T), L_q(\Omega))} \leq C[\mathbf{v}]_T^2$$

for any  $q \in [1, q_2]$ . Since

$$\frac{\partial}{\partial \xi_k} (D_{ij,T}(\mathbf{v})) = \sum_{m=1}^N (a_{mj}(T) \frac{\partial^2 v_i}{\partial \xi_k \partial \xi_m} + a_{mi}(T) \frac{\partial^2 v_j}{\partial \xi_k \partial \xi_m}) + \sum_{m=1}^N ((\frac{\partial}{\partial \xi_m} a_{mj}(T) \frac{\partial v_i}{\partial \xi_m} + \frac{\partial}{\partial \xi_k} a_{mi}(T) \frac{\partial v_j}{\partial \xi_m})),$$

by (36)

$$\langle t \rangle^b \left\| a_{sm}(T) \ell_{is}(T) a_{kj}(T) \frac{\partial}{\partial \xi_k} (D_{ij,T}(\mathbf{v})) \right\|_{L_q(\Omega)} \leq C[\mathbf{v}]_T \{ \langle t \rangle^{b - \frac{N}{2q_2}} \|\nabla^2 \mathbf{v}(\cdot, t)\|_{L_q(\Omega)} + \langle t \rangle^b \|\nabla \mathbf{v}(\cdot, t)\|_{L_\infty(\Omega)} \},$$

so that

$$\langle t \rangle^b \left\| a_{sm}(T) \ell_{is}(T) a_{kj}(T) \frac{\partial}{\partial \xi_k} (D_{ij,T}(\mathbf{v})) \right\|_{L_p((0,T), L_q(\Omega))} \leq C[\mathbf{v}]_T^2$$

for any  $q \in [1, q_2]$ . Analogously, we have

$$\left\| \langle t \rangle^b a_{sm}(T) L_{is} a_{kj} \frac{\partial}{\partial \xi_k} (D_{ij,T}(\mathbf{v})) \right\|_{L_p((0,T), L_q(\Omega))} \leq C[\mathbf{v}]_T^2$$

for any  $q \in [1, q_2]$ . Summing up, we have obtained (27).

Next, we consider  $g$  and  $g$ . To estimate the  $H_p^{\frac{1}{2}}$  norm, we use the following lemma.

### Lemma 5

Let  $f \in H_\infty^1(\mathbb{R}, L_\infty(\Omega))$  and  $g \in H_p^{\frac{1}{2}}(\mathbb{R}, L_q(\Omega))$ . Assume that  $f(x, t) = 0$  for  $(x, t) \notin B_R \times \mathbb{R}$ .

Then,

$$\|f g\|_{H_p^{\frac{1}{2}}(\mathbb{R}, L_q(\Omega))} \leq C \|f\|_{H_\infty^1(\mathbb{R}, L_\infty(\Omega))} \|g\|_{H_p^{\frac{1}{2}}(\mathbb{R}, L_{q_2}(\Omega))} \quad (39)$$

Proof: To prove the lemma, we use the fact that

$$H_p^{\frac{1}{2}}(\mathbb{R}, L_q(\Omega)) = (L_p(\mathbb{R}, L_q(\Omega)), H_p^1(\mathbb{R}, L_q(\Omega)))_{\left[\frac{1}{2}\right]}, \quad (40)$$

where  $(\cdot, \cdot)_{[\sigma]}$  denotes a complex interpolation functor. Let  $q \in [1, q_2]$ . Noting that  $f(x, t) = 0$  for  $(x, t) \notin B_R \times \mathbb{R}$ , we have

$$\|\partial_t(fg)\|_{L_q(\Omega)} \leq \|\partial_t f\|_{L_\infty(\Omega)} \|g\|_{L_{q_2}(\Omega)} + \|f\|_{L_\infty(\Omega)} \|\partial_t g\|_{L_{q_2}(\Omega)},$$

and therefore

$$\|\partial_t(fg)\|_{L_p(\mathbb{R}, L_q(\Omega))} \leq C \|f\|_{H_\infty^1(\mathbb{R}, L_\infty(\Omega))} \|g\|_{H_p^1(\mathbb{R}, L_{q_2}(\Omega))}.$$

for any  $q \in [1, q_2]$ . Moreover, we easily see that

$$\|(fg)\|_{L_p(\mathbb{R}, L_q(\Omega))} \leq C \|f\|_{L_\infty(\mathbb{R}, L_\infty(\Omega))} \|g\|_{L_p(\mathbb{R}, L_{q_2}(\Omega))}.$$

Thus, by (40), we have (39), which completes the proof of Lemma 5.

To use the maximal  $L_p$ - $L_q$  estimate, we have to extend  $g$ ,  $g$  and  $h$  to  $\mathbb{R}$ . For this purpose, we introduce an extension operator  $\tilde{e}_T$ . Let  $f$  be a function defined on  $(0, T)$  such that  $f|_{t=T} = 0$ , and then  $\tilde{e}_T$  is an operator acting on  $f$  defined by

$$[\tilde{e}_T f](t) = \begin{cases} 0 & (t > T), \\ f(t) & (0 < t < T), \\ f(-t) & (-T < t < 0), \\ 0 & (t < -T). \end{cases} \quad (41)$$

### Lemma 6

Let  $1 < p < \infty, 1 \leq q \leq q_2$  and  $0 \leq a \leq b$ . Let  $f \in H_\infty^1(0, T, L_\infty(\Omega))$  and  $g \in H_p^1((0, T), L_{q_2}(\Omega)) \cap L_p((0, T)H_q^2(\Omega))$ . Assume  $f|_{t=T} = 0$  and  $f = 0$  for  $(x, t) \notin B_R \times \mathbb{R}$ . Let  $\langle t \rangle = (1 + t^2)^{1/2}$ . Then we have

$$\begin{aligned} \|\tilde{e}_T \langle t \rangle^a f \nabla g\|_{H_p^{\frac{1}{2}}(\mathbb{R}, L_q(\Omega))} &\leq C \langle t \rangle^{\frac{N}{2q_2}} f \|_{H_\infty^1((0, \infty), L_\infty(\Omega))} \\ &\times \left( \|g\|_{L_p((0, T)H_q^2(\Omega))} + \langle t \rangle^{b - \frac{N}{2q_2}} \|\partial_t g\|_{L_p((0, T), L_{q_2}(\Omega))} + \|g|_{t=0}\|_{B_{q_2, p}^{2(1-1/p)}(\Omega)} \right) \end{aligned} \quad (42)$$

**Proof:** Let  $f_0(t) = \langle t \rangle^{a-b+\frac{N}{2q_2}} f(t)$  and  $g_0(t) = \langle t \rangle^{b+\frac{N}{2q_2}} g(t)$ , and then  $\langle t \rangle^a f \nabla g = f_0 \nabla g_0$ . Let  $\tau(t)$  be the operator given

in (30) and let  $h$  be a function in  $B_{q_2, p}^{2(1-1/p)}(\mathbb{R}^N)$  such that  $h = g|_{t=0}$  in  $\Omega$  and  $h = g|_{t=0} \|h\|_{B_{q_2, p}^{2(1-1/p)}(\Omega)} \leq \|g|_{t=0}\|_{B_{q_2, p}^{2(1-1/p)}(\Omega)}$ . Recall the

operator  $e_T$  defined in (32) and note that  $g_0|_{t=0} = g|_{t=0} = T(t)h|_{t=0}$  in  $\Omega$ . Let  $\tilde{g}(t) = e_T[g_0 - T(\cdot)h](t) + T(t)h$

for  $t > 0$  and let

$$[\iota g](t) = \begin{cases} \tilde{g}(t) & (t > 0), \\ \tilde{g}(-t) & (t < 0), \end{cases} \quad [\iota f](t) = \begin{cases} 0 & (t > T), \\ f_0(t) & (0 < t < T), \\ f_0(-t) & (-T < t < 0), \\ 0 & (t < -T). \end{cases}$$

Since  $\tilde{g}(t) = g_0(t)$  for  $0 < t < T$ , we have

$$\tilde{e}_T \langle t \rangle^a f \nabla g(t) = \begin{cases} 0 & (t > T), \\ f_0(t) \nabla g_0(t) & (0 < t < T), \\ f_0(-t) \nabla g_0(-t) & (-T < t < 0), \\ 0 & (t < -T). \end{cases} = \begin{cases} 0 & (t > T), \\ f_0(t) \nabla \tilde{g}(t) & (0 < t < T), \\ f_0(-t) \nabla \tilde{g}(-t) & (-T < t < 0), \\ 0 & (t < -T). \end{cases}$$

By Lemma 5,

$$= [\iota f](t) \nabla [\iota g](t).$$

$$\|\tilde{e}_T \langle t \rangle^a f \nabla g\|_{H_p^{\frac{1}{2}}(\mathbb{R}, L_q(\Omega))} = \|[\iota f] \nabla [\iota g]\|_{H_p^{\frac{1}{2}}(\mathbb{R}, L_q(\Omega))} \leq C \|[\iota f]\|_{H_\infty^1(\mathbb{R}, L_q(\Omega))} \|\nabla [\iota g]\|_{H_p^{\frac{1}{2}}(\mathbb{R}, L_q(\Omega))}$$

Since,  $f_0|_{t=T} = 0$  we have

$$\|[\iota f]\|_{H_\infty^1(\mathbb{R}, L_q(\Omega))} = 2 \|f_0\|_{H_\infty^1((0, T), L_\infty(\Omega))} \leq \|\langle t \rangle^{\frac{N}{2q_2}} f\|_{H_\infty^1((0, T), L_\infty(\Omega))},$$



Because  $a - b \leq 0$ .

To estimate  $\|\nabla(\iota g)\|_{H_p^{\frac{1}{2}}(\mathbb{R}, L_{q_2}(\Omega))}$ , we use the fact that  $H_p^1(\mathbb{R}, L_{q_2}(\Omega)) \cap L_p(\mathbb{R}, H_{q_2}^2(\Omega))$  is continuously imbedded into  $H_p^2(\mathbb{R}, H_{q_2}^1(\Omega))$  which was proved by Meyries et al.,<sup>52</sup> in case of  $p = q_2$  and by Shibata<sup>30</sup> for any  $1 < p, q_2 < \infty$ . Using this fact and (31), we have

$$\begin{aligned} \|\nabla(\iota g)\|_{H_p^{\frac{1}{2}}(\mathbb{R}, L_{q_2}(\Omega))} &\leq C\|\iota g\|_{H_p^1(\mathbb{R}, L_{q_2}(\Omega))} + \|\iota g\|_{L_p(\mathbb{R}, H_{q_2}^2(\Omega))} \\ &\leq C\left(\|\tilde{g}\|_{H_p^1((0,\infty), L_{q_2}(\Omega))} = \|\tilde{g}\|_{L_p((0,\infty), H_{q_2}^2(\Omega))}\right) \\ &\leq C(\|g_0 \cdot T(\cdot)h\|_{H_p^1((0,T), L_{q_2}(\Omega))} + \|g_0 \cdot T(\cdot)h\|_{L_p((0,T), H_{q_2}^2(\Omega))} \\ &\quad + \|T(\cdot)h\|_{H_p^1((0,T), L_{q_2}(\Omega))} + \|T(\cdot)h\|_{L_p((0,\infty), H_{q_2}^2(\Omega))}) \\ &\leq C(\|<t>^{-\frac{b-N}{2q_2}} \partial_t g\|_{L_p((0,T), L_{q_2}(\Omega))} + \|<t>^{-\frac{b-N}{2q_2}} \partial_t g\|_{L_p((0,T), H_{q_2}^2(\Omega))} + \|g|_{t=0}\|_{B_{q_2,p}^{2(1-1/p)}(\Omega)}) \end{aligned}$$

This completes the proof of Lemma 6.

Recall the definitions of  $g(v)$  and  $h_m(v)$  given in (24) and (25). By Lemma 6 and (36)

$$\begin{aligned} &\|\tilde{e}_T(<t>^a g(v))\|_{H_p^{\frac{1}{2}}(\mathbb{R}, L_q(\Omega))} \\ &\leq \sum_{j,k=1}^N \|<t>^{\frac{N}{2q_2}} (J(T)A_{kj}(\cdot) + T(\cdot)a_{kj}(\cdot))\|_{H_\infty^1((0,T), L_\infty(\Omega))} \\ &\quad \times (\|<t>^{-\frac{b-N}{2q_2}} v\|_{L_p((0,T), H_{q_2}^2(\Omega))} + \|<t>^{-\frac{b-N}{2q_2}} \partial_t v\|_{L_p((0,T), L_{q_2}(\Omega))} + \|u_0\|_{B_{q_2,p}^{2(1-1/p)}(\Omega)}) \\ &\leq C(I + [\mathbf{v}]_T^2) \end{aligned} \quad (43)$$

for any  $a \in [0, b]$  and  $q \in [1, q_2]$ . Analogously, we have

$$\|\tilde{e}_T(<t>^a h(v))\|_{H_p^{\frac{1}{2}}(\mathbb{R}, L_q(\Omega))} \leq C(I + [\mathbf{v}]_T^2) \quad (44)$$

for any  $a \in [0, b]$  and  $q \in [1, q_2]$ . Analogously, we have

Next, by (36), (37) and (41),

$$\begin{aligned} &\|\tilde{e}_T[<t>^a g(v)]\|_{L_p(\mathbb{R}, H_q^1(\Omega))} \\ &\leq \sum_{j,k=1}^N \|<t>^{\frac{N}{2q_2}} (J(T)A_{kj}(\cdot) + J(\cdot)a_{kj}(\cdot))\|_{L_\infty((0,T), L_\infty(\Omega))} \|<t>^{-\frac{b-N}{2q_2}} v\|_{L_p((0,T), H_{q_2}^2(\Omega))} \\ &\leq C[\mathbf{v}]_T^2 \end{aligned} \quad (45)$$

for any  $a \in [0, b]$  and  $q \in [1, q_2]$ . Analogously, we have

$$\|\tilde{e}_T(<t>^b h(v))\|_{L_p((0,T), H_q^1(\Omega))} \leq C[\mathbf{v}]_T^2 \quad (46)$$

for any  $a \in [0, b]$  and  $q \in [1, q_2]$ . Since

$$\partial_t g_k(v) = \sum_{j=1}^N \|(J(T)\partial_t A_{kj}(t) + (\partial_t J(t)a_{kj}(t) + J(t)\partial_t a_{kj}(t))v_j + \sum_{j=1}^N \|(J(T)A_{kj}(t) + J(t)a_{kj}(t))v_j$$

and since  $\|J(T)a_{kj}(t)J^{-1}(t)\|_{L_\infty(\Omega)} \leq C$  as follows from (36), by (37) we have

$$\begin{aligned} & \left\| \tilde{e}_T [ < t >^a \partial_t g_k(v) ] \right\|_{L_p((0,T),L_q(\Omega))} \\ & \leq \sum_{j=1}^N \left( < t >^{\frac{N}{2q_2}} \partial_t (A_{kj}, J, a_{kj}) \right) \|_{L_\infty((0,T),L_\infty(\Omega))} \| < t >^{b-\frac{N}{2q_2}} v \|_{L_p((0,T),L_{q_2}(\Omega))} \\ & + \left( < t >^{\frac{N}{2q_2}} (A_{kj}, J) \right) \|_{L_\infty((0,T),L_\infty(\Omega))} \| < t >^{b-\frac{N}{2q_2}} \partial_t v \|_{L_p((0,T),L_{q_2}(\Omega))}, \end{aligned}$$

which, combined with (36), furnishes that

$$\left\| \tilde{e}_T [ < t >^a g(v) ] \right\|_{L_p(\mathbb{R},L_q(\Omega))} \leq C(I + [v]_T^2) \quad (47)$$

for any  $a \in [0, b]$  and  $q \in [1, q_2]$ .

## Prolongation of local in time solutions

Before proving Theorem 1, we state a unique existence theorem of locally in time solutions to Eq. (5), which can be proved by a standard argumentation based on maximal  $L_p$ - $L_q$  regularity theorem for the Stokes equations with free boundary condition.<sup>29,40</sup>

### Theorem 7

Let  $N < q_2 < \infty$  and  $2 < p < \infty$ . Assume that  $2/p + N/q_2 < 1$ . Then, given  $T > 0$ , there exists an  $\epsilon_0 > 0$  depending on  $T$  such that if initial data  $u_0 \in B_{q_2,p}^{2(1-1/p)}(\Omega)$  satisfies the condition:

$$\|u_0\|_{B_{q_2,p}^{2(1-1/p)}(\Omega)} \leq \epsilon_0 \quad (48)$$

$$v_1 \in H_p^1((T, T+1), L_{q_2}(\Omega)^N) \cap L_p((T, T+1), H_{q_2}^2(\Omega)^N), \quad q_1 \in L_p((T, T+1), H_{q_2}^1(\Omega) \hat{H}_{q_2,0}^1(\Omega))$$

such that  $v_1$  and  $q_1$  are solutions to the equations:

$$\begin{cases} \partial_t v_1 - \operatorname{Div}(\mu \mathbf{D}(v_1) - q_1 \mathbf{I}) = \mathbf{f}(v_1), & \text{in } \Omega \times (T, T+1), \\ \operatorname{div} v_1 = g(v_1) = \operatorname{div} g(v_1) & \text{in } \Omega \times (T, T+1), \\ (\mu \mathbf{D}(v_1) - q_1 \mathbf{I}) \mathbf{n} = \mathbf{h}(v_1) & \text{on } \Gamma \times (0, T), \\ v_1|_{t=T} = v(\cdot, T) & \text{in } \Omega. \end{cases} \quad (51)$$

Here,  $f(v_1)$  is consisting of some linear combinations of nonlinear terms of the forms

$$\begin{aligned} & V_0 \left( \int_0^T (\nabla(\kappa v)) ds + \int_T^t (\nabla(\kappa v_1)) ds \right) \nabla^2 v_1, \quad V_1 \left( \int_0^T (\nabla(\kappa v)) ds + \int_T^t (\nabla(\kappa v_1)) ds \right) \nabla^2 v_1 \right) \partial_t v_1 \\ & V_1 \left( \int_0^T (\nabla(\kappa v)) ds + \int_T^t (\nabla(\kappa v_1)) ds \right) \left( \int_0^T (\nabla(\kappa v)) ds + \int_T^t (\nabla(\kappa v_1)) ds \right) \nabla^2 v_1 \right) \nabla v_1, \end{aligned}$$

$$(1 - \kappa)(\dot{\cdot} \mathbf{A}) - 1 \{v_1(\cdot) \mathbf{A} \nabla v_1\};$$

and  $g(v_1)$ ,  $g(v_1)$  and  $h(v_1)$  have the following forms:

$$\begin{aligned} g(v_1) &= V_3 \left( \int_0^T (\nabla(\kappa v)) ds + \int_T^t (\nabla(\kappa v_1)) ds \right) \nabla v_1; \quad g(v_1) = V_4 \left( \int_0^T (\nabla(\kappa v)) ds + \int_T^t (\nabla(\kappa v_1)) ds \right) v_1; \\ h(v_1) &= V_5 \left( \int_0^T (\nabla(\kappa v)) ds + \int_T^t (\nabla(\kappa v_1)) ds \right) \nabla v_1, \end{aligned}$$

and the compatibility condition (9), then Eq. (5) admits unique solutions  $v$  and  $q$  with

$$v \in H_p^1((0, T), L_{q_2}(\Omega)^N) \cap L_p((0, T), H_{q_2}^2(\Omega)^N), \quad q \in L_p((0, T), H_{q_2}^1(\Omega) + \hat{H}_{q_2,0}^1)$$

Let  $T$  be a positive number  $> 2$  and let  $v$  and  $q$  be solutions of Eq. (5) satisfying (13) and (3). In

view of Theorem 7, such solutions  $v$  and  $q$  exist uniquely provided that

$$\|u_0\|_{B_{q_2,p}^{2(1-1/p)}(\Omega)} \leq \epsilon_0 \quad (49)$$

Thus, we assume that  $0 < \epsilon \leq \epsilon_0$  in Theorem 1. Let  $[v]_T$  and  $I$  be the quantities defined in Theorem 1

in Sect. 1 Introduction. And then, if we prove that there exists a constant  $M > 0$  independent of  $\epsilon$  and

$T$  such that

$$[v]_T \leq M(I + [v]_T^2) \quad (50)$$

then we can prolong  $v$  and  $q$  beyond  $T$ . Namely, there exist  $v_1$  and  $q_1$  with

where  $V_i$  are the same nonlinear functions as in (6) and (7).

In fact, the inequality (50) yields that there exists a small constant  $\varepsilon \in (0, \varepsilon_0]$  such that if  $\mathcal{I} \leq \varepsilon$ , then

$$[\mathbf{v}]_T \leq (2M)^{-1} - \sqrt{(2M)^{-2} - I} = 2MI + O(I^2).$$

Thus, we may assume that

$$[\mathbf{v}]_T \leq 3MI \quad (52)$$

By (29) and (52) we have

$$\|\mathbf{v}(\cdot, T)\|_{B_{q_2, p}^{2(1-1/p)}(\Omega)} \leq M_1 \varepsilon$$

with some positive constant  $M_1$  independent of  $T$ . Thus, noting that

$$\int_0^T \|\nabla \mathbf{v}(\cdot, t)\|_{L_\infty(\Omega)} dt \leq C \left( \int_0^T t^{-bp'} ds \right)^{1/p'} [\mathbf{v}]_T \leq M_2 \delta$$

with some positive constant  $M_2$  independent of  $T$ , choosing  $\varepsilon > 0$  smaller if necessary, we can show the

existence of  $\mathbf{v}_1$  and  $\mathbf{q}_1$ . Thus, setting

$$\mathbf{v}_2(\cdot, t) = \begin{cases} \mathbf{v}(\cdot, t) & 0 < t < T, \\ \mathbf{v}_1(\cdot, t) & T < t < T+1, \end{cases} \quad \mathbf{q}_2 = \begin{cases} \mathbf{q}(\cdot, t) & 0 < t < T, \\ \mathbf{q}_2(\cdot, t) & T < t < T+1, \end{cases}$$

we see that  $\mathbf{v}_2$  and  $\mathbf{q}_2$  satisfy the regularity condition:

$$\mathbf{v}_2 \in H_p^1((0, T+1), L_{q_2}(\Omega)^N) \cap L_p((0, T+1), H_{q_2}^2(\Omega)), \quad \mathbf{q}_2 \in L_p((0, T+1), H_{q_2}^1(\Omega) + \hat{H}_{q_2, 0}^1(\Omega))$$

and Eq. (5) replacing  $T$  by  $T+1$ . Repeating this argument, we can prolong  $\mathbf{v}$  to time interval  $(0, \infty)$ .

This completes the proof of Theorem 1. Therefore, we prove (50).

## A proof of theorem 1

Let  $\mathbf{v}$  and  $\mathbf{q}$  be the same  $N$ -vector of functions and the function as in Sect. 4. We prove that  $\mathbf{v}$  satisfies (50). And, we recall that  $T > 2$ . As was seen in Sect. 2,  $\mathbf{v}$  and  $\mathbf{q}$  satisfy Eq. (26). To estimate  $\mathbf{v}$ , we write  $\mathbf{v}$  by,  $\mathbf{v} = \mathbf{w} + \mathbf{u}$  where  $\mathbf{w}$  is a solution to the equations:

$$\begin{cases} \partial_t \mathbf{w} + \lambda_0 \mathbf{w} - J(T)^{-1} \operatorname{Div} \tilde{\mathbf{S}}(\mathbf{w}, \mathbf{r}) = \mathbf{f}(\mathbf{v}) & \text{in } \Omega \times (0, T), \\ \operatorname{div} \mathbf{w} = \tilde{\mathbf{e}}_T[\mathbf{g}(\mathbf{v})] = \operatorname{div} \tilde{\mathbf{e}}_T[\mathbf{g}(\mathbf{v})] & \text{in } \Omega \times (0, T), \\ \tilde{\mathbf{S}}(\mathbf{w}, \mathbf{r}) = \tilde{\mathbf{e}}_T[\mathbf{h}] & \text{on } \Gamma \times (0, T), \\ \mathbf{w}|_{t=0} = \mathbf{u}_0 & \text{in } \Omega \end{cases} \quad (53)$$

with some pressure term  $\mathbf{r}$ , and  $\mathbf{u}$  is a solution to the equations:

$$\begin{cases} \partial_t \mathbf{u} - J(T)^{-1} \operatorname{Div} \tilde{\mathbf{S}}(\mathbf{u}, \mathbf{p}) = -\lambda_0 \mathbf{w} & \text{in } \Omega \times (0, T), \\ \operatorname{div} \mathbf{u} = 0 & \text{in } \Omega \times (0, T), \\ \tilde{\mathbf{S}}(\mathbf{u}, \mathbf{p}) = 0 & \text{on } \Gamma \times (0, T), \\ \mathbf{u}|_{t=0} = 0 & \text{in } \Omega \end{cases} \quad (54)$$

To estimate  $\mathbf{w}$ , we quote the maximal  $L_p$ - $L_q$  regularity theorem due to Shibata.<sup>33</sup> Let us consider

the equations:

$$\begin{cases} \partial_t \mathbf{w} + \lambda_0 \mathbf{w} - J(T)^{-1} \operatorname{Div} \tilde{\mathbf{S}}(\mathbf{w}, \mathbf{r}) = \mathbf{f} & \text{in } \Omega \times (0, T), \\ \tilde{\operatorname{div}} \mathbf{w} = \mathbf{g} = \operatorname{div} \mathbf{g} & \text{in } \Omega \times (0, T), \\ \tilde{\mathbf{S}}(\mathbf{w}, \mathbf{r}) = \mathbf{h} & \text{on } \Gamma \times (0, T), \\ \mathbf{w}|_{t=0} = \mathbf{w}_0 & \text{in } \Omega \end{cases} \quad (55)$$

And then, we have

## Theorem 8

Let  $\Omega$  be an exterior domain in  $\mathbb{R}^N$  whose boundary  $\Gamma$  is a  $C_2$  hyper surface. Let  $1 < p, q < \infty$  and  $T > 0$ . Assume that

$$\mathbf{w}_0 \in B_{q, p}^{2(1-1/p)}(\Omega), \mathbf{f} \in L_p((0, T), L_q(\Omega)^N), \mathbf{g} \in H_p^{\frac{1}{2}}(\mathbb{R}, L_q(\Omega)) \cap L_p(\mathbb{R}, H_q^1(\Omega)),$$

$$\mathbf{g} \in H_p^1(\mathbb{R}, L_q(\Omega)^N), \mathbf{h} \in H_p^{\frac{1}{2}}(\mathbb{R}, L_q(\Omega)^N) \cap L_p(\mathbb{R}, H_q^1(\Omega)^N)$$

and that  $\mathbf{w}_0$  satisfies the compatibility condition:

$$\mathbf{w}_0 - \mathbf{g}|_{t=0} \in B_{q, p}^{2(1-1/p)}(\Omega)$$

and in addition

$$(\mu \mathbf{D}(\mathbf{w}_0) - \mathbf{h})|_{t=0} = 0 \quad \text{on } \Gamma$$

If  $2/p + 1/q < 1$ , where  $d_\tau = d - \langle d, n \rangle$ . Then, there exists a positive number  $\lambda_0$  such that Eq. (55) admits unique solutions  $\mathbf{w}$  and  $\mathbf{r}$  with

$$\mathbf{w} \in L_p((0, T), H_q^2(\Omega)^N) \cap H_p^1((0, T), L_q(\Omega)^N), \mathbf{r} \in L_p((0, T), H_q^1(\Omega) + \hat{H}_{q, 0}^2(\Omega))$$

Possessing the estimate:

$$\begin{aligned} \|\mathbf{w}\|_{L_p((0, T), H_q^2(\Omega))} + \|\partial_t \mathbf{w}\|_{L_p((0, T), L_q(\Omega))} &\leq C_q \left\{ \|\mathbf{w}_0\|_{B_{q, p}^{2(1-1/p)}(\Omega)} + \|\mathbf{f}\|_{L_p((0, T), L_q(\Omega))} \right. \\ &\quad \left. + \|(g, h)\|_{H_p^{\frac{1}{2}}(\mathbb{R}, L_q(\Omega))} \|(g, h)\|_{L_p(\mathbb{R}, L_q(\Omega))} + \|\partial_t g\|_{L_p(\mathbb{R}, L_q(\Omega))} \right\}, \end{aligned}$$

where  $C_q$  is a constant that depends on  $q$  but is independent of  $T$ . Applying Theorem 8 yields that there exists a large  $\lambda_0 > 0$  such that Eq. (53) admits unique solutions  $\mathbf{w}$  and  $\mathbf{r}$  with

$$\mathbf{w} \in H_p^1((0, T), L_q(\Omega)^N) \cap L_p((0, T), H_q^2(\Omega)^N) \quad \left( q \in \frac{q_1}{2}, q_2 \right)$$

Possessing the estimate:

$$\|\langle t \rangle^b \partial_t \mathbf{w}\|_{L_p((0, T), L_{q_1/2}(\Omega) \cap L_{q_2}(\Omega))} + \|\langle t \rangle^b \partial_t \mathbf{w}\|_{L_p((0, T), H_{q_1/2}^2(\Omega) \cap H_{q_2}^2(\Omega))} \leq C(I + [\mathbf{v}]_T^2). \quad (56)$$

In fact,  $\mathbf{f}(\mathbf{v}), \tilde{\mathbf{e}}_T[\mathbf{g}(\mathbf{v})], \tilde{\mathbf{e}}_T[\mathbf{g}(\mathbf{v})]$  and  $\tilde{\mathbf{e}}_T[\mathbf{h}(\mathbf{v})]$  satisfy (27), (43), (44), (45), (46), and (47), so that we know the existence of  $\mathbf{w}$  possessing the estimate:

$$\partial_t \mathbf{w}\|_{L_p((0, T), L_{q_1/2}(\Omega) \cap L_{q_2}(\Omega))} + \|\langle t \rangle^b \partial_t \mathbf{w}\|_{L_p((0, T), H_{q_1/2}^2(\Omega) \cap H_{q_2}^2(\Omega))} \leq C(I + [\mathbf{v}]_T^2)$$

with some constant  $C$  depending on  $q_1/2$  and  $q_2$ . Let  $a = \min(1, b)$ , and then  $\tilde{\mathbf{w}} := \langle t \rangle^a \mathbf{w}$  satisfies

the equations:

$$\begin{cases} \partial_t \tilde{\mathbf{w}} + \lambda_0 \tilde{\mathbf{w}} - J(T)^{-1} \operatorname{Div} \tilde{\mathbf{S}}(\tilde{\mathbf{w}}, \mathbf{r}) = \langle t \rangle^a \mathbf{f} + \lambda_0 \mathbf{w} & \text{in } \Omega \times (0, T), \\ \tilde{\operatorname{div}} \tilde{\mathbf{w}} = \tilde{\mathbf{e}}_T[\langle t \rangle^a \mathbf{g}(\mathbf{v})] = \operatorname{div} \tilde{\mathbf{e}}_T[\langle t \rangle^a \mathbf{g}(\mathbf{v})] & \text{in } \Omega \times (0, T), \\ \tilde{\mathbf{S}}(\tilde{\mathbf{w}}, \mathbf{r}) = \tilde{\mathbf{e}}_T[\langle t \rangle^a \mathbf{h}] & \text{on } \Gamma \times (0, T), \\ \tilde{\mathbf{w}}|_{t=0} = \mathbf{u}_0 & \text{in } \Omega \end{cases} \quad (57)$$

Since

$$\| \langle t \rangle^{a-2} \mathbf{w} \|_{L_p((0,T), L_q(\Omega))} \leq \| \mathbf{w} \|_{L_p((0,T), L_q(\Omega))} \leq C(I + [\mathbf{v}]_T^2)$$

as follows from the fact that  $a-1 \leq 0$ , we have

$$(\| \langle t \rangle^a \partial_t \mathbf{w} \|_{L_p((0,T), L_{q_1/2}(\Omega) \cap L_{q_2}(\Omega))} + \| \langle t \rangle^a \mathbf{w} \|_{L_p((0,T), H_{q_1/2}^2(\Omega) \cap H_{q_2}^2(\Omega))} \leq C(I + [\mathbf{v}]_T^2).$$

Repeating this argument finite times yields (56). In particular, by (56) we have

$$[\mathbf{w}]_T \leq C(I + [\mathbf{v}]_T^2) \quad (58)$$

Next, we consider  $\mathbf{u}$ . Let  $\{T(t)\}_{t \geq 0}$  be a  $C_0$  analytic semigroup associated with problem (54). Shibata<sup>33</sup> proved the existence of  $\{T(t)\}_{t \geq 0}$  satisfying the estimates:

$$\|T(t)\mathbf{f}\|_{L_p(\Omega)} \leq Ct^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{p})} \|\mathbf{f}\|_{L_q(\Omega)}, \quad \|\Delta T(t)\mathbf{f}\|_{L_p(\Omega)} \leq Ct^{-\frac{1}{2}-\frac{N}{2}(\frac{1}{q}-\frac{1}{p})} \|\mathbf{f}\|_{L_q(\Omega)} \quad (59)$$

for any  $t > 0$  and  $\mathbf{f} \in L_q(\Omega)^N$  provided that  $1 < q \leq p \leq \infty$  and  $q \leq q_2$ . To represent  $\mathbf{u}$  by using  $\{T(t)\}_{t \geq 0}$ , we introduce the solenoidal space  $J_q(\Omega)$  defined by

$$J_q(\Omega) = \left\{ \mathbf{f} \in L_q(\Omega)^N \mid (\mathbf{f}, J^T \mathbf{A} \nabla \varphi)_\Omega = 0 \text{ for any } \varphi \in \hat{H}_{q',0}^1(\Omega) \right\} \quad (60)$$

Here,  $\mathbf{A}$  is the matrix defined in (4) and  $J$  the function given in (15), and

$$\hat{H}_{q',0}^1(\Omega) = \left\{ \varphi \in L_{q',loc}(\Omega) \mid \nabla \varphi|_{\Omega'} \in L_{q'}(\Omega)^N, \varphi|_\Gamma = 0 \right\}.$$

As was proved by Shibata<sup>30</sup> we know that for any  $\mathbf{f} \in L_q(\Omega)^N$  there exists a unique solution  $\psi \in \hat{H}_{q',0}^1(\Omega)$  of the variational equation

$$(\mathbf{A} \nabla \psi, J^T \mathbf{A} \varphi)_\Omega = (\mathbf{f}, J^T \mathbf{A} \nabla \varphi)_\Omega = 0 \text{ for any } \varphi \in \hat{H}_{q',0}^1(\Omega). \quad (61)$$

which possesses the estimate  $\|\nabla \psi\|_{L_q(\Omega)} \leq C_q \|\mathbf{f}\|_{L_q(\Omega)}$ . Here  $C_q$  is a constant that is independent of  $\mathbf{v}$

and  $T$  in view of (36). Given  $\mathbf{f} \in L_q(\Omega)^N$  let  $\psi \in \hat{H}_{q',0}^1(\Omega)$  be a unique solution of Eq.(61), and let  $\mathbf{P}_q$  be an operator acting on  $\mathbf{f}$  defined  $\mathbf{P}_q \mathbf{f} = \mathbf{f} - \nabla \psi$ . And then,  $\mathbf{P}_q \mathbf{f} \in J_q(\Omega)$  and

$$\|\mathbf{P}_q \mathbf{f}\|_{L_q(\Omega)} \leq C_q \|\mathbf{f}\|_{L_q(\Omega)} \quad (62)$$

with some constant  $C_q$  that is independent of  $\mathbf{v}$  and  $T$ . By Proposition 21 in Shibaata,<sup>33</sup> we have

$$\mathbf{u}(\cdot, t) = -\lambda_0 \int_0^t T(t-s)(\mathbf{P} \mathbf{w})(\cdot, s) ds. \quad (63)$$

Combining (59) and (62) yields that

$$\begin{aligned} \|\nabla^j \mathbf{u}(\cdot, t)\|_{L_r(\Omega)} &\leq C_{r, \tilde{q}_1} \int_0^{t-1} (t-s)^{-\frac{j}{2}-\frac{N}{2}(\frac{1}{\tilde{q}_1}-\frac{1}{r})} \|\mathbf{w}(\cdot, s)\|_{L_{\tilde{q}_1}(\Omega)} ds \\ &\quad + C_{r, \tilde{q}_2} \int_0^{t-1} (t-s)^{-\frac{j}{2}-\frac{N}{2}(\frac{1}{\tilde{q}_2}-\frac{1}{r})} \|\mathbf{w}(\cdot, s)\|_{L_{\tilde{q}_2}(\Omega)} ds \end{aligned} \quad (64)$$

for  $j=0, 1$ , for any  $t > 1$  and for any indices  $r, \tilde{q}_1$  and  $\tilde{q}_2$  such that  $1 < \tilde{q}_1, \tilde{q}_2 \leq r \leq \infty$  and

$\tilde{q}_1, \tilde{q}_2 \leq q_2$ , where  $\nabla^0 u = u$  and  $\nabla^1 u = \nabla u$ .

Recall that  $T > 2$ . In what follows, we prove that

$$\left( \int_2^T \langle t \rangle^b \|\mathbf{u}(\cdot, t)\|_{H_\infty^1(\Omega)}^p dt \right)^{1/p} \leq C(I + [\mathbf{v}]_T^2) \quad (65)$$

$$\sup_{2 \leq t \leq T} \langle t \rangle^{\frac{N}{2q_1}} \|\mathbf{u}(\cdot, t)\|_{L_{q_1}} \leq C(I + [\mathbf{v}]_T^2) \quad (66)$$

$$\left( \int_2^T \langle t \rangle^{\frac{b-N}{2q_1}} \|\mathbf{u}(\cdot, t)\|_{H_{q_1}^1(\Omega)}^p dt \right)^{1/p} \leq C(I + [\mathbf{v}]_T^2) \quad (67)$$

$$\left( \int_2^T \langle t \rangle^{\frac{b-N}{2q_2}} \|\mathbf{u}(\cdot, t)\|_{H_{q_2}^1(\Omega)}^p dt \right)^{1/p} \leq C(I + [\mathbf{v}]_T^2) \quad (68)$$

By (64) with  $r = \infty$ ,  $\tilde{q}_1 = q_1/2$  and  $\tilde{q}_2 = q_2$ ,

$$\|\mathbf{u}(\cdot, t)\|_{H_\infty^1(\Omega)} \leq C \int_0^t \|T(t-s)\mathbf{w}(\cdot, s)\|_{H_\infty^1(\Omega)} ds = C(I_\infty(t) + II_\infty(t) + III_\infty(t))$$

With

$$I_\infty(t) = \int_0^{t/2} (t-s)^{-\frac{N}{q_1}} \|\mathbf{w}(\cdot, s)\|_{L_{q_1/2}(\Omega)} ds,$$

$$II_\infty(t) = \int_{t/2}^{t-1} (t-s)^{-\frac{N}{q_1}} \|\mathbf{w}(\cdot, s)\|_{L_{q_1/2}(\Omega)} ds,$$

$$III_\infty(t) = \int_{t-1}^t (t-s)^{-\frac{N}{2q_2}-\frac{1}{2}} \|\mathbf{w}(\cdot, s)\|_{L_{q_2/2}(\Omega)} ds.$$

Since

$$I_{\infty}(t) \leq (t/2)^{-\frac{N}{q_1}} \left( \int_0^{t/2} \langle s \rangle^{-bp'} ds \right)^{1/p'} \left( \int_0^{t/2} \langle s \rangle^b \|\mathbf{w}(\cdot, s)\|_{L_{q_1/2}(\Omega)}^p ds \right)^{1/p}$$

$$\leq C(bp'-1)^{-1/p'} \left( I + [\mathbf{v}]_T^2 \right) t^{\frac{N}{q_1}}$$

as follows from the condition:  $bp' > 1$  in (11), by the condition:  $\left(\frac{N}{q_1} - b\right)p > 1$  in (11), we have

$$\int_2^T (\langle t \rangle^b I_{\infty}(t))^p dt \leq C \int_2^T \langle t \rangle^{-\left(\frac{N}{q_1} - b\right)p} dt \left( I + [\mathbf{v}]_T^2 \right)^p \leq C \left( \left(\frac{N}{q_1} - b\right)p - 1 \right)^{-1} \left( I + [\mathbf{v}]_T^2 \right)^p.$$

By Holder's inequality

$$\langle t \rangle^b II_{\infty}(t) \leq C \int_{t/2}^{t-1} (t-s)^{-\frac{N}{q_1}} \langle s \rangle^b \|\mathbf{w}(\cdot, s)\|_{L_{q_1/2}(\Omega)} ds$$

$$\leq C \left( \int_{t/2}^{t-1} (t-s)^{-\frac{N}{q_1}} ds \right)^{1/p'} \left( \int_{t/2}^{t-1} (t-s)^{-\frac{N}{q_1}} \langle s \rangle^b \|\mathbf{w}(\cdot, s)\|_{L_{q_1/2}(\Omega)}^p ds \right)^{1/p}$$

$$\leq C \left( \frac{N}{q_1} - 1 \right)^{-1/p'} \left( \int_{t/2}^{t-1} (t-s)^{-\frac{N}{q_1}} \langle s \rangle^b \|\mathbf{w}(\cdot, s)\|_{L_{q_1/2}(\Omega)}^p ds \right)^{1/p}$$

Because  $N/q_1 = N/q_2 + 1 > 1$ . By the change of integration order and (56),

$$\int_2^T (\langle t \rangle^b II_{\infty}(t))^p dt \leq C \left( \frac{N}{q_1} - 1 \right)^{-\frac{p}{p'}} \int_2^T dt \int_{t/2}^{t-1} (t-s)^{-\frac{N}{q_1}} (\langle s \rangle^b \|\mathbf{w}(\cdot, s)\|_{L_{q_1/2}(\Omega)})^p ds$$

$$\leq C \left( \frac{N}{q_1} - 1 \right)^{-\frac{p}{p'}} \int_1^{T-1} (\langle s \rangle^b \|\mathbf{w}(\cdot, s)\|_{L_{q_1/2}(\Omega)})^p ds \int_{s+1}^{2s} (t-s)^{-\frac{N}{q_1}} dt$$

$$\leq C \left( \frac{N}{q_1} - 1 \right)^{-p} \left( I + [\mathbf{v}]_T^2 \right)^p.$$

Since  $\frac{N}{2q_2} + \frac{1}{2} < 1$  as follows from  $q_2 > N$ , by Holder's inequality,

$$\langle t \rangle^b III_{\infty}(t) \leq C \int_{t-1}^t (t-s)^{-\frac{N}{2q_2} - \frac{1}{2}} \langle s \rangle^b \|\mathbf{w}(\cdot, s)\|_{L_{q_2}(\Omega)} ds$$

$$\leq C \left( \int_{t-1}^t (t-s)^{-\frac{N}{2q_2} - \frac{1}{2}} ds \right)^{1/p'} \left( \int_{t-1}^t (t-s)^{-\frac{N}{2q_2} - \frac{1}{2}} (\langle s \rangle^b \|\mathbf{w}(\cdot, s)\|_{L_{q_2}(\Omega)})^p ds \right)^{1/p}$$

$$\leq C \left( -\frac{N}{2q_2} - \frac{1}{2} \right)^{-1/p'} \left( \int_{t-1}^t (t-s)^{-\frac{N}{2q_2} - \frac{1}{2}} (\langle s \rangle^b \|\mathbf{w}(\cdot, s)\|_{L_{q_2}(\Omega)})^p ds \right)^{1/p}.$$

By the change of integration order, we have

$$\int_2^T (\langle t \rangle^b III_{\infty}(t))^p dt \leq C \left( 1 - \frac{N}{2q_2} \right)^{-\frac{p}{p'}} \int_2^T dt \int_{t-1}^t (t-s)^{-\frac{N}{2q_2} - \frac{1}{2}} (\langle s \rangle^b \|\mathbf{w}(\cdot, s)\|_{L_{q_2}(\Omega)})^p ds$$

$$\leq C \left( 1 - \frac{N}{2q_2} \right)^{-\frac{p}{p'}} \int_1^T (\langle s \rangle^b \|\mathbf{w}(\cdot, s)\|_{L_{q_2}(\Omega)})^p ds \int_s^{s+1} (t-s)^{-\frac{N}{2q_2} - \frac{1}{2}} dt$$

$$\leq C \left( 1 - \frac{N}{2q_2} \right)^{-p} \left( I + [\mathbf{v}]_T^2 \right)^p.$$

Summing up, we have obtained (65). Next, we prove (66).

By (64) with  $r = q_1$ ,  $\tilde{q}_1 = q_1 / 2$  and  $\tilde{q}_2 = q_1$ ,

$$\|\mathbf{u}(\cdot, t)\|_{L_{q_1}(\Omega)} \leq C(I_{q_1, \infty}(t) + II_{q_1, 1}(t) + III_{q_1, 1}(t))$$

With

$$I_{q_1, 1}(t) = \int_0^{t/2} (t-s)^{-\frac{N}{2q_1}} \|\mathbf{w}(\cdot, s)\|_{L_{q_1/2}(\Omega)} ds,$$

$$II_{q_1, 1}(t) = \int_{t/2}^{t-1} (t-s)^{-\frac{N}{2q_1}} \|\mathbf{w}(\cdot, s)\|_{L_{q_1/2}(\Omega)} ds,$$

$$III_{q_1, 1}(t) = \int_{t-1}^t \|\mathbf{w}(\cdot, s)\|_{L_{q_1}(\Omega)} ds.$$

By (56)

$$\begin{aligned} I_{q_1, 1}(t) &\leq (t/2)^{-\frac{N}{2q_1}} \left( \int_0^{t/2} \langle s \rangle^{-bp'} ds \right)^{1/p'} \left( \int_0^{t/2} \langle s \rangle^b \|\mathbf{w}(\cdot, s)\|_{L_{q_1/2}(\Omega)}^p ds \right)^{1/p} \\ &\leq Ct^{-\frac{N}{2q_1}} \left( I + [\mathbf{v}]_T^2 \right)^p. \end{aligned}$$

Analogously, by Holder's inequality and (56),

$$\begin{aligned} II_{q_1, 1}(t) &= \int_{t/2}^{t-1} (t-s)^{-\frac{N}{2q_1}} \langle s \rangle^{-b} \langle s \rangle^b \|\mathbf{w}(\cdot, s)\|_{L_{q_1/2}(\Omega)} ds, \\ &\leq C \langle t \rangle^{-b} \left( \int_{t/2}^{t-1} (t-s)^{-\frac{Np'}{2q_1}} ds \right)^{1/p'} \left( \int_0^T \langle s \rangle^b \|\mathbf{w}(\cdot, s)\|_{L_{q_1/2}(\Omega)}^p ds \right)^{1/p} \\ &= C \left( 1 - \frac{Np'}{2q_1} \right)^{1/p'} \langle t \rangle^{-b-\frac{N}{2q_1}+\frac{1}{p'}} \left( I + [\mathbf{v}]_T^2 \right) \\ &\leq C \left( 1 - \frac{Np'}{2q_1} \right)^{1/p'} \langle t \rangle^{-\frac{N}{2q_1}} \left( I + [\mathbf{v}]_T^2 \right) \end{aligned}$$

because  $b > \frac{1}{p'}$ . Finally, by (56),

$$\begin{aligned} III_{q_1, 1}(t) &\leq Ct^{-b} \int_{t-1}^t \langle s \rangle^b \|\mathbf{w}(\cdot, s)\|_{L_{q_1/2}(\Omega)} ds \\ &\leq Ct^{-b} \left( \int_{t-1}^t ds \right)^{1/p'} \left( \int_0^T \langle s \rangle^b \|\mathbf{w}(\cdot, s)\|_{L_{q_1/2}(\Omega)}^p ds \right)^{1/p} \\ &\leq Ct^{-b} \left( I + [\mathbf{v}]_T^2 \right). \end{aligned}$$

Summing up, we have obtained (66). Next, we prove (67). By (64),

$$\|\mathbf{u}(\cdot, t)\|_{H_{q_1}^1(\Omega)} \leq C(I_{q_1, 2}(t) + II_{q_1, 2}(t) + III_{q_1, 2}(t))$$

with

$$I_{q_1, 2}(t) = \int_0^{t/2} (t-s)^{-\frac{N}{2q_1}} \|\mathbf{w}(\cdot, s)\|_{L_{q_1/2}(\Omega)} ds,$$

$$II_{q_1, 2}(t) = \int_{t/2}^{t-1} (t-s)^{-\frac{N}{2q_1}} \|\mathbf{w}(\cdot, s)\|_{L_{q_1/2}(\Omega)} ds,$$

$$III_{q_1, 2}(t) = \int_{t-1}^t (t-s)^{-\frac{1}{2}} \|\mathbf{w}(\cdot, s)\|_{L_{q_1}(\Omega)} ds.$$

By (56),

$$\begin{aligned} I_{q_1, 2}(t) &\leq (t/2)^{-\frac{N}{2q_1}} \left( \int_0^{t/2} \langle s \rangle^{-bp'} ds \right)^{1/p'} \left( \int_0^{t/2} \langle s \rangle^b \|\mathbf{w}(\cdot, s)\|_{L_{q_1/2}(\Omega)}^p ds \right)^{1/p} \\ &\leq Ct^{-\frac{N}{2q_1}} \left( I + [\mathbf{v}]_T^2 \right). \end{aligned}$$

so that by the condition:  $(\frac{N}{q_1} - b)p > 1$  in (11)

$$\left( \int_2^T \langle t \rangle^{-b-\frac{N}{2q_1}} I_{q_1, 2}(t)^p dt \right)^{1/p} \leq C \left( \frac{N}{q_1} - b \right)^{p-1} \left( I + [\mathbf{v}]_T^2 \right).$$

By Holder's inequality,

$$\begin{aligned} \langle t \rangle^{-b-\frac{N}{2q_1}} II_{q_1, 2}(t) &\leq C \langle t \rangle^{-\frac{N}{2q_1}} \int_{t/2}^{t-1} (t-s)^{-\frac{N}{2q_1}} \langle s \rangle^b \|\mathbf{w}(\cdot, s)\|_{L_{q_1/2}(\Omega)} ds \\ &\leq C \langle t \rangle^{-\frac{N}{2q_1}} \left( \int_{t/2}^{t-1} (t-s)^{-\frac{Np'}{2q_1}} ds \right)^{1/p'} \left( \int_0^T \langle s \rangle^b \|\mathbf{w}(\cdot, s)\|_{L_{q_1/2}(\Omega)}^p ds \right)^{1/p} \\ &\leq C(1+t)^{-\left(\frac{N}{q_1}-\frac{1}{p'}\right)} \left( I + [\mathbf{v}]_T^2 \right). \end{aligned}$$

Since  $\left(\frac{N-1}{q_1} - \frac{1}{p'}\right)p > 1$  as follows from  $\frac{N}{q_1} = 1 + \frac{N}{q_2} > 1 = \frac{1}{p} + \frac{1}{p'}$ , we have

$$\left( \int_2^T \langle t \rangle^{-b-\frac{N}{q_1}} II_{q_1, 2}(t)^p dt \right)^{1/p} \leq C \left( \frac{N}{q_1} - b \right)^{p-1} \left( I + [\mathbf{v}]_T^2 \right)$$

Since  $q_1/2 < q_1 < q_2$ , we have

$$\|\mathbf{w}(\cdot, t)\|_{L_{q_1}(\Omega)} \leq \|\mathbf{w}(\cdot, t)\|_{L_{q_1/2}(\Omega)}^{\frac{q_2}{N+2q_2}} \|\mathbf{w}(\cdot, t)\|_{L_{q_2}(\Omega)}^{\frac{N+q_2}{N+2q_2}}.$$

Let  $\alpha = \frac{q_2}{N+2q_2}$  and  $\beta = \frac{N+q_2}{N+2q_2}$  then  $\alpha + \beta = 1$ , so that by (56)

and Holder's inequality

$$\begin{aligned} &\|\langle t \rangle^b \mathbf{w}\|_{L_p((0, T), L_{q_1}(\Omega))} \\ &\leq \left( \int_0^T \langle t \rangle^{b\alpha} \|\mathbf{w}(\cdot, t)\|_{L_{q_1/2}(\Omega)}^{p\alpha} \langle t \rangle^{b\beta} \|\mathbf{w}(\cdot, t)\|_{L_{q_2}(\Omega)}^{p\beta} dt \right)^{1/p} \\ &\leq \left( \int_0^T \langle t \rangle^{b\alpha} \|\mathbf{w}(\cdot, t)\|_{L_{q_1/2}(\Omega)}^p dt \right)^{\alpha/p} \left( \int_0^T \langle t \rangle^{b\beta} \|\mathbf{w}(\cdot, t)\|_{L_{q_2}(\Omega)}^p dt \right)^{\beta/p} \\ &\leq C \left( I + [\mathbf{v}]_T^2 \right) \end{aligned} \quad (69)$$

Since

$$\begin{aligned} \langle t \rangle^{-b-\frac{N}{2q_1}} III_{q_1, 2}(t) &\leq \int_{t-1}^t (t-s)^{-\frac{1}{2}} \langle s \rangle^{-b-\frac{N}{2q_1}} \|\mathbf{w}(\cdot, s)\|_{L_{q_1}(\Omega)} ds \\ &\leq \left( \int_{t-1}^t (t-s)^{-\frac{1}{2}} ds \right)^{1/p'} \left( \int_{t-1}^t (t-s)^{-\frac{1}{2}} \langle s \rangle^b \|\mathbf{w}(\cdot, s)\|_{L_{q_1}(\Omega)}^p ds \right)^{1/p}, \end{aligned}$$



by the change of integration order, we have

$$\begin{aligned} \int_2^T \langle t \rangle^{b-\frac{N}{2q_1}} III_{q_1,2}(t)^p dt &\leq 2^{\frac{p}{p'}} \int_2^T dt \int_{t-1}^t (t-s)^{-\frac{1}{2}} \langle s \rangle^b \|\mathbf{w}(\cdot, s)\|_{L_{q_1}(\Omega)}^p ds \\ &\leq 2^{\frac{p}{p'}} \int_0^T \langle s \rangle^b \|\mathbf{w}(\cdot, s)\|_{L_{q_1}(\Omega)}^p ds \int_s^{s+1} (t-s)^{-\frac{1}{2}} dt = 2^p \|\langle t \rangle^b \mathbf{w}\|_{L_p((0,T), L_{q_1}(\Omega))}^p, \end{aligned}$$

which, combined with (69), furnishes that

$$\left( \int_2^T \langle t \rangle^{b-\frac{N}{2q_1}} III_{q_1,2}(t)^p dt \right)^{1/p} \leq C \left( I + [\mathbf{v}]_T^2 \right)$$

Summing up, we have obtained (67).

Finally, we prove (68). By (64) with  $r = q_2$ ,  $\tilde{q}_1 = q_1/2$  and  $\tilde{q}_2 = q_2$ ,

$$\|\mathbf{u}(\cdot, t)\|_{L_{q_2}(\Omega)} \leq C(I_{q_2}(t) + II_{q_2}(t) + III_{q_2}(t))$$

with

$$I_{q_2}(t) = \int_0^{t/2} (t-s)^{-\frac{N}{2}(\frac{2}{q_1} + \frac{1}{q_2})} \|\mathbf{w}(\cdot, s)\|_{L_{q_1/2}(\Omega)} ds,$$

$$II_{q_2}(t) = \int_{t/2}^{t-1} (t-s)^{-\frac{N}{2}(\frac{2}{q_1} + \frac{1}{q_2})} \|\mathbf{w}(\cdot, s)\|_{L_{q_1/2}(\Omega)} ds,$$

$$III_{q_2}(t) = \int_{t-1}^t \|\mathbf{w}(\cdot, s)\|_{L_{q_2}(\Omega)} ds.$$

By Holder's inequality,

$$\begin{aligned} I_{q_2}(t) &\leq (t/2)^{-\frac{N}{2}(\frac{2}{q_1} + \frac{1}{q_2})} \left( \int_0^{t/2} \langle s \rangle^{bp'} ds \right)^{1/p'} \left( \int_0^{t/2} \langle s \rangle^b \|\mathbf{w}(\cdot, s)\|_{L_{q_1/2}(\Omega)}^p ds \right)^{1/p} \\ &\leq C \langle t \rangle^{b-\frac{N}{2}(\frac{2}{q_1} + \frac{1}{q_2})} \left( I + [\mathbf{v}]_T^2 \right). \end{aligned}$$

for  $t \geq 2$ . Since  $\frac{N}{2} \left( \frac{2}{q_1} + \frac{1}{q_2} \right) - \left( b - \frac{N}{2q_2} \right) = \frac{N}{q_1} - b$

by the condition:  $\left( \frac{N}{q_1} - b \right) p > 1$  in (11),

$$\begin{aligned} \left( \int_2^T \langle t \rangle^{b-\frac{N}{2q_2}} I_{q_2}(t)^p dt \right)^{1/p} &\leq C \left( \int_2^T t^{-\frac{(N-b)p}{q_1}} dt \right)^{1/p} \left( I + [\mathbf{v}]_T^2 \right) \\ &\leq C \left( \frac{(N-b)p-1}{q_1} \right)^{-1/p} \left( I + [\mathbf{v}]_T^2 \right). \end{aligned}$$

Since

$$\frac{N}{2} \left( \frac{2}{q_1} - \frac{1}{q_2} \right) = \frac{N}{2} \left( \frac{1}{q_2} + \frac{2}{N} \right) = \frac{N}{2q_2} + 1 > 1,$$

by Holder's inequality

$$\begin{aligned} \langle t \rangle^{b-\frac{N}{2q_2}} II_{q_2}(t) &\leq C \int_{t/2}^{t-1} (t-s)^{-\frac{(N-b)}{2q_2}} \langle s \rangle^b \|\mathbf{w}(\cdot, s)\|_{L_{q_1/2}(\Omega)} ds \\ &\leq C \left( \int_{t/2}^{t-1} (t-s)^{-\frac{(N-b)}{2q_2}} ds \right)^{1/p'} \left( \int_{t/2}^{t-1} (t-s)^{-\frac{(N-b)}{2q_2}} \langle s \rangle^b \|\mathbf{w}(\cdot, s)\|_{L_{q_1/2}(\Omega)}^p ds \right)^{1/p} \\ &\leq C \left( \frac{N}{2q_2} \right)^{-1/p'} \left( \int_{t/2}^{t-1} (t-s)^{-\frac{(N-b)}{2q_2}} \langle s \rangle^b \|\mathbf{w}(\cdot, s)\|_{L_{q_1/2}(\Omega)}^p ds \right)^{1/p}. \end{aligned}$$

so that by the change of integration order and (56)

$$\begin{aligned} \int_2^T \langle t \rangle^{b-\frac{N}{2q_1}} II_{q_2}(t)^p dt &\leq C \left( \frac{N}{2q_2} \right)^{\frac{p}{p'}} \int_2^T dt \int_{t/2}^{t-1} (t-s)^{-\frac{(N-b)}{2q_2}} \langle s \rangle^b \|\mathbf{w}(\cdot, s)\|_{L_{q_1/2}(\Omega)}^p ds \\ &\leq C \left( \frac{N}{2q_2} \right)^{\frac{p}{p'}} \int_0^T \langle s \rangle^b \|\mathbf{w}(\cdot, s)\|_{L_{q_1/2}(\Omega)}^p ds \int_{s+1}^{2s} (t-s)^{-\frac{(N-b)}{2q_2}} dt \leq C \left( \frac{N}{2q_2} \right)^{-p} \left( I + [\mathbf{v}]_T^2 \right). \end{aligned}$$

Analogously, by Holder's inequality

$$\begin{aligned} \langle t \rangle^{b-\frac{N}{2q_2}} III_{q_2}(t) &\leq C \int_{t-1}^t \langle s \rangle^{b-\frac{N}{2q_2}} \|\mathbf{w}(\cdot, s)\|_{L_{q_2}(\Omega)} ds, \\ &\leq C \left( \int_{t-1}^t ds \right)^{1/p'} \left( \int_{t-1}^t \langle s \rangle^b \|\mathbf{w}(\cdot, s)\|_{L_{q_2}(\Omega)}^p ds \right)^{1/p} \\ &\leq \left( \int_{t-1}^t \langle s \rangle^b \|\mathbf{w}(\cdot, s)\|_{L_{q_2}(\Omega)}^p ds \right)^{1/p}, \end{aligned}$$

so that by the change of integration order and (56)

$$\begin{aligned} \int_2^T \langle t \rangle^{b-\frac{N}{2q_2}} III_{q_2}(t)^p dt &\leq C \int_2^T dt \int_{t-1}^t \langle s \rangle^b \|\mathbf{w}(\cdot, s)\|_{L_{q_2}(\Omega)}^p ds \\ &\leq C \int_0^T \langle s \rangle^b \|\mathbf{w}(\cdot, s)\|_{L_{q_2}(\Omega)}^p ds \int_s^{s+1} dt \leq C \left( I + [\mathbf{v}]_T^2 \right). \end{aligned}$$

Summing up, we have obtained (68).

Recalling that  $T \geq 2$ , applying the maximal  $L_p$ - $L_q$  regularity theorem due to Shibata<sup>33</sup> to Eq. (54)

and using (56) give that

$$\|\mathbf{u}\|_{L_p((0,2), H_q^2(\Omega))} + \|\partial_t \mathbf{u}\|_{L_p((0,2), L_q(\Omega))} \leq C_q \|\mathbf{f}_0\|_{L_p((0,2), L_q(\Omega))} \leq C \left( I + [\mathbf{v}]_T^2 \right) \quad (70)$$

For any  $q \in [q_1/2, q_2]$ . Employing the same argumentation as that in proving (29), by real interpolation,

we have

$$\sup_{0 < t < 2} \|\mathbf{u}(\cdot, t)\|_{B_{q,p}^{2(1-1/p)}(\Omega)} \leq C \left( I + [\mathbf{v}]_T^2 \right) \quad (71)$$

for any  $q \in [q_1/2, q_2]$ . Combining (65), (66), (67), (68), (70), (71) and the Sobolev imbedding theorem,

we have

$$\begin{aligned} & \| \langle t \rangle^b u \|_{L_p((0,T)H_\infty^1(\Omega))} + \| \langle t \rangle^{\frac{N}{2q_1}} u \|_{L_\infty((0,T),L_{q_1}(\Omega))} \\ & + \| \langle t \rangle^{\frac{N}{2q_1}} u \|_{L_p((0,T)H_{q_1}^1(\Omega))} + \| \langle t \rangle^{\frac{N}{2q_2}} u \|_{L_p((0,T),L_{q_2}(\Omega))} \leq C(I + [v]_T^2). \end{aligned}$$

From (54),  $u$  satisfies the equations:

$$\begin{cases} \partial_t u + \lambda_0 v J(T)^{-1} \operatorname{Div} \tilde{S}(u, p) = -\lambda_0 w + \lambda_0 u & \text{in } \Omega \times (0, T), \\ \operatorname{div} u = 0 & \text{in } \Omega \times (0, T), \\ \tilde{S}(u, p) = 0 & \text{on } \Gamma \times (0, T), \\ u|_{t=0} = 0 & \text{in } \Omega \end{cases}$$

so that by Theorem 8,

$$\begin{aligned} & \| \langle t \rangle^{\frac{N}{2q_2}} u \|_{L_p((0,T)H_{q_2}^2(\Omega))} + \| \langle t \rangle^{\frac{N}{2q_2}} \partial_t u \|_{L_p((0,T),L_{q_2}(\Omega))} \\ & + C \| \langle t \rangle^{\frac{N}{2q_2}} (u, w) \|_{L_p((0,T),L_{q_2}(\Omega))}, \end{aligned}$$

which, combined with (72), furnishes that

$$[u]_T \leq C(I + [v]_T^2) \quad (73)$$

Since  $v = w + u$ , by (58) and (73), we see that  $v$  satisfies the inequality (50), which completes the proof of Theorem 1.<sup>53–56</sup>

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## Conflicts of interest

Author declares that there is no conflict of interest.

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