Research Article

A comparative study of various estimation methods for modified Lindley distribution

Abstract

The purpose of this study is to examine the finite sample aspects of estimates of the parameters of the weighted Lindley distribution derived by four estimation methods: maximum likelihood, method of moments, ordinary least squares, and weighted least squares, using Monte Carlo simulations. As a comparison criterion, bias and mean-squared error are used. In both small and large samples, the Cramer-von Mises approach is found to be very competing with the maximum likelihood method. To substantiate the conclusion, a statistical analysis of a real data set related to weather is performed.

Keywords: modified Lindley distribution, method of moments, maximum likelihood, bias, mean square error, weighted models

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Introduction

In modelling data related to reliability, one of the most sought out distribution is the Lindley distribution. Lindley distribution was studied by Lindley,[1] with an advantage of having a single parameter, with an increasing hazard rate function (hrf). The Lindley distribution is a linear combination of the Gamma (2, β) and exponential (β) distribution.

Let Z be a random variable following the Lindley distribution with parameter β. The probability density function (pdf) of the Lindley distribution is defined by Lindley[1] as follows

$$g(z) = \frac{\beta^2}{1+\beta}(1+z)e^{-\beta z}, \quad z > 0, \quad \beta > 0$$

The distribution function (df) of the Lindley distribution is also defined as follows

$$G(z) = 1 - \left[1 + \frac{\beta z}{1+\beta}\right]e^{-\beta z}, \quad z > 0, \quad \beta > 0$$

Ghitany[2] discussed the application of the Lindley distribution with a real-world dataset, after which the distribution became popular. Many generalizations of the Lindley distribution were developed. We redirect the readers to Tomy[3] and Chesneau[4] to get a better understanding of the different generalizations and applications of the Lindley distribution.

Among which the modified Lindley (ML) distribution developed by Chesneau[5] is of prime importance. Suppose Y is a random variable following the ML distribution with parameter β, then the pdf of ML distribution is defined as follows

$$g(y) = \frac{\beta}{1+\beta}e^{-2\beta y}\left[(1+\beta)e^{\beta y} + 2\beta y - 1\right], \quad y > 0, \quad \beta > 0 \quad (1)$$

The df of the ML distribution are also defined as follows,

$$G(y) = 1 - \left[1 + \frac{\beta z}{1+\beta}e^{-\beta y}\right]e^{-\beta y}, \quad y > 0, \quad \beta > 0$$

The mean and variance of ML distribution is given as follows,

$$\mu = \frac{5+4\beta}{4\beta(1+\beta)}$$

$$\sigma^2 = \frac{(3+4\beta)(5+4\beta)}{16\beta^2(1+\beta)^2}$$
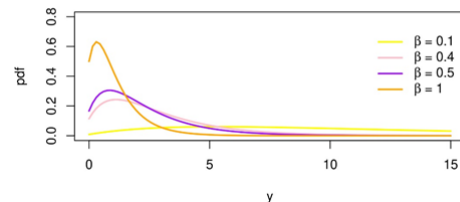

Figure 1 Illustrates the plots of the ML distribution with various values of the parameter β.

Observations –

The shapes of the pdf plots are unimodal, and are decreasing in nature.

In this paper, we discuss the different methods to estimate the unknown single parameter β of the ML distribution. A simulation study is also conducted to compare the efficiency of different estimators.

The structure of the paper is as follows. Section 2 discusses the different methods of estimating β. In Section 3, we compare the efficiency of the estimates produced by the different methods of estimation. Section 4 shows the application of different estimation methods over a real-world dataset.

Inferential aspects

In this section, we concentrate on the estimation of parameter β, with an assumption that β is unknown. Here, we consider estimating the parameter of the ML distribution using the methods, which include Maximum Likelihood (ML), Method of Moment (MOM) estimation, least squares (LS) and Weighted Least squares (WLS) estimation, and Cram´er-von Mises (CVM) estimation.

ML estimation

Consider $(y_1, y_2, ..., y_n)$ to be a random sample from the ML distribution with parameter β, with pdf defined in Equation (1). The log-likelihood (LL) function is given by

$$LL(\beta) = n\log\beta - n\log(1+\beta) - 2\beta\Sigma_{i=1}^n y_i + \log[\Pi_{i=1}^n[(1+\beta)e^{\beta y_i} + 2\beta y_i - 1]$$

The ML estimate of β, $\hat{\beta}$ is defined as

$$\hat{\beta} = \text{argmax}_{\beta\varepsilon(0,\infty)} LL(\beta)$$

The ML estimate of β can be obtained by taking the derivative of $LL(\beta)$ and equating it to zero, which is as follows:

$$\frac{n}{\beta} - \frac{n}{1+\beta} - 2\sum_{i=1}^{n} y_i + \sum_{i=1}^{n} \frac{e^{\beta y_i}(\beta y_i + y_i + 1) + 2y_i}{(1+\beta)e^{\beta y_i} + 2\beta y_i - 1}$$

As there doesn't exist any explicit form of $\tilde{\beta}$, numerical methods are used to get the estimate of β.

MOM estimation

Let $(y_1, y_2, ..., y_n)$ represent n independent ML distribution observations. Chesneau[5] showed that the unique MoM estimate of the pdf defined in Equation, $\tilde{\beta}$, of β is given by

$$f(z) = \frac{-(z-1) + \sqrt{(z-1)^2 + 5z}}{2z} , \quad z > 0$$

LS and WLS estimation

Swain[6] devised the method of least squares. The difference between both the vector of uniformized order statistics and the associated vector of expected values is reduced using this approach. Let $(y_1, y_2, ..., y_n)$ represent a random sample from the ML distribution with parameters. Also, let $y_{(1)}, y_{(2)}, ... y_{(n)}$ be the order values of $y_1, y_2, ..., y_n$ in ascending order.

$$LS(\beta) = \sum_{i=1}^{n} \left[G_0(y_i, \beta) - \frac{i}{(n+1)} \right]^2$$

The LS estimate of β is defined by

$$\hat{\beta} = \underset{\beta \varepsilon(0,\infty)}{\operatorname{argmax}} LS(\beta).$$

After partially differentiating with regard to unknown parameter β, one can construct a nonlinear equation using the df of the ML distribution. The Monte Carlo simulation can be used to find the solution to this nonlinear equation.

The WLS function is given as follows:

$$WLS(\beta) = \eta_i \sum_{i=1}^{n} \left[G_0(y_i, \beta) - \frac{i}{n+1} \right]^2$$

where $\eta_i = \frac{(n+1)^2(n+2)}{[i(n-i+1)]}$

The WLS estimate of β is defined by

$$\hat{\beta} = \underset{\beta \varepsilon(0,\infty)}{\operatorname{argmax}} WLS(\beta)$$

As a result, the WLS estimate of can be derived using a process similar to the LS estimate.

CVM estimation

The CVM approach is identical to the two methods earlier described. The CVM function is described as follows:

$$CVM(\beta) = \frac{1}{12n} + \sum_{i=1}^{n} \left[G_0(y_i, \beta) - \frac{2i-1}{2n} \right]^2$$

The CVM estimate of β is defined by

$$\hat{\beta} = \underset{\beta \varepsilon(0,\infty)}{\operatorname{argmax}} CVM(\beta).$$

As a result, the CVM estimate of follows the same approach as the WL or LS estimations.

Simulation study

We conduct a simulation analysis in this part to assess the efficiency of the ML model parameter estimates reported in Section 3. We employed the Monte Carlo technique in R software, as well as Newton's method and the BFGS (Broyden-Fletcher-Goldfarb-Shanno) algorithm, developed by Broyden[7], Fletcher[8], Goldfarb[9], & Shanno[10].

The bias and Mean Square Error (MSE) were examined in the study,

$$\text{Bias}(\hat{\beta}) = \frac{1}{N\sum_{j=1}^{N}(\hat{\beta}_j - \beta)} \quad \text{and} \quad MSE(\hat{\beta}) = \frac{1}{\sum_{j=1}^{N}(\hat{\beta}_j - \beta)^2}$$

where, for each sample size n, $\hat{\beta}_j$ is the parameter's estimate at the jth iteration using a specific estimation method. R software, version 4.0.5, was used for all calculations.

Tables 1–3 show the biases and mean square error (MSE) of the proposed model's parameters based on N = 5000 replicates. To further analyse the nature of the estimates, various sample sizes (n = 50, 100, 200, and 500) and values of the parameters, $\beta = (0.5, 1, 1.5)$ are employed.

Table 1 Bias and MSE for β = 0.5

n	Estimate	Bias	MSE
			$\hat{\beta}$
50	ML	0.017219	0.004665
	MOM	0.017282	0.004002
	OLS	0.016243	0.004623
	WLS	-0.5	0.25
	CVM	0.016524	0.003894
100	ML	0.013654	0.00199
	MOM	0.013417	0.002018
	OLS	0.013752	0.00229
	WLS	-0.5	0.25
	CVM	0.013256	0.002306
200	ML	0.011259	0.000995
	MOM	0.012063	0.001168
	OLS	0.011815	0.001161
	WLS	-0.5	0.25
	CVM	0.010682	0.000978
500	ML	0.010423	0.000425
	MOM	0.011379	0.000505
	OLS	0.011279	0.000503
	WLS	-0.5	0.25
	CVM	0.010103	0.000422

Table 2 Bias and MSE for β = 1

n	Estimate	Bias	MSE
			$\hat{\beta}$
50	ML	0.03443951	0.0156625
	MOM	0.03748674	0.0162677
	OLS	0.02567615	0.01804344
	WLS	-1	1
	CVM	0.02774687	0.0181966
100	ML	0.02801196	0.008132881
	MOM	0.0302575	0.00823406
	OLS	0.02086761	0.008980337
	WLS	-1	1
	CVM	0.02191171	0.009037173
200	ML	0.02473895	0.003989069
	MOM	0.02688922	0.004123211
	OLS	0.01902113	0.004393245
	WLS	-1	1
	CVM	0.0195463	0.004416063
500	ML	0.0234787	0.001838779
	MOM	0.02486852	0.001912136

Table 3 Bias and MSE for β = 1.5

n	Estimate	Bias	MSE
			$\hat{\beta}$
50	ML	0.1781775	0.07687606
	MOM	0.1962871	0.09591677
	OLS	0.1927908	0.09444386
	WLS	-1.5	2.25
	CVM	0.17634	0.07521178
100	ML	0.1664498	0.04858995
	MOM	0.16478	0.04814568
	OLS	0.1813909	0.05871948
	WLS	-1.5	2.25
	CVM	0.02191171	0.009037173
200	ML	0.1617356	0.03610189
	MOM	0.1784734	0.04441311
	OLS	0.1775813	0.04408753
	WLS	-1.5	2.25
	CVM	0.02113903	0.003554184
500	ML	0.1588387	0.02905477
	MOM	0.1753744	0.03550387
	OLS	0.1750164	0.03537724
	WLS	-1.5	2.25
	CVM	0.0155954	0.00281068

Some observations from Tables 1-3,

1. The estimators of all the parameters are positively biased, except for the WLS estimator.

2. The biases of all the estimators tend to zero for large values of the sample size n.

3. WLS has the largest MSE among the considered four estimators.

4. The CVM estimator has the least bias among all the other estimators.

Real data analysis

We use a well-known real data set to demonstrate the ML model's use in real-life circumstances. To accomplish this, we compare the estimates obtained through the different estimation methods of the ML model. Standard performance validation criteria such the goodness-of-fit testing statistic such as the Kolmogorov Smirnov (K-S), also denoted by D_n, are used to discover the best model.

$$D_n = max_{i=1,2,\dots,n}\left[G_{ML}\left(y_i;\hat{\beta}\right) - \frac{i-1}{n}, \frac{i}{n} - G_{ML}(y_i;\hat{\beta})\right]$$

The p-values for the KS test statistic are also taken into account. The best method of estimation has the smallest KS, as well as the highest p-values. We refer the readers to Kenneth and Anderson[11] for definitions and more insights on these attributes. The data set includes 30 measurements of precipitation in March in Minneapolis, with inch as the unit of measurement. Hinkley[12] has provided this data set.

Table 4 contains the parameter estimates obtained using various estimation methods for the ML distribution in order to complete the comparison of the estimation methods of the ML distribution.

Table 4 $\hat{\beta}$ of different estimation methods for ML distribution with K-S statistic and p-value

Methods	$\hat{\beta}$	K-S	p-value
MOM	0.685563	0.167852	0.366565
OLS	0.613856	0.129532	0.69541
WLS	0.616286	0.130844	0.683348
CVM	0.604734	0.124597	0.740227
MLE	0.6644	0.1567	0.4532

We use the essential metrics to compare the four estimation approaches for the weighted Lindley distribution:

- The K-S test with the highest p-value (which considers the greatest difference between the theoretical and empirical distributions)

Table 4 demonstrates that the CVM approach meets the above-mentioned criteria, namely, the highest p-values for the K–S test. As a result, we can conclude that the CVM estimation approach is the best appropriate among the five estimate methods for the presented data set.

Conclusion

In this study, we compared the estimate of the parameters of the weighted Lindley distribution using four well-known estimation methods: maximum likelihood, method of moments, least-squares, weighted least-squares, and Cram`er-von Mises, utilising extensive simulation trials. In comparison to the maximum likelihood method, simulations reveal that Cram`er-von Mises is a very competing method. This is backed up by the use of a real-world climatic data set.

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Conflicts of interest

The authors declare that there are no conflicts of interest.

References

1. Lindley DV. Fiducial distributions and Bayes theorem. *Journal of the Royal Statistical Society*. 1958;20(1):102–107.

2. Ghitany ME, Atieh B, Nadarajah S. Lindley distribution and its application. *Mathematics and computers in simulation*. 2008;78(4): 493–506.

3. Tomy L. A Retrospective study on Lindley distribution. *Biom Biostat Int J*. 2018;7(2):163–169.

4. Chesneau C, Tomy L, Jose M. A Retrospective Study on Applications of the Lindley Distribution. Journal of Non-linear modelling and Analysis. 2022;4(2):291–309.

5. Chesneau C, Tomy L, Gillariose J. A new modified Lindley distribution with properties and applications. *Journal of Statistics and Management Systems*. 2021;24(7):1383–1403.

6. Swain JJ, Venkatraman S, Wilson JR. Least-squares estimation of distribution functions in Johnson's translation system. *J Stat Comput Simul*. 1988;29(4):271–297.

7. Broyden CG. The convergence of a class of double-rank minimization algorithms 1. general considerations. *IMA J App Math*. 1970;6:76–90.

8. Fletcher R. A new approach to variable metric algorithms. *Comput J*. 1970;13(3):317–322.

9. Goldfarb D. A family of variable-metric methods derived by variational means. *Math Comput*. 1970;24(109):23–26.

10. Shanno DF. Conditioning of quasi-Newton methods for function minimization. *Math Comput*. 1970;24(111):647–656.

11. Kenneth PB, Anderson DR. Model Selection and Multi-Model Inference. *A Practical Information-Theoretic Approach*. 2nd edn. 2002.

12. Hinkley D. On quick choice of power transformations. *Applied Statistics*. 1977;26(1):67–69.