Research Article

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Reliability estimation of type-II generalized log-logistic distribution

Abstract

In this article, a lifetime distribution named as Type-II generalized log-logistic distribution (TGLLD) is considered and its failure rate of products with different shape parameters used to find out ageing criteria. An attempt has been made to derive the statistical and reliability properties of TGLLD. Parameters are evaluated using maximum likelihood estimation and obtained the reliability of the distribution. A simulation study also conducted to know the performance of the estimators. The estimates obtained are validated with the use of live data.

Keywords: type-ii generalized log-logistic distribution, reliability function, hazard rate, reverse hazard rate, moments, maximum likelihood estimation

MATHEMATICS SUBJECT CLASSIFICATION: 62F10; 62F15; 62N05

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Introduction

Over a period of time, statistical literature witnessed the origin of many continuous univariate distributions. However, in the present era, these distributions are extended by introducing the additional parameters in order to cater the requirements from different areas such as lifetime analysis, finance, engineering industries, insurance etc. The present distribution dealt in this article is one such distribution introduced by Rosaiah *et al.*[1] When a distribution is introduced, one may keen to know the behavior for its characterization. The same can be achieved by finding its statistical properties viz., mean, median, mode, variance, quantiles, moments, cumulants, order statistics, ML estimates, confidence intervals etc. Distinguished authors have made their efforts in estimating such properties for different distributions viz. Balakrishnan[2,3] for half logistic and generalized logistic, Mudholkar and Srivastava[4] Mudholkar[5] for exponentiated Weibull, Gupta *et al.*[6] for log-logistic, Nadarajah[7] for exponentiated Gumbel, Nadarajah and Gupta[8] for exponentiated gamma, Abouammoh and Alshingiti[9] for inverted exponential distribution, Rosaiah *et. al.*[10] for odds exponential log logistic Distribution and many more.

Log-logistic distribution (LLD) has proven its importance in quality control, mainly in analyzing the lifetime data. Many authors have made their contribution in developing the various features of this distribution by creating some extensions to the original distribution, Type-II generalized log-logistic distribution (TGLLD) is one such distribution. In this article an effort to derive mathematical properties of TGLLD. The rest of the article is organized as follows. In Section 2, the cumulative distribution function, probability density function, reliability function and hazard function of TGLLD are given. Also, the key properties, moments of TGLLD and i^{th} order statistic are obtained. In Section 3, ML estimators, asymptotic confidence intervals are derived. Fitting reliability data, computation of ML estimates, statistics such as -2logL, AIC and BIC are presented in section 4. Lastly in Section 5, the concluding remarks are given.

Type-II generalized log logistic distribution

Log-logistic distribution (LLD) has proven its importance in quality control. Different authors developed properties and types of acceptance sampling plans for LLD viz., Ashkar and Mahdi.[11] The cumulative distribution function (CDF) of the log-logistic distribution (LLD) is

$$F(t;\sigma,\theta)=\frac{(t/\sigma)^{\lambda}}{\left[1+(t/\sigma)^{\lambda}\right]};t>0,\sigma>0,\lambda>1 \qquad (1)$$

Since the practical pertinence of generalized log-logistic distribution (GLLD) in diverse sectors, various authors have paid their attention in developing some extensions for effective and wide use of log-logistic distribution. One such extension to this distribution named as Type-II generalized log-logistic distribution (TGLLD) introduced by Rosaiah *et al.*,[1] its cumulative distribution function (cdf) is

$$F(t;\sigma,\theta,\lambda)=1-\left[1+(t/\sigma)^{\lambda}\right]^{-\theta};t>0,\sigma>0,\theta>0,\lambda>1 \qquad (2)$$

It may be noted that the distribution given in (2) is defined through the reliability oriented generalization of log-logistic distribution. In short, we call this as the Type-II generalized log-logistic distribution [Type-I generalized (exponentiated) log-logistic distribution is dealt by Rosaiah *et al.*[12] The corresponding probability density function (PDF) is given by

$$f(t;\sigma,\theta,\lambda)=\frac{\lambda\theta}{\sigma}\frac{(t/\sigma)^{\lambda-1}}{\left[1+(t/\sigma)^{\lambda}\right]^{\theta+1}};t>0,\sigma>0,\theta>0,\lambda>1 \qquad (3)$$

where σ is the scale parameter, λ and θ are shape parameters.

Rao *et al.*[13,14] developed the reliability test plans for this distribution. The reliability function and hazard (failure rate) function of type-II generalized log-logistic distribution are respectively given by

$$R(t)=\left[1+(t/\sigma)^{\lambda}\right]^{-\theta} \qquad (4)$$

$$h(t)=\frac{\lambda\theta(t/\sigma)^{\lambda-1}}{\left[1+(t/\sigma)^{\lambda}\right]};t>0,\sigma>0,\theta>0,\lambda>1 \qquad (5)$$

The three-parameter TGLLD will be denoted by TGLLD$\left(\sigma,\theta,\lambda\right)$. If $\theta=1$, then Eq. (3) becomes log-logistic distribution and if $\lambda=1$ then TGLLD becomes reduced to Pareto type-II distribution. Figures 1-4 depicted that the PDF, CDF, reliability function and hazard function curves of TGLLD for various parametric combinations.

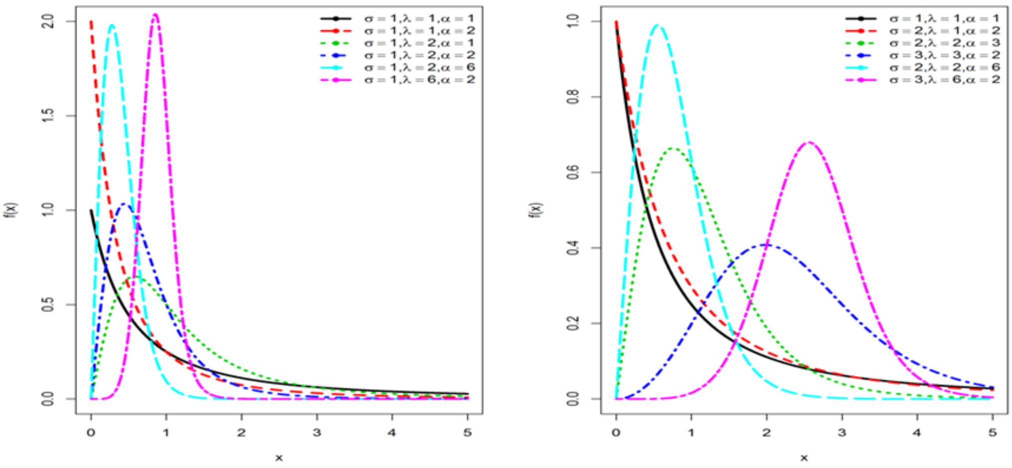

Figure 1 The probability density function of TGLLD for different λ and θ at σ.

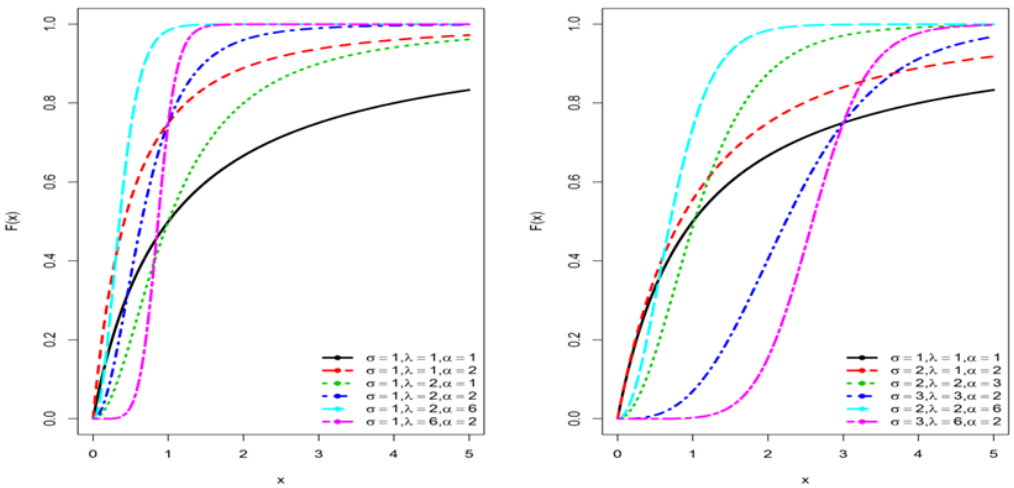

Figure 2 The cumulative density function of TGLLD for different λ and θ at σ.

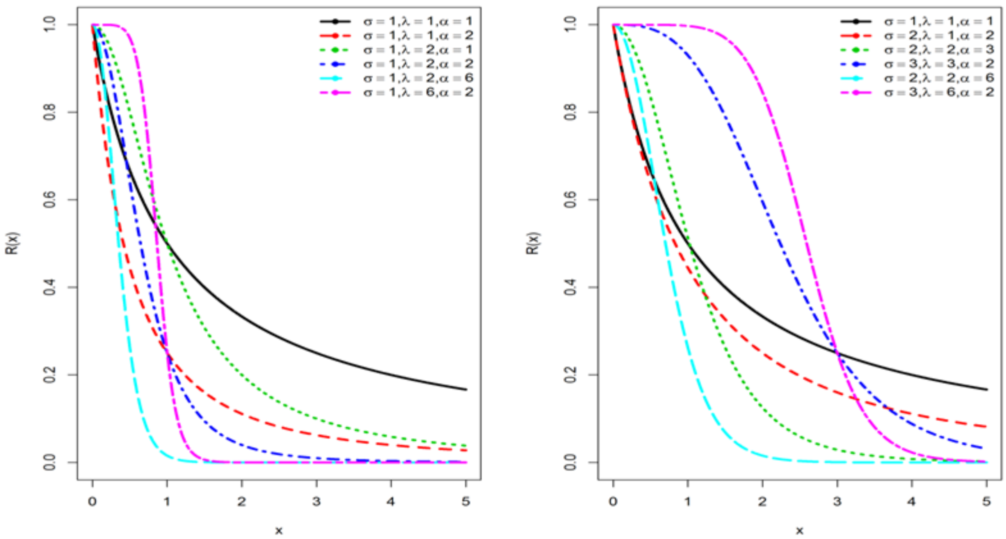

Figure 3 The reliability function of TGLLD for different λ and θ at σ.

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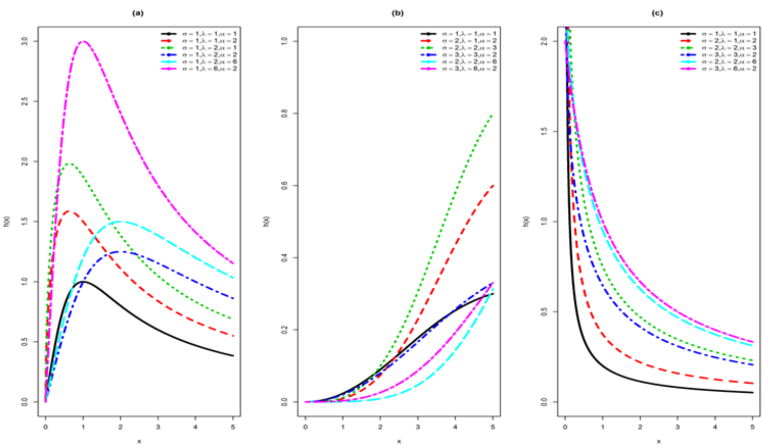

Figure 4 The hazard function of TGLLD for different λ and θ at σ. (a) Upside-down bathtub (b) Increasing (c) Decreasing.

Properties of the TGLLD

Limits of the distribution function

$$F(t;\sigma,\theta,\lambda)=1-\left[1+(t/\sigma)^{\lambda}\right]^{-\theta};t>0,\sigma,\theta>0,\lambda>1$$

$$\lim_{t\to 0}F(t;\sigma,\theta,\lambda)=\lim_{t\to 0}\left\{1-\left[1+(t/\sigma)^{\lambda}\right]^{-\theta}\right\}=0$$

$$\lim_{t\to \infty}F(t;\sigma,\theta,\lambda)=\lim_{t\to \infty}\left\{1-\left[1+(t/\sigma)^{\lambda}\right]^{-\theta}\right\}=1$$

Reverse hazard function of a non-negative random variable is

$$r(t)=\frac{f(t)}{F(t)}=\frac{\lambda\theta}{\sigma}\frac{(t/\sigma)^{\lambda-1}}{\left[1+(t/\sigma)^{\lambda}\right]^{\theta}\left[\left[1+(t/\sigma)^{\lambda}\right]^{\theta}-1\right]} \quad (6)$$

The odd function of a random variable can be derived from

$$O(t)=\frac{F(t)}{h(t)}=\frac{\lambda\theta(t/\sigma)^{\lambda-1}\left[1+(t/\sigma)^{\lambda}\right]^{\theta-1}}{\left[1+(t/\sigma)^{\lambda}\right]^{\theta}-1} \quad (7)$$

Mean of TGLLD is derived as

$$\mu=\int_0^{\infty}t\frac{\lambda\theta}{\sigma}\frac{(t/\sigma)^{\lambda-1}}{\left[1+(t/\sigma)^{\lambda}\right]^{\theta+1}}dt=\frac{\sigma}{\lambda}\frac{\Gamma\left(\frac{1}{\lambda}\right)\Gamma\left(\theta-\frac{1}{\lambda}\right)}{\Gamma(\theta)} \quad (8)$$

Percentile and median

The $100p^{th}$ percentile of a random variable T is denoted by t_p and is defined as

$$t_p=\inf\{t\in\Re:F(t)\geq p\}, \text{ where}$$

$$F(t;\sigma,\theta,\lambda)=1-\left[1+(t/\sigma)^{\lambda}\right]^{-\theta}$$

If $t\in\Re;t_p$ is unique for each $p\in(0,1),F^{-1}(p)$ is an inverse function, then $t_p=F^{-1}(p)$

$$100p^{th}\text{ Percentile, } p=F(t)=1-\left[1+(t/\sigma)^{\lambda}\right]^{-\theta}$$

$$\Rightarrow t_p=\sigma\left[(1-p)^{-1/\theta}-1\right]^{1/\lambda} \quad (9)$$

Quantile function $Q(p)=\sigma\left[(1-p)^{-1/\theta}-1\right]^{1/\lambda}$

Median (M) (50th percentiles) is

$$M=\sigma\left[2^{1/\theta}-1\right]^{1/\lambda} \quad (10)$$

Mode of TGLLD is obtained as the value of x for which

$$\frac{\partial\log f(t)}{\partial t}=0 \text{ and } \frac{\partial^2\log f(t)}{\partial t^2}\leq 0.$$

$$\log f(t)==\log\lambda+\log\theta-\log\sigma+(\lambda-1)\log(t/\sigma)-(\theta+1)\log\left[1+(t/\sigma)^{\lambda}\right]$$

$$\frac{\partial\log f(t)}{\partial t}=(\lambda-1)(1/t)-(\theta+1)\frac{1}{\left[1+(t/\sigma)^{\lambda}\right]}\lambda(t/\sigma)^{\lambda-1}$$

$$\frac{\partial^2\log f(t)}{\partial t^2}=-\left\{\frac{(\lambda-1)}{t^2}+(\theta+1)\left[\frac{\lambda^2}{\sigma}\left(\frac{t}{\sigma}\right)^{\lambda-1}\frac{-\lambda(t/\sigma)^{\lambda}}{\left(1+(t/\sigma)^{\lambda}\right)^2}\right]\right\}\leq 0$$

Hence mode is the solution of the non-linear equation

$$\frac{(\lambda-1)}{t}-\lambda(\theta+1)\frac{(t/\sigma)^{\lambda-1}}{\left[1+(t/\sigma)^{\lambda}\right]}=0$$

Moments of TGLLD

Moments are the useful tools which can be used to derive the key features of the distribution. Here we tried to derive r^{th} moment of the random variable T, where $T\sim TGLLD(\theta,\lambda,\sigma)$.

The r^{th} moment of T is denoted by $\mu_r^{'}$ and is defined as

$$\mu_r^{'}=E(t^r)=\int_0^{\infty}t^r\frac{\lambda\theta}{\sigma}\frac{(t/\sigma)^{\lambda-1}}{\left[1+(t/\sigma)^{\lambda}\right]^{\theta+1}}dt=\frac{r}{\lambda}\sigma^r\frac{\Gamma\left(\frac{r}{\lambda}\right)\Gamma\left(\theta-\frac{r}{\lambda}\right)}{\Gamma(\theta)} \quad (11)$$

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Variance

$$=\mu_2=\frac{\sigma^2}{\lambda\Gamma(\theta)}\left(\frac{2\Gamma\left(\frac{2}{\lambda}\right)\Gamma\left(\theta-\frac{2}{\lambda}\right)\Gamma(\theta)-\frac{1}{\lambda}\left(\Gamma\left(\frac{1}{\lambda}\right)\Gamma\left(\theta-\frac{1}{\lambda}\right)\right)^2}{\Gamma(\theta)}\right)$$

$$\mu_3=\frac{\sigma^3}{\lambda\Gamma(\theta)}\left(3\Gamma\left(\frac{3}{\lambda}\right)\Gamma\left(\theta-\frac{3}{\lambda}\right)-\frac{\frac{6}{\lambda}\Gamma\left(\frac{1}{\lambda}\right)\Gamma\left(\frac{2}{\lambda}\right)\Gamma\left(\theta-\frac{1}{\lambda}\right)\Gamma\left(\theta-\frac{2}{\lambda}\right)}{\Gamma(\theta)}+\frac{\frac{2}{\lambda^2}\left(\Gamma\left(\frac{1}{\lambda}\right)\Gamma\left(\theta-\frac{1}{\lambda}\right)\right)^3}{\left(\Gamma(\theta)\right)^2}\right)$$

$$\mu_4=\frac{\sigma^4}{\lambda\Gamma(\theta)}\left[\begin{array}{l}4\Gamma\left(\frac{4}{\lambda}\right)\Gamma\left(\theta-\frac{4}{\lambda}\right)-\frac{\frac{12}{\lambda}\Gamma\left(\frac{1}{\lambda}\right)\Gamma\left(\frac{3}{\lambda}\right)\Gamma\left(\theta-\frac{1}{\lambda}\right)\Gamma\left(\theta-\frac{3}{\lambda}\right)}{\Gamma(\theta)}\\+\frac{\frac{12}{\lambda^2}\Gamma\left(\frac{2}{\lambda}\right)\Gamma\left(\theta-\frac{2}{\lambda}\right)\left[\Gamma\left(\frac{1}{\lambda}\right)\Gamma\left(\theta-\frac{1}{\lambda}\right)\right]^2}{\left(\Gamma(\theta)\right)^2}-\frac{\frac{3}{\lambda^3}\left[\Gamma\left(\frac{1}{\lambda}\right)\Gamma\left(\theta-\frac{1}{\lambda}\right)\right]^4}{\left(\Gamma(\theta)\right)^3}\end{array}\right]$$

Mean, median, skewness and kurtosis of TGLLD for various combinations of λ, θ and σ are given in Table 1.

Table 1 Mean, Median, Skewness and Kurtosis of TGLLD for various combinations of $\lambda,\theta\, and\, \sigma$

λ	θ	σ	**Mean**	**Median**	**Skewness**	**Kurtosis**
1.5	1.5	1	1.1498	0.7014	4.0543	3.0912
2.5	2.5	1	0.6985	0.6336	2.5365	9.9632
3	3	1	0.6718	0.6382	0.8453	5.1318
3.5	3.5	1	0.6657	0.648	0.3195	3.8556
4	4	1	0.6682	0.6595	0.112	3.3635
4.5	4.5	1	0.6744	0.6714	0.0283	3.1535
5	5	1	0.6824	0.6831	0.0016	3.07
5.5	5.5	1	0.6911	0.6942	0.0039	3.0504
6	6	1	0.6999	0.7047	0.0215	3.0653
2	3	2	1.1781	1.0196	3.6429	12.4635
3	2	3	2.4184	2.2363	2.5253	10.8095
2	6	2	0.7731	0.6999	1.1978	5.1176
6	2	3	2.618	2.5902	0.1888	4.1057

Moment Generating function (MGF) is given by

$$M_t(z)=E\left(e^{tz}\right)=\sum_{r=0}^{\infty}\frac{z^r}{r!}\sigma^{r-1}\frac{\left(\frac{r}{\lambda}\right)!\left(\theta-\frac{r}{\lambda}-1\right)!}{(\theta-1)!} \quad (12)$$

Characteristic Function is given by

$$\phi_t(z)=E\left(e^{izt}\right)=\sum_{r=0}^{\infty}\frac{(iz)^r}{r!}\sigma^{r-1}\frac{\left(\frac{r}{\lambda}\right)!\left(\theta-\frac{r}{\lambda}-1\right)!}{(\theta-1)!} \quad (13)$$

Cumulative Generating function is defined by

$$K_t(z)=\ln\left(M_t(z)\right)=K_t(z)=\ln\left(\sum_{r=0}^{\infty}\frac{z^r}{r!}\sigma^{r-1}\frac{\left(\frac{r}{\lambda}\right)!\left(\theta-\frac{r}{\lambda}-1\right)!}{(\theta-1)!}\right) \quad (14)$$

Order statistics of TGLLD

Let $T_{1:n}\le T_{2:n}\le\dots\dots\dots\le T_{n:n}$ denotes the order statistics obtained from a random sample of size *n* drawn from TGLLD (θ,λ,σ). The probability density function of i^{th} order statistic is given by

$$f_{i:n}(t;\theta,\lambda,\sigma)=\frac{1}{\beta(i,n-i+1)}\left(F(t)\right)^{i-1}\left(1-F(t)\right)^{n-i}f(t) \quad (15)$$

Since $0<F(t;\theta,\lambda,\sigma)<1$ for $t>0$, then using binomial expansion

$$\left(F(t)\right)^{i-1}=\sum_{j=0}^{i-1}\binom{i-1}{j}(-1)^j\left(1-F(t)\right)^j \quad (16)$$

Then

$$f_{i:n}(t;\theta,\lambda,\sigma)=\frac{1}{\beta(i,n-i+1)}f(t)\sum_{j=0}^{i-1}\binom{i-1}{j}(-1)^j\left(1-F(t)\right)^{n+j-i}$$

$$\left(1-F(t)\right)^{n+j-i}=\left[1+(t/\sigma)^\lambda\right]^{-\theta(n+j-i)} \quad (17)$$

Now take

$$f(t)\left[1-f(t)\right]^{n+j-i}=\frac{\lambda\theta(t/\sigma)^{\lambda-1}}{\sigma\left[1+(t/\sigma)^\lambda\right]^{\theta+1}}\left[1+(t/\sigma)^\lambda\right]^{-\theta(n+j-i)}$$

$$=\frac{\lambda\theta}{\sigma}(t/\sigma)^{\lambda-1}\left[1+(t/\sigma)^\lambda\right]^{-\theta(n+j-i+1)-1}$$

Hence

$$f_{i:n}(t;\lambda,\theta,\sigma)=\sum_{j=0}^{i-1}(-1)^j\frac{n!}{j!(n-i)!(i-j-1)!}f\left(t;\lambda,\theta,\sigma,(n+j-i+1)\right) \quad (18)$$

The distribution function of i^{th} order statistic $T_{(i)}$ is

$F_{i:n}\left(t;\sigma,\theta,\lambda\right)=\sum_{j=i}^{n}\binom{n}{i}F^{j}\left(t\right)\left[1-F\left(t\right)\right]^{n-j}$, using (16), it can be expressed as

$$F_{i:n}\left(t;\lambda,\theta,\sigma\right)=\sum_{j=i}^{n}\sum_{k=0}^{j}\binom{n}{i}\binom{j}{k}(-1)^{k}\left(1+\left(t/\sigma\right)^{\lambda}\right)^{-\theta(n+k-j)} \quad (19)$$

Distribution function of first order statistic $T_{(1)}$ is

$$F_{1}\left(t\right)=1-\left(1-F\left(t\right)\right)^{n}=1-\left(1+\left(t/\sigma\right)^{\lambda}\right)^{-n\theta} \quad (20)$$

Distribution function of n^{th} order statistic $T_{(n)}$ is

$$F_{n}\left(t\right)=\left(F\left(t\right)\right)^{n}=\left(1-\left(1+\left(t/\sigma\right)^{\lambda}\right)^{-\theta}\right)^{n} \quad (21)$$

Parameter estimation and inference

For estimating the parameters of $TGLLD(\sigma,\theta,\lambda)$, we considered two known methods viz., maximum likelihood method of estimation and least square method. It is observed that the estimates obtained from both methods for the unknown parameters cannot be expressed in closed form and hence the estimates are obtained using simulation study.

Maximum likelihood estimators (MLEs)

Let $t_1,t_2.....t_n$ be a random sample of size n drawn from TGLLD $\left(T;\theta,\lambda,\sigma\right)$, then likelihood function L of the sample is

$$L=\prod_{i=1}^{n}f\left(t_i;\theta,\lambda,\sigma\right)=\prod_{i=1}^{n}\frac{\lambda\theta}{\sigma}\frac{\left(t_i/\sigma\right)^{\lambda-1}}{\left[1+\left(t_i/\sigma\right)^{\lambda}\right]^{\theta+1}}$$

The log-likelihood function is

$$\log L=n\log\lambda+n\log\theta-n\log\sigma+\left(\lambda-1\right)\sum_{i=1}^{n}\log\left(t_i/\sigma\right)-\left(\theta+1\right)\sum_{i=1}^{n}\log\left[1+\left(t_i/\sigma\right)^{\lambda}\right] \quad (22)$$

The MLE's of θ,λ and σ are obtained as

$$\frac{\partial\log L}{\partial\sigma}=0\Rightarrow\frac{-n\lambda}{\sigma}+\frac{\lambda\left(\theta+1\right)}{\sigma}\sum_{i=1}^{n}\frac{\left(t_i/\sigma\right)^{\lambda}}{\left[1+\left(t_i/\sigma\right)^{\lambda}\right]}=0 \quad (23)$$

$$\frac{\partial\log L}{\partial\lambda}=0\Rightarrow\frac{n}{\lambda}+\sum_{i=1}^{n}\log\left(t_i/\sigma\right)-\left(\theta+1\right)\sum_{i=1}^{n}\frac{\left(t_i/\sigma\right)^{\lambda}\log\left(t_i/\sigma\right)}{\left[1+\left(t_i/\sigma\right)^{\lambda}\right]}=0 \quad (24)$$

$$\frac{\partial\log L}{\partial\theta}=\frac{n}{\theta}-\sum_{i=1}^{n}\log\left[1+\left(t_i/\sigma\right)^{\lambda}\right]=0\Rightarrow\hat{\theta}=\frac{n}{\sum_{i=1}^{n}\log\left[1+\left(t_i/\sigma\right)^{\lambda}\right]} \quad (25)$$

Using Eq. (25) in Eqs. (23) and (24) we get two equations in terms of σ and λ, these equations cannot be solved analytically, so they need to be solved numerically. Iterative techniques can be applied for obtaining the estimators of the parameters. Let $\hat{\sigma},\hat{\lambda}$ and $\hat{\theta}$ are ML estimates of the parameters σ,λ and θ respectively. Using invariance property of the MLE, the MLE of reliability function can be obtained by

$$\hat{R}(t)=\left[1+\left(t/\hat{\sigma}\right)^{\hat{\lambda}}\right]^{-\hat{\theta}} \quad (26)$$

Asymptotic confidence interval

Here, an attempt has been made to derive the asymptotic confidence intervals of the unknown parameters θ,λ and σ. Using large sample approach and assume that the MLE's of $(\hat{\theta},\hat{\lambda}$ and $\hat{\sigma})$ are approximately multivariate normal with mean $\left(\theta,\lambda,\sigma\right)$ and Variance-covariance matrix I^{-1}, where I^{-1} is observed information matrix which is defined as

$$I^{-1}=\begin{bmatrix}\frac{\partial^2\log L}{\partial\theta^2} & \frac{\partial^2\log L}{\partial\lambda\partial\theta} & \frac{\partial^2\log L}{\partial\sigma\partial\theta}\\ \frac{\partial^2\log L}{\partial\theta\partial\lambda} & \frac{\partial^2\log L}{\partial\lambda^2} & \frac{\partial^2\log L}{\partial\sigma\partial\lambda}\\ \frac{\partial^2\log L}{\partial\theta\partial\sigma} & \frac{\partial^2\log L}{\partial\lambda\partial\sigma} & \frac{\partial^2\log L}{\partial\sigma^2}\end{bmatrix}=$$

$$\begin{bmatrix}var\left(\hat{\theta}\right) & cov\left(\hat{\lambda},\hat{\theta}\right) & cov\left(\hat{\sigma},\hat{\theta}\right)\\ cov\left(\hat{\theta},\hat{\lambda}\right) & var\left(\hat{\lambda}\right) & cov\left(\hat{\sigma},\hat{\lambda}\right)\\ cov\left(\hat{\theta},\hat{\sigma}\right) & cov\left(\hat{\lambda},\hat{\sigma}\right) & var\left(\hat{\sigma}\right)\end{bmatrix} \quad (27)$$

Now, the second order partial derivatives of the parameters given in I^{-1} are

$$\frac{\partial^2\log L}{\partial\theta^2}=\frac{\partial}{\partial\theta}\left[\frac{\partial\log L}{\partial\theta}\right]=-\frac{n}{\theta^2} \quad (28)$$

$$\frac{\partial^2\log L}{\partial\lambda\partial\theta}=\frac{\partial}{\partial\lambda}\left[\frac{\partial\log L}{\partial\theta}\right]=-\sum_{i=1}^{n}\frac{\left(t_i/\sigma\right)^{\lambda}\log\left(t_i/\sigma\right)}{\left[1+\left(t_i/\sigma\right)^{\lambda}\right]} \quad (29)$$

$$\frac{\partial^2\log L}{\partial\sigma\partial\theta}=\frac{\partial}{\partial\sigma}\left[\frac{\partial\log L}{\partial\theta}\right]=-\sum_{i=1}^{n}\frac{\frac{\lambda}{\sigma}\left(t_i/\sigma\right)^{\lambda}}{\left[1+\left(t_i/\sigma\right)^{\lambda}\right]} \quad (30)$$

$$\frac{\partial^2\log L}{\partial\theta\partial\lambda}=\frac{\partial}{\partial\theta}\left[\frac{\partial\log L}{\partial\lambda}\right]=-\sum_{i=1}^{n}\frac{\left(t_i/\sigma\right)^{\lambda}\log\left(t_i/\sigma\right)}{\left[1+\left(t_i/\sigma\right)^{\lambda}\right]} \quad (31)$$

$$\frac{\partial^2\log L}{\partial\lambda^2}=\frac{\partial}{\partial\lambda}\left[\frac{\partial\log L}{\partial\lambda}\right]=\left[-\frac{n}{\lambda^2}-\left(\theta+1\right)\sum_{i=1}^{n}\left\{\frac{\left(t_i/\sigma\right)^{\lambda}\left[1-\left(t_i/\sigma\right)^{\lambda}\right]\left[\log\left(t_i/\sigma\right)\right]^2}{\left[1+\left(t_i/\sigma\right)^{\lambda}\right]^2}\right\}\right] \quad (32)$$

$$\frac{\partial^2\log L}{\partial\sigma\partial\lambda}=\frac{\partial}{\partial\sigma}\left[\frac{\partial\log L}{\partial\lambda}\right]-\frac{n}{\sigma}-\left(\theta+1\right)\sum_{i=1}^{n}\left\{\frac{\frac{t_i^{\lambda}}{\sigma^{(\lambda+1)}}\left[\lambda\log\left(t_i/\sigma\right)-\frac{1}{t_i}-\frac{1}{t_i}\left(t_i/\sigma\right)^{\lambda}\right]}{\left[1+\left(t_i/\sigma\right)^{\lambda}\right]^2}\right\} \quad (33)$$

$$\frac{\partial^2\log L}{\partial\theta\partial\sigma}=\frac{\partial}{\partial\theta}\left[\frac{\partial\log L}{\partial\sigma}\right]=\frac{\lambda}{\sigma^{(\lambda+1)}}\sum_{i=1}^{n}\frac{t_i^{\lambda}}{\left[1+\left(t_i/\sigma\right)^{\lambda}\right]} \quad (33)$$

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$$\frac{\partial^2 \log L}{\partial\lambda\partial\sigma}=\frac{\partial}{\partial\lambda}\left[\frac{\partial \log L}{\partial\sigma}\right]=-\frac{n}{\sigma}+\frac{1}{\sigma^{(\lambda+1)}}\sum_{i=1}^{n}\frac{t_i^{\lambda}}{\left[1+(t_i/\sigma)^{\lambda}\right]}\left[\frac{(1-\lambda\log\sigma)+\lambda\log t_i\left[1-(t_i/\sigma)^{\lambda}\right]}{\left[1+(t_i/\sigma)^{\lambda}\right]}\right] \quad (34)$$

$$\frac{\partial^2 \log L}{\partial\sigma^2}=\frac{\partial}{\partial\sigma}\left[\frac{\partial \log L}{\partial\sigma}\right]=\frac{n\lambda}{\sigma^2}+\frac{t_i^{\lambda}}{\sigma^{(\lambda+1)}\left[1+(t_i/\sigma)^{\lambda}\right]}\left\{\frac{-(\lambda+1)}{\sigma}+\frac{\lambda\left(t_i^{\lambda}\right)}{\sigma^{(\lambda+1)}\left[1+(t_i/\sigma)^{\lambda}\right]}\right\} \quad (35)$$

The Asymptotic $(1-\alpha)$ 100% confidence interval of $(\hat{\theta},\hat{\lambda}\,\text{and}\,\hat{\sigma})$ are $\hat{\theta}\pm Z_{\frac{\alpha}{2}}\sqrt{Var(\hat{\theta})}$, $\hat{\lambda}\pm Z_{\frac{\alpha}{2}}\sqrt{Var(\hat{\lambda})}$ and $\hat{\sigma}\pm Z_{\frac{\alpha}{2}}\sqrt{Var(\hat{\sigma})}$ respectively, where $Z_{\frac{\alpha}{2}}$ is the upper $\left(\frac{\alpha}{2}\right)^{th}$ percentile of the standard normal distribution.

To obtain the asymptotic confidence interval for $\hat{R}(t)$, we proceed as follows.

The asymptotic variance of the MLEs are given by

$$V(\hat{\lambda})=\left[E\left(-\frac{\partial^2 L}{\partial\lambda^2}\right)\right]^{-1}=E\left[\frac{n}{\lambda^2}+(\theta+1)\sum_{i=1}^{n}\left\{\frac{(t_i/\sigma)^{\lambda}\left[1-(t_i/\sigma)^{\lambda}\right]\left[\log(t_i/\sigma)\right]^2}{\left[1+(t_i/\sigma)^{\lambda}\right]^2}\right\}\right]^{-1} \quad (36)$$

$$V(\hat{\theta})=\left[E\left(-\frac{\partial^2 L}{\partial\theta^2}\right)\right]^{-1}=E\left[\frac{n}{\theta^2}\right]^{-1}=\frac{\theta^2}{n} \quad (37)$$

$$V(\hat{\sigma})=\left[E\left(-\frac{\partial^2 L}{\partial\sigma^2}\right)\right]^{-1}=E\left[-\frac{n\lambda}{\sigma^2}-\frac{t_i^{\lambda}}{\sigma^{(\lambda+1)}\left[1+(t_i/\sigma)^{\lambda}\right]}\left\{\frac{-(\lambda+1)}{\sigma}+\frac{\lambda\left(t_i^{\lambda}\right)}{\sigma^{(\lambda+1)}\left[1+(t_i/\sigma)^{\lambda}\right]}\right\}\right]^{-1} \quad (38)$$

Now $$\frac{\partial R(t)}{\partial\lambda}=-\theta\left(1+(t/\sigma)^{\lambda}\right)^{-1-\theta}(t/\sigma)^{\lambda}\log[t/\sigma] \quad (39)$$

$$\frac{\partial R(t)}{\partial\theta}=-\left(1+(t/\sigma)^{\lambda}\right)^{-\theta}\log\left[1+(t/\sigma)^{\lambda}\right] \quad (40)$$

$$\frac{\partial R(t)}{\partial\sigma}=\frac{t\lambda\theta\left(1+(t/\sigma)^{\lambda}\right)^{-1-\theta}(t/\sigma)^{-1+\lambda}}{\sigma^2} \quad (41)$$

The asymptotic variance (A V) of an estimate of $\hat{R}(t)$ which is a function of parameter estimates (say) $\hat{\lambda},\hat{\theta}\,and\,\hat{\sigma}$ is given by Rao[15]

$$AV=V(\hat{\lambda})\left(\frac{\partial R(t)}{\partial\lambda}\right)^2+V(\hat{\theta})\left(\frac{\partial R(t)}{\partial\theta}\right)^2+V(\hat{\sigma})\left(\frac{\partial R(t)}{\partial\sigma}\right)^2 \quad (42)$$

Thus the asymptotic variance of $\hat{R}(t)$ can be obtained after substituting equations (36) to (41) in equation (42), which will be analytically solved thereafter.

Samples are generated at varying sizes n = 20, 30, 40, 50 and 100 for different assumed values of shape parameters λ and θ when $\sigma = 30$. MLEs, mean square error (*MSE*) and bias of λ when θ is assumed as 1.5, 2 and 2.5 are presented in Table 2. Similarly MLEs, *MSE* and bias of θ when λ is assumed as 1.5, 2 and 2.5 are presented in Table 3. Table 4 represents the values of MLEs, mean square error (*MSE*) and bias of reliability function (*R*) for the parametric combinations of (λ, θ) = (1.5, 1.5), (1.5, 2), (2, 1.5), (2, 2), (2, 2.5), (2.5, 2) and (2.5, 2.5).

It can be seen from Tables 2 and 3, the bias and *MSE* of both λ and θ decrease as the sample size increases. This reflects the asymptotic consistency property of the MLEs. From Table 2, when the parameter λ increases the concerned bias and *MSE* are also observed as increases and the rate of this increment is high at smaller sample sizes especially till sample size *n* reaches 50. Similar pattern noticed for the parameter θ from Table 3.

Table 2 MLE of λ when σ = 30 for TGLLD

	n = 20	***n = 30***	***n = 40***	***n = 50***	***n = 100***
(θ = 2)					
λ	1.5000	1.5000	1.5000	1.5000	1.5000
$\hat{\lambda}$	1.5968	1.5625	1.5471	1.5372	1.5182
Bias	0.0968	0.0625	0.0471	0.0372	0.0182
MSE	0.0991	0.0576	0.0410	0.0319	0.0148
(θ =1.5)					
λ	2.0000	2.0000	2.0000	2.0000	2.0000
$\hat{\lambda}$	2.1374	2.0885	2.0668	2.0527	2.0257
Bias	0.1374	0.0885	0.0668	0.0527	0.0257
MSE	0.2027	0.1166	0.0828	0.0641	0.0297
(θ = 2)					
λ	2.0000	2.0000	2.0000	2.0000	2.0000
$\hat{\lambda}$	2.1290	2.0833	2.0628	2.0496	2.0242
Bias	0.1290	0.0833	0.0628	0.0496	0.0242
MSE	0.1761	0.1025	0.0730	0.0566	0.0263

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Table Continued

	n = 20	*n = 30*	*n = 40*	*n = 50*	*n = 100*
(θ =2.5)					
λ	2.0000	2.0000	2.0000	2.0000	2.0000
$\hat{\lambda}$	2.1264	2.0816	2.0614	2.0485	2.0237
Bias	0.1264	0.0816	0.0614	0.0485	0.0237
MSE	0.1648	0.0959	0.0682	0.0530	0.0246
(θ =2.5)					
λ	2.5000	2.5000	2.5000	2.5000	2.5000
$\hat{\lambda}$	2.6580	2.6019	2.5768	2.5606	2.5296
Bias	0.1580	0.1019	0.0768	0.0606	0.0296
MSE	0.2574	0.1499	0.1066	0.0828	0.0385

Table 3 MLE of θ when σ = 30 for TGLLD

	n = 20	*n = 30*	*n = 40*	*n = 50*	*n =100*
(λ = 1.5)					
θ	1.5000	1.5000	1.5000	1.5000	1.5000
$\hat{\theta}$	1.5771	1.5487	1.5347	1.5279	1.5133
Bias	0.0771	0.0487	0.0347	0.0279	0.0133
MSE	0.1664	0.0962	0.0679	0.0524	0.0246
(λ = 1.5)					
θ	2.0000	2.0000	2.0000	2.0000	2.0000
$\hat{\theta}$	2.1380	2.0863	2.0619	2.0494	2.0235
Bias	0.1380	0.0863	0.0619	0.0494	0.0235
MSE	0.3282	0.1794	0.1236	0.0940	0.0433
(λ = 2)					
θ	2.0000	2.0000	2.0000	2.0000	2.0000
$\hat{\theta}$	2.1380	2.0863	2.0619	2.0494	2.0235
Bias	0.1380	0.0863	0.0619	0.0494	0.0235
MSE	0.3282	0.1794	0.1236	0.0940	0.0433
(λ = 2.5)					
θ	2.0000	2.0000	2.0000	2.0000	2.0000
$\hat{\theta}$	2.1380	2.0863	2.0619	2.0494	2.0235
Bias	0.1380	0.0863	0.0619	0.0494	0.0235
MSE	0.3282	0.1794	0.1236	0.0940	0.0433
(λ = 2)					
θ	2.5000	2.5000	2.5000	2.5000	2.5000
$\hat{\theta}$	2.7155	2.6334	2.5959	2.5761	2.5361
Bias	0.2155	0.1334	0.0959	0.0761	0.0361
MSE	0.6020	0.3118	0.2096	0.1575	0.0708

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Table Continued

	n = 20	n = 30	n = 40	n = 50	n =100
(λ = 2.5)					
θ	2.5000	2.5000	2.5000	2.5000	2.5000
$\hat{\theta}$	2.7155	2.6334	2.5959	2.5761	2.5361
Bias	0.2155	0.1334	0.0959	0.0761	0.0361
MSE	0.6020	0.3118	0.2096	0.1575	0.0708

Table 4 MLE of *R* when σ = 30 for TGLLD

	n = 20	n = 30	n = 40	n = 50	n =100
(λ = 1.5, θ = 1.5)					
$\hat{R}$	0.9977	0.9977	0.9977	0.9977	0.9977
Bias	-0.0009	-0.0006	-0.0004	-0.0004	-0.0002
MSE	0.0005	0.0004	0.0004	0.0003	0.0002
(λ = 1.5, θ = 2)					
$\hat{R}$	0.9978	0.9978	0.9978	0.9978	0.9978
Bias	-0.0008	-0.0005	-0.0004	-0.0003	-0.0002
MSE	0.0008	0.0007	0.0007	0.0004	0.0002
(λ = 2, θ = 1.5)					
$\hat{R}$	0.9996	0.9996	0.9996	0.9996	0.9996
Bias	0.0003	-0.0002	-0.0002	-0.0001	-0.0001
MSE	0.0008	0.0006	0.0005	0.0003	0.0002
(λ = 2, θ = 2)					
$\hat{R}$	0.9996	0.9996	0.9996	0.9996	0.9996
Bias	-0.0003	-0.0002	-0.0001	-0.0001	-0.0001
MSE	0.0009	0.0007	0.0006	0.0005	0.0003
(λ = 2, θ = 2.5)					
$\hat{R}$	0.9996	0.9996	0.9996	0.9996	0.9996
Bias	-0.0003	-0.0002	-0.0001	-0.0001	0.0000
MSE	0.0009	0.0008	0.0007	0.0006	0.0003
(λ = 2.5, θ = 2)					
$\hat{R}$	0.9999	0.9999	0.9999	0.9999	0.9999
Bias	-0.0001	-0.0001	0.0000	0.0000	0.0000
MSE	0.0009	0.0008	0.0006	0.0005	0.0003
(λ = 2.5, θ = 2.5)					
$\hat{R}$	0.9999	0.9999	0.9999	0.9999	0.9999
Bias	-0.0001	-0.0001	0.0001	0.0001	0.0002
MSE	0.0009	0.0008	0.0007	0.0006	0.0004

From Table 4, it is observed that when both parameters are increases, the $\hat{R}(t)$ resulted more accurate values and the absolute bias decreases as the sample size increases. Also mse of $\hat{R}(t)$ observed as very small values. From Table 5 we noticed that average length of 95% confidence interval is decreases as sample size and parametric values are increases furthermore the coverage probabilities for λ and θ are very close to 95% that we expected. The results shows that estimation and confidence limits of shape parameter λ and θ are performed well using maximum likelihood estimation.

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Table 5 Average length and coverage probability of λ and θ when σ =30

n	λ	θ	Average length		Coverage Probability	
			λ	θ	λ	θ
20	2.0	1.5	1.5510	1.4140	0.9410	0.9373
30	2.0	1.5	1.2360	1.1310	0.9455	0.9423
40	2.0	1.5	1.0590	0.9701	0.9461	0.9433
50	2.0	1.5	0.9403	0.8634	0.9471	0.9449
100	2.0	1.5	0.6559	0.6043	0.9474	0.9484
20	1.5	1.5	1.1640	1.4140	0.9410	0.9373
30	1.5	1.5	0.9270	1.1310	0.9456	0.9423
40	1.5	1.5	0.7942	0.9701	0.9461	0.9433
50	1.5	1.5	0.7052	0.8634	0.9470	0.9449
100	1.5	1.5	0.4919	0.6043	0.9474	0.9485
20	2.0	2.0	1.4550	1.9070	0.9406	0.9375
30	2.0	2.0	1.1620	1.5080	0.9453	0.9422
40	2.0	2.0	0.9965	1.2870	0.9463	0.9434
50	2.0	2.0	0.8855	1.1420	0.9471	0.9445
100	2.0	2.0	0.6185	0.7951	0.9479	0.9476
20	2.5	2.0	1.8180	1.9070	0.9406	0.9375
30	2.5	2.0	1.4520	1.5080	0.9453	0.9422
40	2.5	2.0	1.2460	1.2870	0.9463	0.9434
50	2.5	2.0	1.1070	1.1420	0.9471	0.9445
100	2.5	2.0	0.7732	0.7951	0.9479	0.9476
20	2.5	2.5	1.7570	2.4850	0.9403	0.9367
30	2.5	2.5	1.4040	1.9410	0.9449	0.9418
40	2.5	2.5	1.2040	1.6480	0.9461	0.9440
50	2.5	2.5	1.0700	1.4580	0.9470	0.9432
100	2.5	2.5	0.7476	1.0100	0.9481	0.9482
20	1.5	2.0	1.0910	1.9070	0.9406	0.9375
30	1.5	2.0	0.8715	1.5080	0.9453	0.9422
40	1.5	2.0	0.7474	1.2870	0.9463	0.9434
50	1.5	2.0	0.6641	1.1420	0.9471	0.9445
100	1.5	2.0	0.4639	0.7951	0.9479	0.9476
20	2.0	2.5	1.4060	2.4850	0.9403	0.9367
30	2.0	2.5	1.1230	1.9410	0.9449	0.9418
40	2.0	2.5	0.9633	1.6480	0.9461	0.9440
50	2.0	2.5	0.8560	1.4580	0.9471	0.9432
100	2.0	2.5	0.5980	1.0100	0.9482	0.9482

Fitting reliability data

In this section, we considered a real data set to compare the ML estimates of TGLLD given in (3), with the log-logistic (LL), McDonald log-logistic (McLL) studied by Tahir *et al.,*[16] beta log-logistic (BeLL) by Lemonte,[17] Kumaraswamy log-logistic (KwLL) discussed by de Santana *et al*.,[18] Marshal-Olkin log-logistic (MoLL) developed by Gui.[19] The corresponding densities functions of the above distributions are reproduced below.

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LL: $$f(t)=\left(\frac{\lambda}{\sigma}\right)\frac{(t/\sigma)^{\lambda-1}}{\left[1+(t/\sigma)^{\lambda}\right]^{2}},t>0,\sigma>0,\lambda>1$$

McLL:

$$f(t)=\frac{c}{B\left(ac^{-1},b\right)}(\lambda/\sigma)(t/\sigma)^{a\lambda-1}\left[1+(t/\sigma)^{\lambda}\right]^{-(a+1)}\left[1-\left\{1-\left[1+(t/\sigma)^{\lambda}\right]^{-1}\right\}^{c}\right]^{b-1},t>0$$

BeLL: $$f(t)=\frac{1}{B(a,b)}(\lambda/\sigma)(t/\sigma)^{a\lambda-1}\left[1+(t/\sigma)^{\lambda}\right]^{-(a+b)}$$

KwLL:

$$f(t)=ab(\lambda/\sigma)(t/\sigma)^{a\lambda-1}\left[1+(t/\sigma)^{\lambda}\right]^{-(a+1)}\left[1-\left\{1-\frac{1}{1+(t/\sigma)^{\lambda}}\right\}^{a}\right]^{b-1}$$

MoLL: $$f(t)=\frac{(\alpha\lambda/\sigma)(t/\sigma)^{\lambda-1}}{\left[\alpha+(t/\sigma)^{\lambda}\right]^{2}},0\le t\le\infty;\alpha,\sigma>0;\lambda>1$$

We will be considering the single fibers of 20mm in gauge length, with sample sizes n = 69 respectively. For easy reference, this data is presented below.

1.312, 1.314, 1.479, 1.552, 1.700, 1.803, 1.861, 1.865, 1.944, 1.958, 1.966, 1.997, 2.006, 2.021, 2.027, 2.055, 2.063, 2.098, 2.140, 2.179, 2.224, 2.240, 2.253, 2.270, 2.272, 2.274, 2.301, 2.301, 2.359, 2.382, 2.382, 2.426, 2.434, 2.435, 2.478, 2.490, 2.511, 2.514, 2.535, 2.554, 2.566, 2.570, 2.586, 2.629, 2.633, 2.642, 2.648, 2.684, 2.697, 2.726, 2.770, 2.773, 2.800, 2.809, 2.818, 2.821, 2.848, 2.880, 2.954, 3.012, 3.067, 3.084, 3.090, 3.096, 3.128, 3.233, 3.433, 3.585, 3.585.

Table 6 describes the fitted data of MLEs of the unknown parameters of different log-logistic distributions, it is observed that TGLLD throws better results and found more suitable in analyzing the data. Plots of empirical and fitted PDFs drawn for the observed data are shown in Figure 5.

Table 6 The MLEs (SEs in parentheses) and statistics of the distribution parameters

Model	Estimates					-2logL	AIC	BIC
TGLLD (λ, σ, θ)	**6.509 (1.0967)**	**3.119 (0.6536)**	**3.595 (3.5802)**			**97.84**	**103.8**	**110.5**
LL (λ, σ)	8.484 (0.8531)	2.431 (0.0598)				101.56	105.6	110
McLL (λ, σ, a, b, c)	5.493 (18.802)	2.655 (0.07716)	1.282 (5.6039)	4.353 (30.5138)	4.42 (11.27)	97.82	107.8	119
BeLL (λ, σ, a, b)	5.624 (5.441)	3.298 (1.755)	1.227 (1.617)	5.129 (14.521)		97.78	105.8	114.7
KwLL (λ, σ, a, b)	4.002 (10.091)	3.436 (1.811)	1.778 (5.328)	11.064 (70.15)		97.74	105.7	114.7
MoLL (λ, σ, α)	8.484 (0.8532)	2.055 (2.3428)	4.162 (40.2466)			101.56	107.6	114.3

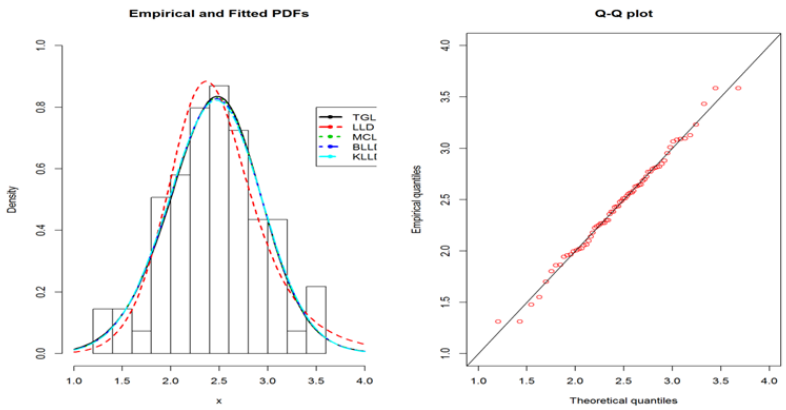

Figure 5 Fitted PDFs of different log-logistic distributions along with TGLLD and Q-Q plot for the observed data.

Conclusion

In this article, a lifetime distribution named as Type-II generalized log-logistic distribution (TGLLD) is considered. We obtained the properties of the distribution viz. mean, percentile, median, quantile function, moments, variance, skewness, kurtosis and distribution function of i^{th} order statistics. Also derived the MLEs of unknown parameters, reliability function and obtained their asymptotic variances. The estimated values are presented through simulation. With the use of real data, the model values are validated and the results compared with other log-logistic distributions by finding the statistics such as -2logL, AIC and BIC. With observed results, it is noticed that this model proven to be more suitable for lifetime models.

Authors' contributions

GSR contributed to study design, data analysis and drafting of the manuscript. SV and KR contributed to this work by data acquisition and revising of the manuscript. All co-authors have given the final approval of the version to be published.

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