Research Article

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Weighted Adya distribution with properties and application

Abstract

In this paper a weighted version of Adya distribution which includes Adya distribution has been suggested for modeling lifetime data. The natures of descriptive statistics including coefficients of variation, skewness, kurtosis, and index of dispersion have been studied. The reliability measures including hazard rate function, reversed hazard rate function, mean residual life function and stochastic ordering have been studied. Method of maximum likelihood estimation has been discussed for estimating the parameters. A simulation study has been presented to know the performance of maximum likelihood estimates of parameters. The goodness of fit of the proposed distribution has been explained with a real lifetime data.

Keywords: Adya distribution, moments, reliability properties, maximum likelihood estimation, application

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Rama Shanker,[1] Kamlesh Kumar Shukla[2]
[1]Department of Statistics, Assam University, Silchar, Assam, India
[2]Department of Community Medicine, Noida International Institute of Medical Science, India

Correspondence: Kamlesh Kumar Shukla, Department of Community Medicine, Noida International Institute of Medical Science, India, Email kkshukl22@gmail.com

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Introduction

As we know that the basic purpose of distribution theory is to determine a reasonable distributional model for the data arising from different fields of knowledge. And once the distributional model of the data is determined, characterization of distribution, computation of confidence intervals of parameters and critical regions for hypothesis tests can easily be done. It has been observed that the discrete or continuous data that we are getting are stochastic in nature and the existing distributions are not able to explain the true nature of data and this is the reasons for the search for new distributions. Further, it has been observed that many times the true nature of data can be better explained by weighted distribution with appropriate weight function. The concept of weighted distributions was firstly introduced by Fisher[1] to model ascertainment biases which were later reformulated by Rao[2] in a unifying theory for problems where the observations fall in the category of non-experimental, non-replicated and non-random. When observations are recorded by an investigator in the nature according to some stochastic model, the distribution of the recorded observations will not have the original distribution unless every observation is given an equal chance of being recorded. For instance, let the original observation x_0 comes from a distribution having probability density function (pdf) $f(x,\theta)$, where θ may be a parameter vector and the observation x is recorded according to a probability re-weighted by weight function $w(x,\alpha)>0$, α being a new parameter vector, then x comes from a distribution having pdf

$$f_w(x;\theta,\alpha)=\frac{w(x;\alpha)f(x;\theta)}{E(W(X,\alpha))} \tag{1.1}$$

Note that such types of distribution are known as weighted distributions. The weighted distributions with weight function $w(x,\alpha)=x$ are called length biased distributions or simple size-biased distributions. Patil et al.[3,4] have examined some general probability models leading to weighted probability distributions and their applications and showed the occurrence of $w(x;\alpha)=x$ in a natural way in problems relating to sampling.

The study of weighted distributions is useful in distribution theory because it provides a new understanding of the existing standard probability distributions and it provides methods for extending existing standard probability distributions for modeling lifetime data due to introduction of additional parameter in the model which creates flexibility in their behavior. Weighted distributions occur in modeling datasets having clustered sampling, heterogeneity, and extraneous variation.

Shanker et al.[5] introduced a one parameter lifetime distribution named Adya distribution (AD) having pdf and cdf

$$f_1(x;\theta)=\frac{\theta^3}{\theta^4+2\theta^2+2}(\theta+x)^2 e^{-\theta x} \quad ;x>0,\ \theta>0 \tag{1.2}$$

$$F_1(x,\theta)=1-\left[1+\frac{\theta x\left(\theta x+2\theta^2+2\right)}{\theta^4+2\theta^2+2}\right]e^{-\theta x} \quad ;x>0,\theta>0 \tag{1.3}$$

Obviously Adya distribution is a convex combination of an exponential (θ) distribution, a gamma $(2,\theta)$ distribution and a gamma $(3,\theta)$ distribution with their mixing proportions $\frac{\theta^4}{\theta^4+2\theta^2+2}$, $\frac{2\theta^2}{\theta^4+2\theta^2+2}$ and $\frac{2}{\theta^4+2\theta^2+2}$, respectively.

Shanker et al.[5] discussed its statistical properties including coefficient of variation, skewness, kurtosis, index of dispersion, hazard rate function, mean residual life function, stochastic ordering, mean deviations from the mean and the median, Bonferroni and Lorenz curves, stress-strength reliability along with estimation of parameter using maximum likelihood estimation and applications to model lifetime data from engineering and biomedical sciences.

The main purpose of the present paper is to derive weighted version of Adya distribution and discuss its important statistical properties. Its statistical properties including behaviour of pdf, cdf, hazard rate function, mean residual life function and moments based descriptive measures. Maximum likelihood estimation has been discussed to estimate parameters of the distribution. The simulation study to know the performance of maximum likelihood estimates is presented. Finally, an application of the proposed weighted Adya distribution has been presented to test its goodness of fit with other distributions.

Weighted Adya distribution

Considering the weight function $w(x;\alpha)=x^{\alpha-1}$ in (1.1) and using the pdf of Adya distribution from (1.2), the pdf of weighted Adya distribution (WAD) can be expressed as

$$f_2(x;\theta,\alpha)=\frac{\theta^{\alpha+2}}{\theta^4+2\theta^2\alpha+\alpha(\alpha+1)}\frac{x^{\alpha-1}}{\Gamma(\alpha)}(\theta+x)^2 e^{-\theta x}\;;x>0,\theta>0,\alpha>0, \quad (2.1)$$

where $\Gamma(\alpha)$ is the complete gamma function defined as

$$\Gamma(\alpha)=\int_0^\infty e^{-y}y^{\alpha-1}dy\;;y>0,\alpha>0 \quad (2.2)$$

It can be easily shown that Adya distribution is a particular case of WAD at $\alpha=1$. Like Adya distribution, the pdf of Weighted Adya distribution can be easily expressed as a convex combination of gamma (θ,α), gamma $(\theta,\alpha+1)$ and gamma $(\theta,\alpha+2)$ distributions. We have

$$f_2(x;\theta,\alpha)=p_1 g_1(x;\theta,\alpha)+p_2 g_2(x;\theta,\alpha+1)+(1-p_1-p_2)g_3(x;\theta,\alpha+2), \quad (2.3)$$

where

$$p_1=\frac{\theta^4}{\theta^4+2\theta^2\alpha+\alpha(\alpha+1)},\; p_2=\frac{2\theta^2\alpha}{\theta^4+2\theta^2\alpha+\alpha(\alpha+1)}$$

$$g_1(x;\theta,\alpha)=\frac{\theta^\alpha}{\Gamma(\alpha)}e^{-\theta x}x^{\alpha-1},\; g_2(x;\theta,\alpha+1)=\frac{\theta^{\alpha+1}}{\Gamma(\alpha+1)}e^{-\theta x}x^{\alpha+1-1} \text{ and}$$

$$g_3(x;\theta,\alpha+2)=\frac{\theta^{\alpha+2}}{\Gamma(\alpha+2)}e^{-\theta x}x^{\alpha+2-1}.$$

The survival (reliability) function of WAD can be obtained as

$$S(x;\theta,\alpha)=P(X>x)=\int_x^\infty f_3(t;\theta,\alpha)dt=\frac{\theta^{\alpha+2}}{\left[\theta^4+2\theta^2\alpha+\alpha(\alpha+1)\right]\Gamma(\alpha)}\int_x^\infty t^{\alpha-1}\left(\theta^2+2\theta t+t^2\right)e^{-\theta t}dt$$

$$=\frac{\left[\theta^4+2\theta^2\alpha+\alpha(\alpha+1)\right]\Gamma(\alpha,\theta x)+(\theta x)^\alpha\left(\theta x+2\theta^2+\alpha+1\right)e^{-\theta x}}{\left[\theta^4+2\theta^2\alpha+\alpha(\alpha+1)\right]\Gamma(\alpha)}, \quad (2.5)$$

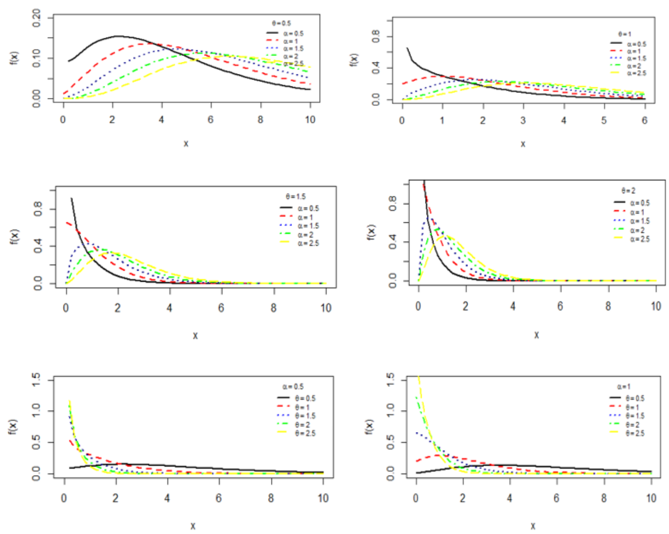
where $\Gamma(\alpha,z)$ the upper incomplete gamma function defined as

$$\Gamma(\alpha,z)=\int_z^\infty e^{-y}y^{\alpha-1}dy\;;y\ge 0,\alpha>0 \quad (2.6)$$

Thus, the cdf of WAD can thus be given by

$$F_1(x;\theta,\alpha)=1-S(x;\theta,\alpha)=1-\frac{\left[\theta^4+2\theta^2\alpha+\alpha(\alpha+1)\right]\Gamma(\alpha,\theta x)+(\theta x)^\alpha\left(\theta x+2\theta^2+\alpha+1\right)e^{-\theta x}}{\left[\theta^4+2\theta^2\alpha+\alpha(\alpha+1)\right]\Gamma(\alpha)}$$

Graphs of the pdf and the cdf of WAD for varying values of the parameters θ and α are shown in Figures 1 & 2 respectively.



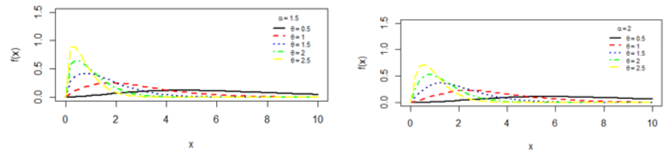


Figure 1 pdf of WAD for varying values of θ and α.

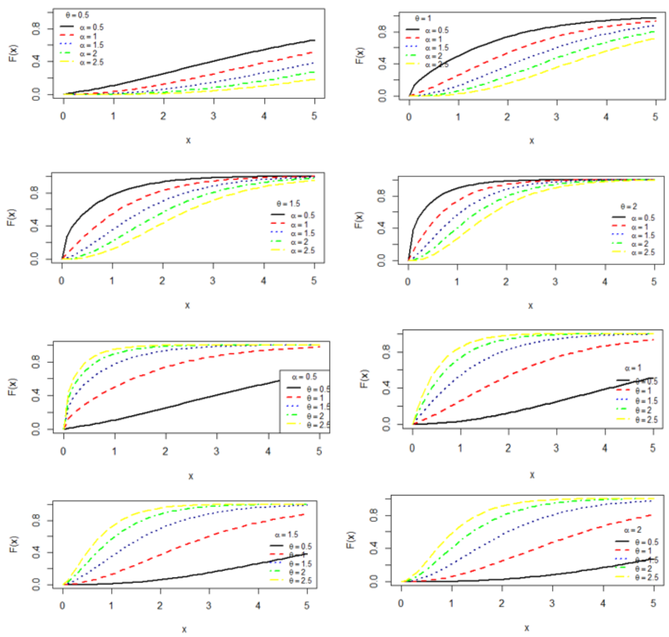


Figure 2 cdf of WAD for varying values of θ and α.

Moments based measures

The r th moment about origin μ_r' of WAD can be obtained as

$$\mu_r'=E\left(X^r\right)=\frac{\Gamma(\alpha+r)}{\Gamma(\alpha)}\frac{\theta^4+2(\alpha+r)\theta^2+(\alpha+r)(\alpha+r+1)}{\theta^r\left\{\theta^4+2\theta^2\alpha+\alpha(\alpha+1)\right\}};r=1,2,3,... \quad (3.1)$$

The first four moments about origin of WAD thus can be obtained as

$$\mu_1'=\frac{\alpha\left\{\theta^4+2(\alpha+1)\theta^2+(\alpha+1)(\alpha+2)\right\}}{\theta\left\{\theta^4+2\theta^2\alpha+\alpha(\alpha+1)\right\}}$$

$$\mu_2'=\frac{\alpha(\alpha+1)\left\{\theta^4+2(\alpha+2)\theta^2+(\alpha+2)(\alpha+3)\right\}}{\theta^2\left\{\theta^4+2\theta^2\alpha+\alpha(\alpha+1)\right\}}$$

$$\mu_3'=\frac{\alpha(\alpha+1)(\alpha+2)\left\{\theta^4+2(\alpha+3)\theta^2+(\alpha+3)(\alpha+4)\right\}}{\theta^3\left\{\theta^4+2\theta^2\alpha+\alpha(\alpha+1)\right\}}$$

$$\mu_4'=\frac{\alpha(\alpha+1)(\alpha+2)(\alpha+3)\left\{\theta^4+2(\alpha+4)\theta^2+(\alpha+4)(\alpha+5)\right\}}{\theta^4\left\{\theta^4+2\theta^2\alpha+\alpha(\alpha+1)\right\}}.$$

Using the relationship between central moments and moments about origin, the central moments of WAD can be obtained as

$$\mu_2=\frac{\alpha\left\{\theta^8+(4\alpha+4)\theta^6+\left(6\alpha^2+12\alpha+6\right)\theta^4+\left(4\alpha^3+12\alpha^2+8\alpha\right)\theta^2+\left(\alpha^4+4\alpha^3+5\alpha^2+2\alpha\right)\right\}}{\theta^2\left\{\theta^2+2\theta^2\alpha+\alpha(\alpha+1)\right\}^2}$$

$$\mu_3=\frac{2\alpha\left\{\begin{array}{l}\theta^{12}+(6\alpha+6)\theta^{10}+\left(15\alpha^2+27\alpha+12\right)\theta^8+\left(20\alpha^3+50\alpha^2+30\alpha\right)\theta^6\\+\left(15\alpha^4+48\alpha^3+39\alpha^2+6\alpha\right)\theta^4+\left(6\alpha^5+24\alpha^4+30\alpha^3+12\alpha^2\right)\theta^2\\+\left(\alpha^6+5\alpha^5+9\alpha^4+7\alpha^3+2\alpha^2\right)\end{array}\right\}}{\theta^3\left\{\theta^2+2\theta^2\alpha+\alpha(\alpha+1)\right\}^3}$$

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$$\mu_4=\frac{3\alpha\left\{\begin{array}{l}(\alpha+2)\theta^{16}+\left(8\alpha^2+24\alpha+16\right)\theta^{14}+\left(28\alpha^3+112\alpha^2+124\alpha+40\right)\theta^{12}\\+\left(56\alpha^4+280\alpha^3+416\alpha^2+192\alpha\right)\theta^{10}+\left(70\alpha^5+420\alpha^4+782\alpha^3+488\alpha^2+56\alpha\right)\theta^8\\+\left(56\alpha^6+392\alpha^5+888\alpha^4+760\alpha^3+208\alpha^2\right)\theta^6\\+\left(28\alpha^7+224\alpha^6+608\alpha^5+696\alpha^4+324\alpha^3+40\alpha^2\right)\theta^2\\+\left(8\alpha^8+72\alpha^7+232\alpha^6+344\alpha^5+240\alpha^4+64\alpha^3\right)\theta\\+\left(\alpha^9+10\alpha^8+38\alpha^7+72\alpha^6+73\alpha^5+38\alpha^4+8\alpha^3\right)\end{array}\right\}}{\theta^4\left\{\theta^2+2\theta\alpha+\alpha(\alpha+1)\right\}^4}$$

Thus the coefficient of variation (C.V), coefficient of skewness $\left(\sqrt{\beta_1}\right)$, coefficient of kurtosis (β_2), and index of dispersion (γ) of WAD are obtained as

$$C.V.=\frac{\sigma}{\mu_1'}=\frac{\sqrt{\theta^8+(4\alpha+4)\theta^6+\left(6\alpha^2+12\alpha+6\right)\theta^4+\left(4\alpha^3+12\alpha^2+8\alpha\right)\theta^2+\left(\alpha^4+4\alpha^3+5\alpha^2+2\alpha\right)}}{\sqrt{\alpha}\left\{\theta^4+2(\alpha+1)\theta^2+(\alpha+1)(\alpha+2)\right\}}$$

$$\sqrt{\beta_1}=\frac{\mu_3}{\mu_2^{3/2}}=\frac{2\left\{\begin{array}{l}\theta^{12}+(6\alpha+6)\theta^{10}+\left(15\alpha^2+27\alpha+12\right)\theta^8+\left(20\alpha^3+50\alpha^2+30\alpha\right)\theta^6\\+\left(15\alpha^4+48\alpha^3+39\alpha^2+6\alpha\right)\theta^4+\left(6\alpha^5+24\alpha^4+30\alpha^3+12\alpha^2\right)\theta^2\\+\left(\alpha^6+5\alpha^5+9\alpha^4+7\alpha^3+2\alpha^2\right)\end{array}\right\}}{\sqrt{\alpha}\left\{\theta^8+(4\alpha+4)\theta^6+\left(6\alpha^2+12\alpha+6\right)\theta^4+\left(4\alpha^3+12\alpha^2+8\alpha\right)\theta^2+\left(\alpha^4+4\alpha^3+5\alpha^2+2\alpha\right)\right\}^{3/2}}$$

$$\beta_2=\frac{\mu_4}{\mu_2^2}=\frac{3\left\{\begin{array}{l}(\alpha+2)\theta^{16}+\left(8\alpha^2+24\alpha+16\right)\theta^{14}+\left(28\alpha^3+112\alpha^2+124\alpha+40\right)\theta^{12}\\+\left(56\alpha^4+280\alpha^3+416\alpha^2+192\alpha\right)\theta^{10}+\left(70\alpha^5+420\alpha^4+782\alpha^3+488\alpha^2+56\alpha\right)\theta^8\\+\left(56\alpha^6+392\alpha^5+888\alpha^4+760\alpha^3+208\alpha^2\right)\theta^6\\+\left(28\alpha^7+224\alpha^6+608\alpha^5+696\alpha^4+324\alpha^3+40\alpha^2\right)\theta^2\\+\left(8\alpha^8+72\alpha^7+232\alpha^6+344\alpha^5+240\alpha^4+64\alpha^3\right)\theta\\+\left(\alpha^9+10\alpha^8+38\alpha^7+72\alpha^6+73\alpha^5+38\alpha^4+8\alpha^3\right)\end{array}\right\}}{\alpha\left\{\theta^8+(4\alpha+4)\theta^6+\left(6\alpha^2+12\alpha+6\right)\theta^4+\left(4\alpha^3+12\alpha^2+8\alpha\right)\theta^2+\left(\alpha^4+4\alpha^3+5\alpha^2+2\alpha\right)\right\}^2}$$

$$\gamma=\frac{\sigma^2}{\mu_1'}=\frac{\left\{\theta^8+(4\alpha+4)\theta^6+\left(6\alpha^2+12\alpha+6\right)\theta^4+\left(4\alpha^3+12\alpha^2+8\alpha\right)\theta^2+\left(\alpha^4+4\alpha^3+5\alpha^2+2\alpha\right)\right\}}{\theta\left\{\theta^4+2\theta^2\alpha+\alpha(\alpha+1)\right\}\left\{\theta^4+2(\alpha+1)\theta^2+(\alpha+1)(\alpha+2)\right\}}.$$

It should be noted that these moments of WAD reduce to the corresponding moments of Adya distribution at $\alpha=1$. Behaviors of coefficient of variation (C.V), coefficient of Skewness (S.K), coefficient of kurtosis (S.K.) and index of dispersion (I.D) of WAD are drawn for varying values of parameters θ and α and are shown in Figures 3–6 respectively.

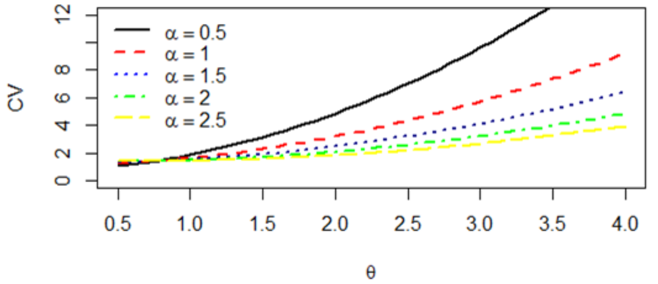

Figure 3 Graphs of C.V of WAD for varying values of parameters θ and α.

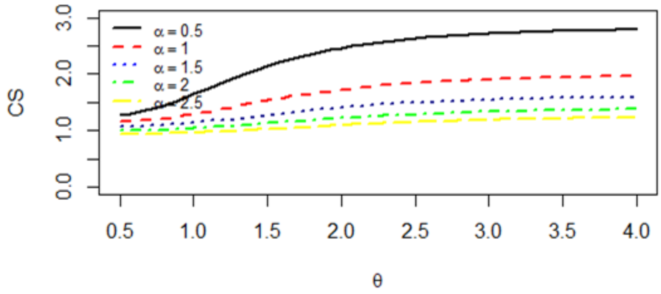

Figure 4 Graphs of C.S of WAD for varying values of parameters θ and α.

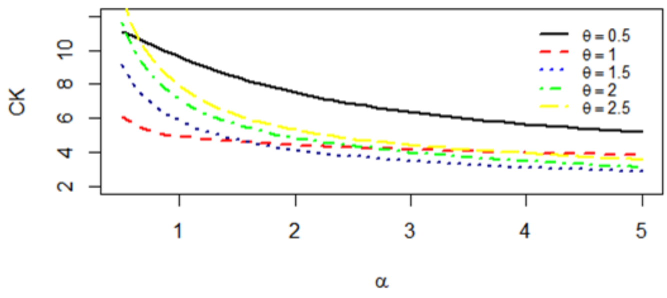

Figure 5 Graphs of C.K of WAD for varying values of parameters θ and α.

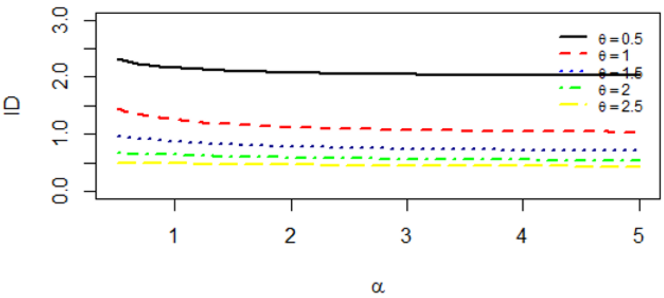

Figure 6 Graphs of I.D of WAD for varying values of parameters θ and α.

Moment generating function

The moment generating function of WAD can be obtained as

$$M_X(t)=\sum_{j=0}^{\infty}\frac{t^j}{j!}\mu_j'$$

$$=\sum_{j=0}^{\infty}\frac{t^j}{j!}\frac{\Gamma(\alpha+j)}{\Gamma(\alpha)}\frac{\theta^4+2(\alpha+j)\theta^2+(\alpha+j)(\alpha+j+1)}{\theta^j\left\{\theta^4+2\theta^2\alpha+\alpha(\alpha+1)\right\}}.$$

$$=\frac{1}{\left\{\theta^4+2\theta^2\alpha+\alpha(\alpha+1)\right\}\Gamma(\alpha)}\sum_{j=0}^{\infty}\frac{t^j}{j!}\frac{\Gamma(\alpha+j)\left\{\theta^4+2(\alpha+j)\theta^2+(\alpha+j)(\alpha+j+1)\right\}}{\theta^j}.$$

It can be easily verified that the coefficient of $\frac{t^j}{j!}$ in $M_X(t)$ gives the same μ_r' as given by (3.1).

Harmonic mean

The harmonic mean of WAD can be obtained as

$$HM=E\left(\frac{1}{X}\right)=\frac{\theta^{\alpha+2}}{\left\{\theta^4+2\theta^2\alpha+\alpha(\alpha+1)\right\}\Gamma(\alpha)}\int_0^{\infty}\frac{1}{x}x^{\alpha-1}(\theta+x)^2e^{-\theta x}\,dx$$

$$=\frac{\theta^{\alpha+2}}{\left\{\theta^4+2\theta^2\alpha+\alpha(\alpha+1)\right\}\Gamma(\alpha)}\int_0^{\infty}x^{\alpha-2}\left(\theta^2+2\theta x+x^2\right)e^{-\theta x}\,dx$$

$$=\frac{\theta^{\alpha+2}}{\left\{\theta^4+2\theta^2\alpha+\alpha(\alpha+1)\right\}\Gamma(\alpha)}\left[\theta^2\int_0^{\infty}e^{-\theta x}x^{\alpha-1-1}dx+2\theta\int_0^{\infty}e^{-\theta x}x^{\alpha-1}dx+\int_0^{\infty}e^{-\theta x}x^{\alpha+1-1}dx\right]$$

$$=\frac{\theta^{\alpha+2}}{\left\{\theta^4+2\theta^2\alpha+\alpha(\alpha+1)\right\}\Gamma(\alpha)}\left[\theta^2\frac{\Gamma(\alpha-1)}{\theta^{\alpha-1}}+2\theta\frac{\Gamma(\alpha)}{\theta^{\alpha}}+\frac{\Gamma(\alpha+1)}{\theta^{\alpha+1}}\right]$$

$$=\frac{\theta\left\{\theta^4\Gamma(\alpha-1)2\theta^2\Gamma(\alpha)+\Gamma(\alpha+1)\right\}}{\left\{\theta^4+2\theta^2\alpha+\alpha(\alpha+1)\right\}\Gamma(\alpha)};\alpha\geq 1.$$

Reliability measures

There are some important reliability measures of a distribution namely, the hazard rate function, reverse hazard rate function, Mills ratio and inverse Mills ratio, the mean residual life function and

Stochastic ordering. In this section these reliability measures for WAD have been discussed.

Hazard rate function

The hazard (or instantaneous failure rate function) plays a crucial role in reliability and survival analysis, as it defines the conditional probability of failure of an item in the next very small units of time Δx, given that it did not fail before x. Suppose X is a random variable with cdf $F(x)=P(X\le x)$. If $F(x)$ is absolutely continuous, then the random variable X has a probability density function $f(x)$. The hazard rate (HR) function $h(x)$ of X is defined as

$$h(x)=\frac{f(x)}{1-F(x)}=\frac{f(x)}{S(x)}$$

The hazard (or failure rate) function, $h(x)$ of WAD can be obtained as

$$h(x;\theta,\alpha)=\frac{f_3(x;\theta,\alpha)}{S(x;\theta,\alpha)}=\frac{\theta^{\alpha+2}x^{\alpha-1}(\theta+x)^2e^{-\theta x}}{\left\{\theta^4+2\theta^2\alpha+\alpha(\alpha+1)\right\}\Gamma(\alpha,\theta x)+(\theta x)^{\alpha}\left(\theta x+2\theta^2+\alpha+1\right)e^{-\theta x}}.$$

Graphs of $h(x)$ of WAD for varying values of parameters θ and α are shown in Figure 7. It is obvious that for varying values of parameters, the shapes of hazard rate function of WAD are changing and it can be used for data of various nature.

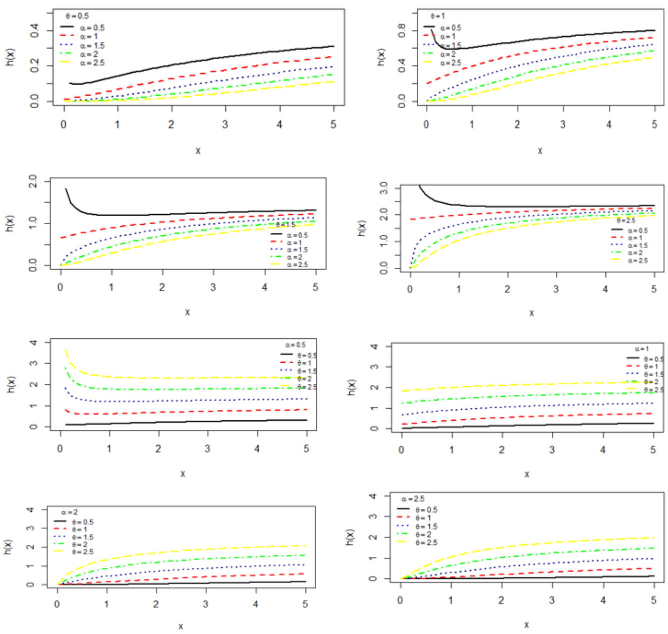

Figure 7 $h(x)$ of WAD for varying values of parameters θ and α.

Reverse hazard rate function

A function closely related to the hazard rate function is the reverse hazard rate function which was firstly introduced by Barlow et al. It is the dual of the hazard rate function and is defined as

$$r(x)=\frac{f(x)}{F(x)}.$$

Thus, the corresponding reverse hazard rate function of WAD can be obtained as

$$r(x)=\frac{\theta^{\alpha+2}x^{\alpha-1}(\theta+x)^2e^{-\theta x}}{\left\{\theta^4+2\theta^2\alpha+\alpha(\alpha+1)\right\}\Gamma(\alpha)-\left[\left\{\theta^4+2\theta^2\alpha+\alpha(\alpha+1)\right\}\Gamma(\alpha,\theta x)+(\theta x)^{\alpha}\left(\theta x+2\theta^2+\alpha+1\right)e^{-\theta x}\right]}.$$

Mills ratio and inverse mills ratio

The Mills ratio is defined as the ratio of the complementary cdf to the pdf of a random variable X and is denoted as $m(x)$ and defined as:

$$m(x)=\frac{1-F(x)}{f(x)}=\frac{S(x)}{f(x)}=\frac{\left\{\theta^4+2\theta^2\alpha+\alpha(\alpha+1)\right\}\Gamma(\alpha,\theta x)+(\theta x)^{\alpha}\left(\theta x+2\theta^2+\alpha+1\right)e^{-\theta x}}{\theta^{\alpha+2}x^{\alpha-1}(\theta+x)^2e^{-\theta x}}.$$

It is also related to the hazard rate function as

$$m(x)=\frac{1}{h(x)}$$

The inverse Mills ratio is the ratio of the pdf to the complementary cdf of a random variable X.

Mean residual life function

The mean residual life function $m(x)=E(X-x\,|\,X>x)$ of WAD can be obtained as

$$m(x;\theta,\alpha)=\frac{1}{S(x;\theta,\alpha)}\int_x^{\infty}t\,f_1(t;\theta,\alpha)\,dt-x$$

$$=\frac{1}{S(x;\theta,\alpha)}\left[\frac{\theta^{\alpha+2}}{\left\{\theta^4+2\theta^2\alpha+\alpha(\alpha+1)\right\}\Gamma(\alpha)}\int_x^{\infty}t^{\alpha}\left(\theta+2\theta t+t^2\right)e^{-\theta t}dt\right]-x$$

$$=\frac{(\theta x)^{\alpha}\left[\theta x+\theta^4+2(\alpha+1)\theta^2+(\alpha+1)(\alpha+2)\right]e^{-\theta x}+\left[\alpha\theta^4+\alpha(\alpha+1)\left(2\theta^2+\alpha+2\right)-\theta x\left\{\theta^4+2\theta^2\alpha+\alpha(\alpha+1)\right\}\right]\Gamma(\alpha,\theta x)}{\theta\left[(\theta x)^{\alpha}\left(\theta x+2\theta^2+\alpha+1\right)e^{-\theta x}+\left\{\theta^4+2\theta^2\alpha+\alpha(\alpha+1)\right\}\Gamma(\alpha,\theta x)\right]}$$

It can be easily shown that $m(0;\theta,\alpha)=\dfrac{\alpha\left\{\theta^4+2(\alpha+1)\theta^2+(\alpha+1)(\alpha+2)\right\}}{\theta\left\{\theta^4+2\theta^2\alpha+\alpha(\alpha+1)\right\}}=\mu_1'$.

Graphs of $m(x)$ of WAD for varying values of parameters θ and α are shown in Figure 8.

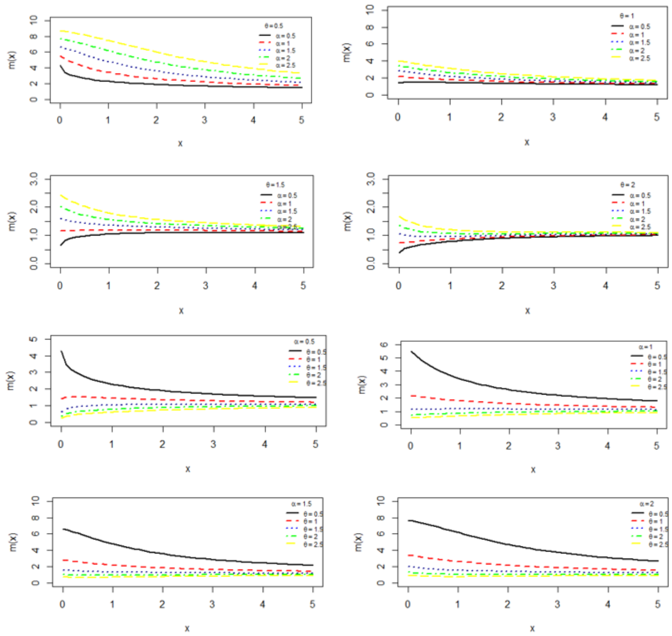

Figure 8 $m(x)$ of WAD for varying values of parameters θ and α.

Stochastic ordering

The stochastic ordering of positive continuous random variables is an important tool for judging their comparative behavior. A random variable X is said to be smaller than a random variable Y in the

i. stochastic order $(X\le_{st}Y)$ if $F_X(x)\ge F_Y(x)$ for all x
ii. hazard rate order $(X\le_{hr}Y)$ if $h_X(x)\ge h_Y(x)$ for all x
iii. mean residual life order $(X\le_{mrl}Y)$ if $m_X(x)\le m_Y(x)$ for all x

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iv. Likelihood ratio order $\left(X \le_{lr} Y\right)$ if $\frac{f_X(x)}{f_Y(x)}$ decreases in x .

The following interrelationships due to Shaked & Shanthikumar[6] are well known for establishing stochastic ordering of distributions

$$X \le_{lr} Y \Rightarrow X \le_{hr} Y \Rightarrow X \le_{mrl} Y$$
$$\Downarrow$$
$$X \le_{st} Y$$

It can be easily shown that WAD is ordered with respect to the strongest 'likelihood ratio' ordering. The stochastic ordering of WAD has been explained in the following theorem:

Theorem: Suppose $X \sim \text{WAD}(\theta_1,\alpha_1)$ and $Y \sim \text{WAD}(\theta_2,\alpha_2)$. If $\alpha_1 \le \alpha_2$ and $\theta_1 > \theta_2$ (or $\alpha_1 < \alpha_2$; $\theta_1 \ge \theta_2$), then $X \le_{lr} Y$ and hence $X \le_{hr} Y$, $X \le_{mrl} Y$ and $X \le_{st} Y$.

Proof: We have

$$\frac{f_X(x;\theta_1,\alpha_1)}{f_Y(x;\theta_2,\alpha_2)} = \left[\frac{\theta_1^{\alpha_1+2}\left\{\theta_2^4 + 2\theta_2^2\alpha_2 + \alpha_2(\alpha_2+1)\right\}\Gamma(\alpha_2)}{\theta_2^{\alpha_2+2}\left\{\theta_1^4 + 2\theta_1^2\alpha_1 + \alpha_1(\alpha_1+1)\right\}\Gamma(\alpha_1)}\right]\left(\frac{\theta_1+x}{\theta_2+x}\right)^2 x^{\alpha_1-\alpha_2}\, e^{-(\theta_1-\theta_2)x};\, x>0$$

Now, taking logarithm both sides, we get

$$\log\frac{f_X(x;\theta_1,\alpha_1)}{f_Y(x;\theta_2,\alpha_2)} = \log\left[\frac{\theta_1^{\alpha_1+2}\left\{\theta_2^4 + 2\theta_2^2\alpha_2 + \alpha_2(\alpha_2+1)\right\}\Gamma(\alpha_2)}{\theta_2^{\alpha_2+2}\left\{\theta_1^4 + 2\theta_1^2\alpha_1 + \alpha_1(\alpha_1+1)\right\}\Gamma(\alpha_1)}\right] + 2\log\left(\frac{\theta_1+x}{\theta_2+x}\right) + (\alpha_1-\alpha_2)\log x - (\theta_1-\theta_2)x$$

This gives $\frac{d}{dx}\ln\frac{f_X(x;\theta_1,\alpha_1)}{f_Y(x;\theta_2,\alpha_2)} = \frac{2(\theta_2-\theta_1)}{(\theta_1+x)(\theta_2+x)} + \frac{\alpha_1-\alpha_2}{x} - (\theta_1-\theta_2)$.

Thus for $\alpha_1 \le \alpha_2$ and $\theta_1 > \theta_2$ (or $\alpha_1 < \alpha_2$ and $\theta_1 \ge \theta_2$), $\frac{d}{dx}\ln\frac{f_X(x;\theta_1,\alpha_1)}{f_Y(x;\theta_2,\alpha_2)} < 0$. This means that $X \le_{lr} Y$ and hence $X \le_{hr} Y$, $X \le_{mrl} Y$ and $X \le_{st} Y$.

Maximum likelihood estimation

Suppose $(x_1, x_2, x_3, \dots, x_n)$ be a random sample of size n from WAD. The log-likelihood function, $\log L$ of WAD can be obtained as

$$\log L = \sum_{i=1}^{n} \log f_2(x_i;\theta,\alpha)$$

$$= n\left[(\alpha+2)\log\theta - \log\left(\theta^4 + 2\theta^2\alpha + \alpha^2 + \alpha\right) - \log\Gamma(\alpha)\right] + (\alpha-1)\sum_{i=1}^{n}\log(x_i) + 2\sum_{i=1}^{n}\log(\theta+x_i) - n\theta\bar{x}$$

The maximum likelihood estimates (MLE's) $\left(\hat{\theta}, \hat{\alpha}\right)$ of the parameters (θ, α) of WAD are the solutions of the following log likelihood equations

$$\frac{\partial \log L}{\partial\theta} = \frac{n(\alpha+2)}{\theta} - \frac{4n\left(\theta^3+\theta\alpha\right)}{\theta^4 + 2\theta^2\alpha + \alpha^2 + \alpha} + 2\sum_{i=1}^{n}\frac{1}{\theta+x_i} - n\,\bar{x} = 0$$

$$\frac{\partial \log L}{\partial\alpha} = n\ln\theta - \frac{n\left(2\theta^2 + 2\alpha + 1\right)}{\theta^4 + 2\theta^2\alpha + \alpha^2 + \alpha} - n\,\psi(\alpha) + \sum_{i=1}^{n}\log(x_i) = 0$$

where $\psi(\alpha) = \frac{d}{d\alpha}\log\Gamma(\alpha)$ is the digamma function.

Since these two log likelihood equations cannot be expressed in closed forms, they cannot be solved analytically. However, these two log likelihood equations can be solved using R-software.

A simulation study

In this section, a simulation study has been carried to check the performance of maximum likelihood estimates by taking sample sizes (n=20, 40, 60, 80 for values of $\theta = 0.5, 1.0, 1.5, 2.0$ and $\alpha = 0.5$ & 1.5. Acceptance and rejection method is used to generate random number for data simulation using R-software. The process was repeated 1,000 times for the calculation of Average Bias error (ABE) and MSE (Mean square error) of parameters θ and α and are presented in Tables1 & 2 respectively. For the WAD decreasing trend has been observed in ABE and MSE as the sample size increases and this shows that the performance of maximum likelihood estimators is quite good and consistent.

Table 1 ABE and MSE of parameters at fixed value $\alpha = 0.5$

n	θ	ABE(θ)	MSE (θ)	ABE (α)	MSE (α)
20	0.5	-0.0037	0.0003	-0.0560	0.0007
20	1.0	-0.0287	0.01649	-0.0310	0.0193
20	1.5	-0.0537	0.0577	-0.0560	0.0628
20	2.0	-0.0787	0.1239	-0.0810	0.1313
40	0.5	0.0022	0.0002	-0.0225	0.0003
40	1.0	-0.0103	0.0042	-0.0099	0.0039
40	1.5	-0.0227	0.0207	-0.02249	0.0202
40	2.0	-0.0353	0.0497	-0.0349	0.0489
60	0.5	0.001	0.0006	-0.0142	0.0003
60	1.0	-0.0073	0.0032	-0.0059	0.0021
60	1.5	-0.0156	0.0147	-0.0142	0.0121
60	2.0	-0.0239	0.0345	-0.0225	0.0305
80	0.5	0.0001	0.0000	-0.0117	0.0004
80	1.0	-0.0061	0.0030	-0.0055	0.0024
80	1.5	-0.0124	0.0123	-0.0117	0.0111
80	2.0	-0.0186	0.0277	-0.0180	0.0360

Table 2 ABE and MSE of parameters at fixed value $\alpha = 0.5$

n	θ	ABE(θ)	MSE (θ)	ABE (α)	MSE (α)
20	0.5	0.0298	0.0177	0.0052	0.0608
20	1.0	0.0048	0.0005	0.0302	0.0182
20	1.5	-0.0202	0.0081	0.0052	0.0005
20	2.0	-0.0452	0.0408	-0.0198	0.0078
40	0.5	0.0166	0.0112	0.0162	0.0678
40	1.0	0.0041	0.0007	0.0287	0.0329
40	1.5	-0.0084	0.0028	0.0162	0.0104
40	2.0	-0.0208	0.0174	0.0036	0.0005
60	0.5	0.0084	0.0094	0.0067	0.0329
60	1.0	0.0001	0.0000	0.0151	0.0136
60	1.5	-0.0082	0.0040	0.0067	0.0027
60	2.0	-0.0165	0.0164	-0.0015	0.0001
80	0.5	0.0070	0.0039	0.0066	0.0293
80	1.0	0.0007	0.0004	0.0129	0.0133
80	1.5	-0.0055	0.0024	0.0066	0.0035
80	2.0	-0.0117	0.0109	0.0004	0.0001

A numerical example

In this section application and the goodness of fit of WAD has been discussed with the following real lifetime data relating to the lifetime of a certain device reported by Sylwia[7] and the observations are 0.0094, 0.0500, 0.4064, 4.6307, 5.1741, 5.8808, 6.3348, 7.1645, 7.2316, 8.2604, 9.2662, 9.3812, 9.5223, 9.8783, 9.9346, 10.0192, 10.4077, 10.4791, 11.0760, 11.3250, 11.5284, 11.9226, 12.0294, 12.0740, 12.1835, 12.3549, 12.5381, 12.8049, 13.4615, 13.8530.

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For this data set, WAD has been fitted along with one parameter exponential and Adya distributions and two-parameter distributions including weighted Lindley distribution (WLD) introduced by Ghitany et al.,[8] quasi Lindley distribution (QLD) of Shanker & Mishra,[9] a two-parameter Lindley distribution (TPLD-I) of Shanker & Mishra,[10] a two-parameter Lindley distribution (TPLD-II) of Shanker et al.[11] and Weibull distribution introduced by Weibull.[12] The pdf and cdf of the considered distributions: WLD, Weibull, TPLD-I, TPLD-II, QLD and exponential distribution are given in Table 3. The ML estimates, values of $-2\ln L$, Akaike Information criteria (AIC), K-S statistics and p-value of the fitted distributions are presented in Table 4. The AIC and K-S Statistics are computed using the following formulae: $AIC=-2\ln L+2k$ and $\text{K-S}=\underset{x}{\text{Sup}}\left|F_n(x)-F_0(x)\right|$, where k = the number of parameters, n = the sample size, $F_n(x)$ is the empirical (sample) cumulative distribution function, and $F_0(x)$ is the theoretical cumulative distribution function. The best distribution is the distribution corresponding to lower values of $-2\ln L$, AIC, and K-S statistics.

The variance-covariance matrix and the 95% confidence intervals (CI's) of the ML estimates of the parameters θ and α of WAD are presented in Table 5. From the goodness of fit in Table 4 and the fitted plots of the distribution for the considered datasets in Figure 9 shows that WAD gives much closure fit as compared to other considered distributions.

The profile of likelihood estimates of parameters $\hat{\theta}$ and $\hat{\alpha}$ of WAD for the given dataset is shown in Figure 9. Also, the fitted plots of the considered dataset for WAD are shown in Figure 10.

Table 3 The pdf and the cdf of fitted distributions

Distributions	Pdf	Cdf
WLD	$f(x;\theta,\alpha)=\frac{\theta^{\alpha+1}}{\theta+\alpha}\frac{x^{\alpha-1}}{\Gamma(\alpha+1)}(1+x)e^{-\theta x}$	$F(x;\theta,\alpha)=1-\frac{(\theta+\alpha)\Gamma(\alpha,\theta x)+(\theta x)^{\alpha}e^{-\theta x}}{(\theta+\alpha)\Gamma(\alpha)}$
Weibull	$f(x;\theta,\alpha)=\theta\alpha x^{\alpha-1}e^{-\theta x^{\alpha}}$	$F(x;\theta,\alpha)=1-e^{-\theta x^{\alpha}}$
TPLD-I	$f(x;\theta,\alpha)=\frac{\theta^2}{\theta\alpha+1}(\alpha+x)e^{-\theta x};x>0,\theta>0,\theta\alpha>-1$	$F(x;\theta,\alpha)=1-\left(1+\frac{\theta x}{\alpha\theta+1}\right)e^{-\theta x}$
TPLD-II	$f(x;\theta,\alpha)=\frac{\theta^2}{\theta+\alpha}(1+\alpha x)e^{-\theta x}\;;x>0,\theta>0,\alpha>0$	$F(x;\theta,\alpha)=1-\left(1+\frac{\alpha\theta x}{\theta+\alpha}\right)e^{-\theta x}$
QLD	$F(x;\theta,\alpha)=1-\left(1+\frac{\alpha\theta x}{\theta+\alpha}\right)e^{-\theta x}$	$F(x;\theta,\alpha)=1-\left(1+\frac{\theta x}{\alpha+1}\right)e^{-\theta x}$
Exponential (ED)	$f(x;\theta)=\theta e^{-\theta x}$	$F(x;\theta)=1-e^{-\theta x}$

Table 4 MLE's, - 2ln L, AIC, K-S statistics and p-values of the fitted distributions

Distribution	ML Estimates		$-2\ln L$	AIC	K-S	P-value
	$\hat{\theta}$	$\hat{\alpha}$				
WAD	0.2303	0.132	179.31	183.31	0.276	0.010
WLD	0.1733	0.7537	183	187	0.281	0.012
Weibull	0.0258	1.6168	185.45	189.45	0.278	0.014
TPLD-I	0.2001	1.1774	183.74	187.74	0.275	0.016
TPLD-II	0.2002	0.849	183.74	187.74	0.275	0.016
QLD	0.2001	0.2356	183.74	187.74	0.275	0.016
AD	0.3314	-------	187.43	189.43	0.338	0.001
Exponential	0.1106	--------	192.09	194.09	0.314	0.004

Table 5 Variance-Covariance matrix and 95% confidence intervals (CI's) of the ME estimates $(\hat{\theta},\hat{\alpha})$ of parameters (θ,α) for the given dataset

Parameters	Variance-covariance matrix		95 % CI	
	$\hat{\theta}$	$\hat{\alpha}$	Lower	Upper
$\hat{\theta}$	0.001109402	0.00279572	0.17450148	0.3121785
$\hat{\alpha}$	0.002795720	0.01862830	0.01988068	0.6389076

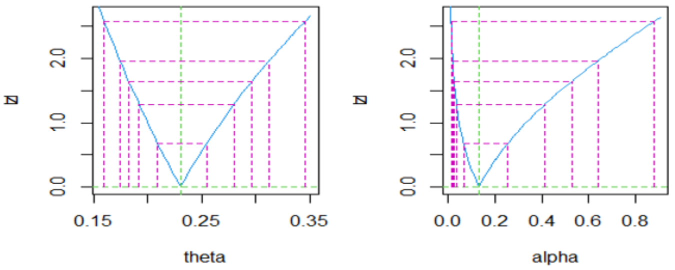

Figure 9 Profile of the likelihood estimates θ and α of WAD for the given dataset.

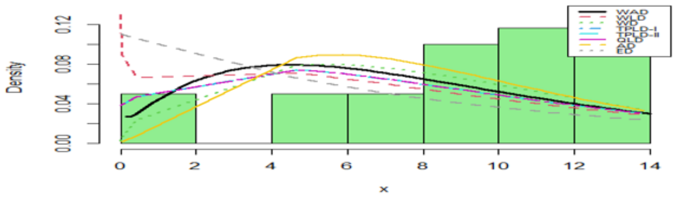

Figure 10 Fitted plots of the two-parameter distributions for the given dataset.

Concluding remarks

In this paper a two-parameter weighted Adya distribution (WAD) which includes one parameter Adya distribution as special case has been suggested for modeling lifetime data. Some of its properties including shapes of probability density function, moments-based measures including coefficients of variation, skewness, kurtosis, and index of dispersion; hazard rate function, mean residual life function and stochastic ordering have been studied. Method of maximum likelihood estimation has been discussed for estimating the parameters. The simulation study has been presented to know the performance of parameters. The goodness of the proposed distribution has been explained with a real lifetime data and the fit has been found to be quite satisfactory over one parameter and two parameter lifetime distributions.

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Conflicts of interest

The authors declare that there are no conflicts of interest.

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