Research Article

The seven-parameter lindley distribution

Abstract

In this paper, a seven parameter Lindley Distribution (SPL) distribution is proposed as a new generalization of the basic Lindley distribution. The structural properties of the new distribution are investigated. These include the compounding representation of the distribution, reliability analysis and statistical measures. Expressions for Lorenz and Bonferroni curves and Renyi entropy as a measure for uncertainty reduction are derived. Maximum likelihood estimation is used to evaluate the parameters. The new model contains twelve lifetime distributions as special cases such as the Lindley, Quasi Lindley, gamma, and exponential distributions, among others. This model has the advantage of being capable of modeling various shapes of aging and failure criteria. Finally, the usefulness of the new model for modeling reliability data is illustrated using a real data set.

Keywords: lindley distribution, mixture, reliability analysis, moment generating function, order statistics, maximum likelihood estimation

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Introduction

In many applied sciences such as medicine, engineering, and finance, amongst others, modeling and analyzing lifetime data are crucial. Several lifetime distributions have been used to model such kinds of data. The quality of the procedures used in a statistical analysis depends heavily on the assumed probability model or distribution along with relevant statistical methodologies. However, there remain many important problems where the real data does not follow any of the classical or standard probability models.

The one parameter family of distributions with density function

$$f(x;\theta) = \frac{\theta^2}{\theta+1}(1+x)e^{-\theta x}; x > 0, \theta > 0, \qquad (1)$$

is used by Lindley[1] to illustrate the difference between fiducial distribution and posterior distribution. Sankaran[2] introduced the discrete Poisson-Lindley distribution by combining the Poisson and Lindley distributions. Ghitany et al.[3] have discussed various properties of (4.1). Another discrete version of this distribution has been suggested by Deniz and Ojeda[4] with applications in count data related to insurance. Ghitany et al.[5] obtained size-biased and zero-truncated version of Poisson-Lindley distribution with various properties and applications.

The aim of this chapter is to introduce a new generalization of Lindley[1] distribution. This generalization is flexible enough to model different types of lifetime data having different forms of failure rate. The new distribution can accommodate both decreasing and increasing failure rates as its antecessors, as well as unimodal and bathtub shaped failure rates. The Lindley distribution is generalized by mixing. Several authors have considered versions from usual density functions by following this idea. For instance, Zakerzadeh and Dolati[6] considered a generalized Lindley distribution as a generalization of the usual Lindley distribution, Rama and Mishra[7] studied quasi-Lindley distribution, Rama et al.[8] introduced a new distribution for generalizing the Lindley model which called Janardan distribution, and Elbatal et al.[9] have suggested a new generalized Lindley distribution.

The introduced model will be named Seven Parameter Lindley Distribution (SPL) distribution. The basic idea behind this generalization is to use a unified approach that accommodates all the proceeding generalizations of Lindley distribution abovementioned, and to add new model that could offer a better fit to lifetime data. The proposed distribution includes nine models as special cases plus three new models (all special cases are shown in section 2). The procedure used here is based on certain mixtures of two gamma distributions with various weights. The research examines various properties of the new distribution.

The rest of paper is organized as follows: Section 2 introduces the definition of the probability density function (pdf) of the SPL distribution including its cumulative distribution function (cdf) and the sub-models of the new suggested model. The reliability analysis including the survival function, the hazard (or failure) rate function, the reversed hazard rate function, the cumulative hazard rate function, and the mean residual lifetime is explored in Section 3. The statistical properties of the new distribution such as the moments, the moment generating function, and the distribution of order statistics are investigated in Section 4, with a proposed algorithm for generating random data from the new distribution in this section. Section 5 introduces Lorenz and Bonferroni curves and Renyi entropy as measures of inequality and uncertainty, respectively. Section 6 discusses the estimation of parameters by using maximum likelihood estimation. Finally, Section 7 provides an application for modeling real data sets to illustrate the performance of the new distribution.

Generalization and related sub-models

In this section, we introduce the pdf and the cdf of the five parameter Lindley distribution and then the special cases of the SPL distribution are mentioned.

Generalization

Let

$$f_1(x;\alpha,\theta) = \frac{\theta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\theta x}; x > 0 \qquad (2)$$

and

$$f_2(x;\beta,\theta) = \frac{\theta^\beta}{\Gamma(\beta)}x^{\beta-1}e^{-\theta x}; x > 0, \theta, \beta > 0 \qquad (3)$$

be gamma (α,θ) and gamma (β,θ) densities, respectively.

We define a new seven parameter Lindley distribution as a mixture of (2) and (3) with Probabilities $p = \dfrac{\theta^{\phi}k}{\eta^{\sigma}+\theta^{\phi}k}$ and (1-p), respectively, as follows:

$$f(x,\theta,\alpha,\beta,k,\eta,\phi,\sigma) = pf_1(x;\alpha,\theta) + (1-p)f_2(x;\beta,\theta); p \in (0,1)$$

Therefore, the pdf of the seven Parameter Lindley Distribution (SPL) distribution, is defined as

$$f(x;\theta,\alpha,\beta,\kappa,\eta,\phi,\sigma) = \frac{\theta^{\phi}k}{\eta^{\sigma}+\theta^{\phi}k}\left[\frac{k\theta^{\phi-1}(\theta x)^{\alpha-1}}{\Gamma(\alpha)} + \frac{\eta^{\sigma}(\theta x)^{\beta-1}}{\theta\Gamma(\beta)}\right]e^{-\theta x}, x>0$$

We note that $f(x;\theta,\alpha,\beta,\kappa,\eta,\phi,\sigma)$ incorporates seven parameters namely $\theta>0,\alpha>0,\beta>0,\kappa\geq0,\phi>0,\sigma>0$ and $\eta\geq0$ subject to k and η are not allowed to be simultaneously zeros. The corresponding cumulative distribution function (cdf) of the SPLD is

$$F(x;\theta,\alpha,\beta,\kappa,\eta,\phi,\sigma) = \frac{\left[\theta^{\phi}k\gamma_{\alpha}(\theta x)+\eta^{\sigma}\gamma_{\beta}(\theta x)\right]}{\eta^{\sigma}+\theta^{\phi}k}; x>0,\theta,\alpha,\beta>0,k,\eta\geq0$$

where

$$Y_{\alpha}(b) = \frac{Y(a,b)}{\Gamma(\alpha)}\int_0^b t^{a-1}e^{-t}dt$$

is known as the lower incomplete gamma function ratio. Also, the upper incomplete gamma function ratio is given by

$$\Gamma_a(b) = \frac{\Gamma(a,b)}{\Gamma(a)} = \frac{1}{\Gamma(a)}\int_b^{\infty} t^{a-1}e^{-t}dt$$

Figures 1&2 illustrate some of the possible shapes of the pdf and the cdf, respectively, of the SPLD for different values of the parameters $\theta,\alpha,\beta,k,\phi,\sigma$ and η chosen from the ranges specified in Equation (4).
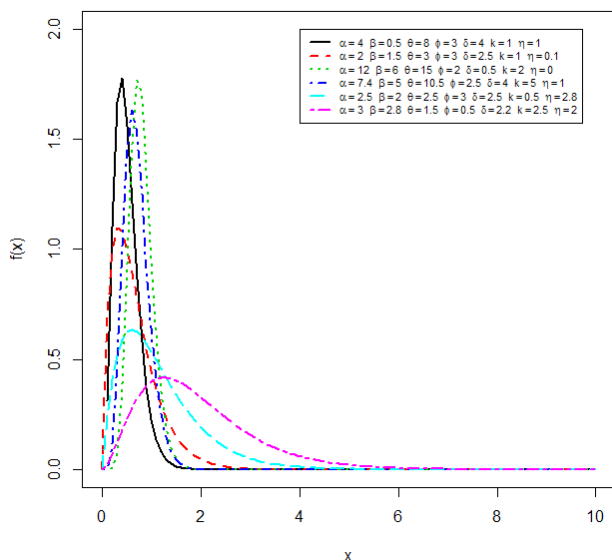
Probability density function

Figure 1 Different shapes of the pdf for the SPLD.
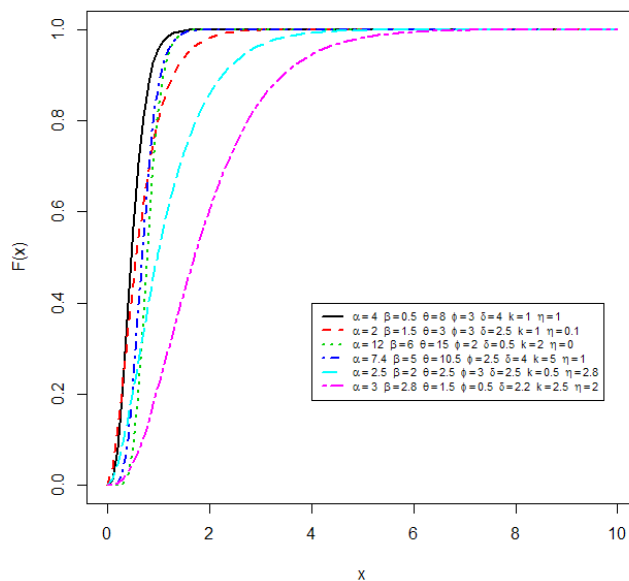
Cumulative distribution function

Figure 2 The distribution function (cdf) of the SPLD.

Sub-models of the SPLD

It is clear that the seven parameter Lindley distribution is very flexible. Assigning particular numerical values of some subsets of the parameters yields several special generalizations of Lindley distribution. The special cases include nine distributions namely; the new generalized Lindley distribution (NGLD) introduced by Elbatal et al.,[9] generalized Lindley distribution (GLD) introduced by Zakerzadeh and Dolati,[6] quasi Lindley distribution (QLD) introduced by Rama and Mishra,[7] Lindley distribution (LD) by Lindley,[1] Erlang distribution, Janardan distribution introduced by Rama et al.,[8] gamma distribution, the exponential distribution (ED), and Chi-square distribution. In addition to yield all the previous distributions, our generalization model allowed us to create new three distributions namely, the 4-parameter Lindley type I (4-p L type I) distribution, the 4-parameter Lindley type II (4-p L type II) distribution and the 2-parameter Lindley (2-p L) distribution.

Reliability analysis

In this section, we present the survival function, the hazard rate function, the reversed hazard rate function, the cumulative hazard rate function and the mean residual lifetime for the seven parameter Lindley distribution.

The survival function

The survival function $R(x)$ which is the probability of an item not failing prior to some time is defined by $R(x)=1-F(x)$. Therefore, the survival function of the SPL distribution is given by

$$R(x) = 1 - \frac{1}{\eta^{\sigma}+\theta^{\phi}k}\left[\theta^{\phi}k\gamma_{\alpha}(\theta x)+\eta^{\sigma}\gamma_{\beta}(\theta x)\right]; x>0 \qquad (6)$$

The hazard rate function

The other characteristic of interest of a random variable is the hazard rate function, *h(x)* the hazard rate function of the SPLD is given by

$$h(x) = \frac{\theta^2 \left[\frac{k\theta^{\phi-1}(\theta x)^{\alpha-1}}{\Gamma(\alpha)} + \frac{\eta^\sigma (\theta x)^{\beta-1}}{\theta \Gamma(\beta)} \right] e^{-\theta x}}{\eta^\sigma + \theta^\phi k - \left[\theta^\phi k \gamma_\alpha (\theta x) + \eta^\sigma \gamma_\beta (\theta x) \right]} \quad (7)$$

We note that $h(x)$ might be constant, increasing, decreasing, or bathtub shaped depending on the values of the parameters involved. For example, if $\eta = 0$ and $\alpha = 1$ then $h(x) = \theta$, a constant, while for $\alpha \geq 1$ and $\beta \geq 2$ it will be increasing, and it is going to be decreasing if $\beta \geq 2\alpha \leq 1, \beta \leq 2, \eta = 0$, and the bathtub-type curve appears for $\alpha < 1, \beta < 2$, and $\eta > 0$.

The next result describes some particular cases for the hazard rate function arising from the five parameter Lindley distribution by assigning relevant values of the parameters.

Theorem 1:

The hazard rate function of the particular cases from the five parameter Lindley distribution are given by

i. If $\alpha = k = \eta = 1$ and $\beta = 2$ the failure rate is same as the $LD(\theta)$.

ii. If $\alpha = 1, \beta = 2$, and $\eta = \theta$ the failure rate is same as the $QLD(k,\theta)$

iii. If $\alpha = 1, \beta = 2$, and $\eta = \theta$ the failure rate is same as the $ED(\theta)$.

iv. If $\alpha = k = 1, \theta = (\theta/\eta)$, and $\beta = 2$ the failure rate is same as the $JD(\theta, \eta)$.

v. If $k = \eta = 1$ the failure rate is same as the $NGLD(\theta, \alpha, \beta)$.

vi. If $\eta = \theta$ the failure rate is same as the $Gamma\ (\theta, \alpha)$.

Proof:

i. If $\alpha = 1, \beta = 2$, and $\eta = \theta$ the failure rate is same as the $QLD(k,\theta)$. $h(x) = \frac{\theta^2 (1+x)}{\theta + \theta x + 1}$.

ii. If $\alpha = 1, \beta = 2$, and $\eta = \theta$ the failure rate is same as the $QLD(k,\theta)$. $h(x) = \frac{\theta(k+\theta x)}{k + \theta x + 1}$.

iii. If $\alpha = 1, \beta = 2$, and $\eta = \theta$ the failure rate is same as the $ED(\theta)$ $/ h(x) = \theta$.

iv. If $\alpha = k = 1, \theta = (\theta/\eta)$, and $\beta = 2$ the failure rate is same as the $JD(\theta, \eta)$. $h(x) = \frac{\theta^2 (1+\eta x)}{\eta(\theta + \eta^2) + \theta \eta^2 x}$.

v. If $k = \eta = 1$ the failure rate is same as the $NGLD(\theta, \alpha, \beta)$.

$$h(x) = \frac{\theta^2 \left[\frac{(\theta x)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(\theta x)^{\beta-1}}{\theta \Gamma(\beta)} \right] e^{-\theta x}}{1 + \theta - \left[\theta Y_\alpha (\theta x) + Y_{\hat{a}} (\theta x) \right]}.$$

The reversed hazard rate function

The reversed hazard rate function $r(x)$, of a random variable distributed according to the $Spl(x; \theta, \alpha, \beta, k, \eta, \phi, \sigma)$ after some simplifications is given by

$$r(x) = \theta^2 \left[\frac{\theta \phi k (\theta x)^{\alpha-1} \Gamma(\beta) + \eta^\sigma (\theta x)^{\beta-1} \Gamma(\alpha)}{\theta^\phi k \gamma(\alpha, \theta x) \Gamma(\beta) + \eta^\sigma \gamma(\beta, \theta x) \Gamma(\alpha)} \right] e^{-\theta x} \quad (8)$$

The cumulative hazard rate function

Many generalized models have been proposed in reliability literature through the relationship between the reliability function $R(x)$ and its cumulative hazard rate function $H(x)$, given by $H(x) = -\ln R(x)$. The cumulative hazard rate function of the SPL distribution is given by

$$H(x) = -\ln \ln \left\{ 1 - \frac{1}{\eta^\sigma + \theta^\phi k} \left[\theta^\phi k \gamma_\alpha (\theta x) + \eta^\sigma \gamma_\beta (\theta x) \right] \right\} \quad (9)$$

where $H(x)$, is the total number of failure or deaths over an interval of time, and $H(x)$, is a non-decreasing function of satisfying.

$$H(0) = 0, (b) \lim_{x \to \infty} H(x) = \infty$$

The mean residual lifetime

The additional lifetime given that the component has survived up to time x is called the residual life function of the component, the nth expectation of the random variable X_x that represent the remaining lifetime is called the mean residual lifetime (MRL) and is given by

$$m(x) = E(X_x) = E(X - xX > x) = \frac{\int_x^\infty R(t) dt}{R(x)},$$

or equivalently

$$m(x) = \frac{\int_x^\infty t f(t) dt}{R(x)} - x$$

While the hazard rate function $h(x)$ provides information about a small interval after time x (just after x), the MRL considers information about the whole interval after x (all after x). The MRL as well as the hazard rate function or the reliability function is very important as each of them can be used to characterize a unique corresponding lifetime distribution.

The MRL function $m(x)$ for SPL random variable is given by

$$m(x) = \frac{\theta^\phi k \Gamma_{\alpha+1} (\theta x) + \beta \eta^\sigma \Gamma_{\beta+1} (\theta x)}{\{\eta^\sigma + \theta^\phi k - \left[\theta^\oslash k Y_\alpha (\theta x) + \eta^\sigma Y_\beta (\theta x) \right]\}} - x \quad (10)$$

The MRL function given in Equation (4.10) satisfies the following properties.

i. $m((x) \geq 0$,

ii. $\frac{dm(x)}{d(x)} \geq -1$,

iii. $\int_0^\infty [m(x)]^{-1} dx = \infty$, and

iv. $m(0) = \mu_1' = \frac{\alpha \theta^\phi k + \beta \eta^\sigma}{\theta^\phi (\eta^\sigma + \theta x)}$,

where μ_1' is the first non-central moment of the SPL Distribution (the mean of the distribution).

Statistical properties

This section investigates the statistical properties of the SPL Distribution as the moments (non- central and central), the moment generating function and an algorithm for random number generating.

The moment generating function

The following theorem gives the moment generating function (mgf) of SPL Distribution ($x;\theta,\alpha,\beta,k,\eta,\phi,\sigma$).

Theorem 2:

If X has the SPL Distribution ($x;\theta,\alpha,\beta,k,\eta,\phi,\sigma$), then the mgf of X say $M_X(t)$ *is* given as follows

$$M_X(t) = \frac{1}{\eta^\sigma + \theta^\phi k}\{\sum_{r=0}^{\infty}\frac{(-1)^r}{\theta^r}[\theta^\phi k\binom{-\alpha}{\rho} + \eta^\sigma\binom{-\beta}{r}]t^r\};t < \theta \quad (11)$$

Proof:

$$M_X(t) = E\left(e^{tX}\right) = \int_0^\infty e^{tx} f(x;\theta,\alpha,\beta,k,\eta,\phi,\sigma)dx.$$

$$M_X(t) = \frac{\theta^2}{\eta^\sigma + \theta^\phi k}[\frac{k\theta^{\phi+\alpha-2}}{\Gamma(\alpha)}\int_0^\infty x^{\alpha-1}e^{-(\theta-t)x}dx + \frac{\eta^\sigma\theta^{\phi+\beta-2}}{\Gamma(\beta)}\int_0^\infty x^{\beta-1}e^{-(\theta-t)x}dx].$$

$$M_X(t) = \frac{1}{\eta^\sigma + \theta^\phi k}\left[\frac{k\theta^{\phi+\alpha}}{(\theta\phi-t)^\alpha} + \frac{\eta^\sigma\theta^\beta}{(\theta^\phi-t)^\beta}\right].$$

$$M_X(t) = \frac{1}{\eta^\sigma + \theta^\phi k}[\theta^\phi k(1-\frac{t}{\theta^\phi})^{-\alpha} + \eta^\sigma(1-\frac{t}{\theta^\phi})^{-\beta}.$$

using the expansion $(1-Z)^{-d} = \sum_0^\infty (-1)^j\binom{-d}{j}Z^j$, one has

$$M_X(t) = \frac{1}{\eta^\sigma + \theta^\phi k}\left\{\sum_{r=0}^{\infty}\frac{(-1)^r}{\theta^r}\left[\theta^\phi k\binom{-\alpha}{r} + \eta^\sigma\binom{-\beta}{r}\right]t^r\right\};t < \theta.$$

This completes the proof.

Depending on the previous theorem, we can conclude the basic statistical properties as follows:

i. The rth non-central moments μ_r' are the coefficients of $\frac{t^r}{r!}$. In Equation (11), for $r = 0,1,2,......$. Therefore, the mean μ_r' and the variance $Var(X)$ of the SPL random variable X are, respectively, given by

$$\mu_1' = \frac{a\theta^\phi k + \text{â}\eta^\sigma}{\theta(\eta^\sigma + \theta^\phi k)} \quad (12)$$

and

$$Var(X) = \mu_2' - (\mu_1')^2 \quad (13)$$

Where μ_2' is the second non-central moment which is given by

$$\mu_2' = \frac{a\theta^\phi k(a+1) + \beta\eta^\sigma(\beta+1)}{\theta^2(\eta^\sigma + \theta^\phi k)} \quad (14)$$

ii. The nth central moments μ_n can be obtained easily from the rth moments through the relation

$$\mu_n = \sum_{r=0}^{n}\binom{n}{r}(-\mu)^{n-r}\mu_r'$$

Where $\mu \equiv \mu_1'$

Then the nth central moments of the SPL distribution are given by

$$\mu_n = \sum_{r=0}^{n}\binom{n}{r}(-1)^{n-r}(\frac{a\theta^t k + \text{â}\eta^\sigma}{\theta^2(\eta^\sigma + \theta^\phi k)})^{n-r}\mu_r' \quad (15)$$

iii. Finally, the coefficient of variation (ρ), the coefficient of skewness (ρ_1), and the coefficient of kurtosis (ρ_2) of SPLdistribution are, respectively, obtained according the following relations

$$\rho = \frac{\sqrt{\mu_2}}{\mu_1'} = \frac{\sqrt{\mu_2' - \mu_1'^2}}{\mu_1'},(16)$$

$$\rho_1 = \frac{\mu_3}{[\mu_2]^{\frac{3}{2}}} = \frac{\mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3}{[\mu_1' - \mu_1'^2]^{\frac{3}{2}}},(17)$$

$$\rho_2 = \frac{\mu_4}{[\mu_2]^2} = \frac{\mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^3 - 3\mu_1'^2}{[\mu_2' - \mu_1'^2]^2}.(18)$$

Distribution of order statistics

Let $X_1, X_2, ... X_n$. denote n independent random variables from a distribution function $F_X(x)$ with pdf $f_X(x)$, and then the pdf of $X(j)$ (the j order sample arrangement) is given by

$$f_{X(j)}(x) = \frac{n!}{(j-1)!(n-j)!}f_X(x)\left[F_X(x)\right]^{j-1}.\left[1 - F_X(x)\right]^{n-j}, j = 1,2,...,n \quad (19)$$

Using Equations (4) and (5) into Equation (19), then the pdf of $X_{(j)}$ according to the SPL distribution is given by

$$f_{X(j)}(x) = \frac{n!}{(j-1)!(n-j)!}\{\frac{\theta^2}{\eta^\sigma + \theta^\phi k}\left[\frac{k(\theta x)^{a-1}}{\Gamma(a)} + \frac{\eta^\sigma(\theta x)^{\beta-1}}{\theta^\oslash\Gamma(\beta)}\right]e^{-\theta x}$$

$$\times\{\frac{1}{\eta^\sigma + \theta^\phi k}\left[\theta^\phi k\gamma_\alpha(\theta x) + \eta^\sigma\gamma_\beta(\theta x)\right]\}^{j-1} \times \{1$$

$$-\frac{1}{\eta^\sigma + \theta^\phi k}\left[\theta^\phi k\gamma_\alpha(\theta x) + \eta^\sigma\gamma_\beta(\theta x)\right]\}^{j-1}.$$

$$(20)$$

Hence, the pdf of the largest order statistic $X_{(n)}$ and the smallest order statistic $X_{(1)}$ are, respectively, given by

$$f_{X_{(n)}}(x) = n\left\{\frac{\theta^2}{\eta^\sigma + \theta^\phi k}\left[\frac{k(\theta x)^{a-1}}{\Gamma(a)} + \frac{\eta^\sigma(\theta x)^{\beta-1}}{\theta^\oslash\Gamma(\beta)}\right]e^{-\theta x}\right\}$$

$$\times\{\frac{1}{\eta^\sigma + \theta^\phi k}\left[\theta^\phi k\gamma_\alpha(\theta x) + \eta^\sigma\gamma_\beta(\theta x)\right]\}^{n-1}, \quad (21)$$

and

$$f_{X_{(1)}}(x) = n\left\{\frac{\theta^2}{\eta^\sigma + \theta^\phi k}\left[\frac{k(\theta x)^{a-1}}{\Gamma(a)} + \frac{\eta^\sigma(\theta x)^{\beta-1}}{\theta^\phi\Gamma(\beta)}\right]e^{-\theta x}\right\} \times$$

$$-\frac{1}{\eta^\sigma + \theta^\phi k}\left[\theta^\phi k\gamma_\alpha(\theta x) + \eta^\sigma\gamma_\beta(\theta x)\right]\}^{n-1} \quad (22)$$

Random variates generation

The probability density function of the SPL distribution can be expressed in terms of the gamma density function as follows

$$f\left(x;\theta,\alpha,\beta,k,\eta,\phi,\sigma\right)=\frac{\theta^{\phi}k}{\eta^{\sigma}+\theta^{\phi}k}f_1\left(x;\alpha,\theta\right)+\frac{\eta^{\sigma}}{\eta^{\sigma}+\theta^{\phi}k}f_1\left(x;\beta,\theta\right)$$

To generate random variates X_i, for $i=1,\ldots,n$, from SPL $\left(\theta,\alpha,\beta,k,\eta,\phi,\sigma\right)$, we can use the following algorithm:

1. Generate $U_i, i=1,\ldots,n$, from $\text{Uniform}\left(0,1\right)$ distribution.

2. Generate $Y_{1i}, i=1,\ldots,n$, from $\text{Gamma}\left(\alpha,\theta\right)$.

3. Generate $Y_{2i}, i=1,\ldots,n$, from $\text{Gamma}\left(\hat{a},\theta\right)$.

4. If $U_i\leq\dfrac{\theta^{\phi}k}{\eta^{\sigma}+\theta^{\phi}k}$, then the set of random variates $X_i=Y_{1i}$; otherwise Set $X_i=Y_{2i}, i=1,\ldots,n$

Measures of inequality and uncertainty

In this section Lorenz and Bonferroni curves are introduced as measures of inequality. Also, Renyi entropy will be mentioned as an important measure of uncertainty.

Lorenz and bonferroni curves

Lorenz and Bonferroni curves are the most widely used inequality measures in income and wealth distribution.[10]

In fact, Lorenz and Bonferroni curves are depending on the length-biased distribution with pdf $f^*\left(x\right)$ defined by

$$f^*\left(x\right)=\frac{xf\left(x\right)}{E\left(x\right)} \quad (23)$$

Where $f\left(x\right)$ is the pdf of the base distribution with mean $E\left(x\right)$.

Accordingly, Lorenz and Bonferroni curves denoted by $L\left(x\right)$ and $B\left(x\right)$ respectively, defined by

$$l\left(x\right)=\frac{F^*\left(x\right)}{E\left(x\right)} \text{ and } B\left(x\right)=\frac{L\left(x\right)}{F\left(x\right)} \quad (24)$$

where $F^*\left(x\right)$ is the cdf of the length-biased distribution. Now, we shall derive the expressions of $L\left(x\right)$ and $B\left(x\right)$ based on $f^*\left(x\right)$ and $F^*\left(x\right)$ for SPLD.

It is easily shown that the pdf of the length-biased distribution can be obtained as Follows

$$f^*\left(x\right)=\frac{\theta^{\phi}\left[\frac{\alpha\theta^{\phi}k(\theta x)^{\alpha}}{\Gamma\left(\alpha+1\right)}+\frac{\beta\eta^{\sigma}(\theta x)^{\beta}}{\Gamma\left(\beta+1\right)}\right]e^{-\theta x}}{\alpha\theta^{\phi}k+\beta\eta^{\sigma}} \quad (25)$$

With cdf defined by

$$F^*\left(x\right)=\frac{\alpha\theta^{\varnothing}kY_{\alpha+1}\left(\theta x\right)+\beta\eta^{\sigma}Y_{\beta+1}\left(\theta x\right)}{\alpha\theta^{\phi}k+\beta\eta^{\sigma}} \quad (26)$$

It follows from (12), (24), and (26) that $L\left(x\right)$ and $B\left(x\right)$ are

$$l\left(x\right)=\frac{\theta^{\phi}(\theta^{\phi}k+\eta^{\sigma})\left[\alpha\theta^{\phi}kY_{\alpha+1}\left(\theta x\right)+\beta\eta^{\sigma}Y_{\beta+1}\left(\theta x\right)\right]}{(\alpha\theta^{\phi}k+\beta\eta^{\sigma})^2} \quad (27)$$

and

$$B\left(x\right)=\frac{\theta^{\phi}(\theta^{\phi}k+\eta^{\sigma})^2\left[\alpha\theta^{\phi}kY_{\alpha+1}\left(\theta x\right)+\beta\eta^{\sigma}Y_{\beta+1}\left(\theta x\right)\right]}{(\alpha\theta^{\phi}k+\beta\eta^{\sigma})^2\left[\theta^{\varnothing}kY_{\alpha}\left(\theta x\right)+\eta^{\sigma}Y_{\beta}\left(\theta x\right)\right]} \quad (28)$$

Renyi entropy

If X is a random variable having an absolutely continuous cdf $F\left(x\right)$ and pdf $f\left(x\right)$, then the basic uncertainty measure for distribution F (called the entropy of F) is defined as $Y\left(x\right)=E\left[-\ln\left(f\left(x\right)\right)\right]$. Statistical entropy is a probabilistic measure of uncertainty or ignorance about the outcome of a random experiment and is a measure of a reduction in that uncertainty. Abundant entropy and information indices, among them the Renyi entropy, have been developed and used in various disciplines and contexts. Information theoretic principles and methods have become integral parts of probability and statistics and have been applied in various branches of statistics and related fields.

Renyi entropy is an extension of Shannon entropy. Renyi entropy of the SPLD is defined to be

$$Y_\nu\left\{f\left(x;\theta,\alpha,\beta,k,\eta,\phi,\sigma\right)\right\}=\frac{1}{\left(1-\nu\right)}\times\ln\left(\int_0^\infty f_{SPL}^\nu\left(x;\theta,\alpha,\beta,k,\eta,\phi,\sigma\right)dx\right) \quad (29)$$

Where $\nu>0$ and $\nu\neq1$. Renyi entropy tends to Shannon entropy as $\nu\to1$. Now,

$$\int_0^\infty f_{SPL}^\nu\left(x;\theta,\alpha,\beta,k,\eta,\phi,\sigma\right)dx=\left(\frac{\theta^2}{\eta^{\sigma}+\theta^{\phi}k}\right)^\nu\int_0^\infty\left(\frac{k(\theta x)^{a-1}}{\Gamma(a)}+\frac{\eta^{\sigma}(\theta x)^{\beta-1}}{\theta^{\phi}\Gamma(\beta)}\right)^\nu e^{-\nu\theta x}dx \quad (30)$$

Using $A=\int_0^\infty\left(\dfrac{k(\theta x)^{a-1}}{\Gamma(a)}+\dfrac{\eta^{\sigma}(\theta x)^{\beta-1}}{\theta^{\varnothing}\Gamma(\beta)}\right)^\nu e^{-\nu\theta x}dx$. then one has

$$A=\int_0^\infty(1+\frac{\eta^{\sigma}\Gamma(a)\times(\theta x)^{\beta-a}}{\theta^{\varnothing}k\Gamma(\beta)})^\nu+\left(\frac{k(\theta x)^{a-1}}{\Gamma(a)}\right)^\nu e^{-\nu\theta x}dx \quad (31)$$

Using the expansion: $(1+z)^n=\sum_{r=0}^n\binom{n}{j}z^j$ one can have

$$(1+\frac{\eta^{\sigma}\Gamma(a)\times(\theta x)^{\beta-a}}{\theta^{\phi}k\Gamma(\beta)})^\nu=\sum_{j=0}^\nu\binom{\nu}{j}(\frac{\eta^{\sigma}\Gamma(a)\times(\theta x)^{\beta-a}}{\theta^{\phi}k\Gamma(\beta)})^j \quad (32)$$

$$A=(\frac{k}{\Gamma(a)})^\nu\sum_{j=0}^\nu\binom{\nu}{j}(\frac{\eta^{\sigma}\Gamma(a)}{\theta^{\phi}k\Gamma(\beta)})^j\int_0^\infty(\theta x)^{\nu(a-1)+(\beta-\alpha)}e^{-\nu\theta x}dx \quad (33)$$

Using the gamma function to evaluate the integral in (33) and collecting the entire above evaluations then substitute into (29), the Renyi entropy of the SPLD can be written as

$$Y_\nu=\frac{1}{1-\nu}[\ln c++\ln\left\{\sum_{j=0}^\nu\binom{\nu}{j}(\frac{\eta^{\sigma}\Gamma(a)}{\theta^{\phi}k\Gamma(\beta)})^j\times\frac{\Gamma(\nu(a-1)+j(\beta-a)+1)}{\nu^{j(\beta-a)}}\right\}], (34)$$

Where C is a constant as

$$C=[\frac{k\theta^{\frac{2\nu-1}{\nu}}}{\nu^a\left(\eta^{\sigma}+\theta^{\phi}k\right)\Gamma(a)}]^\nu$$

Estimation of the parameters

In this section, we use the method of likelihood to estimate the parameters involved and use them to create confidence intervals for the unknown parameters.

Let $X_1, X_2, \ldots X_n$ be a sample size n from SPL distribution. Then the likelihood function (l) is given by

$$l = \prod_{i=1}^{n} f_i(x) = \left(\frac{\theta^2}{\eta^\sigma + \theta^\phi k}\right)^n \prod_{i=1}^{n} \left[\frac{k\theta^{\phi+a-1}(\theta x_i)^{a-1}}{\Gamma(a)} + \frac{\eta^\sigma \theta^{\beta-2} x_i^{\beta-1}}{\Gamma(\beta)}\right] a$$

Then,

$$l = \left(\frac{\theta^2}{\eta^\sigma + \theta^\phi k}\right)^n \exp\left[-\theta\sum_{i=1}^{n} x_i\right] \left[\Gamma(a)\Gamma(\beta)\right]^{-n} \prod_{i=1}^{n} k\Gamma(\beta)\theta^{\phi+a-2}\left(x_i\right)^{a-1} + \eta^\sigma \Gamma(a)\theta^{\beta-2}x_i^{\beta-1}$$

(35)

Hence, the log-likelihood function $\mathcal{L} = \ln l$ becomes

$$\mathcal{L} = 2n\ln\theta - n\ln\ln\left(\eta^\sigma + \theta^\phi k\right) - \theta\sum_{i=1}^{n} x_i - n\ln\Gamma(a) - n\ln\Gamma(\beta)$$

$$+ \sum_{i=1}^{n} \log[[k\Gamma(\beta)\theta^{\phi+a-2}x_i^{a-1} + \eta^\sigma\Gamma(a)\theta^{\beta-2}x_i^{\beta-1}]$$

(36)

Therefore, the maximum likelihood estimators (MLEs) of $\theta, \alpha, \beta, k, \phi, \sigma$ and η are derived from the derivatives of $\mathcal{L}$.

They should satisfy the following equations

$$\frac{\partial\mathcal{L}}{\partial\theta} = \frac{2n}{\theta} - \frac{\phi nk\theta^{\phi-1}}{\eta^\sigma + \theta^\phi k} - \sum_{i=1}^{n} x_i$$

$$+ \sum_{i=1}^{n}\left[\frac{k(\phi+a-2)\Gamma(\beta)\theta^{\phi+a-3}x_i^{a-1} + \eta^\sigma(\beta-2)\Gamma(a)\theta^{\beta-3}x_i^{\beta-1}}{k\Gamma(\beta)\theta^{\phi+a-2}x_i^{a-1} + \eta^\sigma\Gamma(a)\theta^{\beta-2}x_i^{\beta-1}}\right] = 0$$

(37)

$$\frac{\partial\mathcal{L}}{\partial\alpha} = -n\psi(\alpha) + \sum_{i=1}^{n}\left[\frac{k\Gamma(\beta)\theta^{\phi+a-2}x_i^{a-1}\log\log(\theta x_i) + \eta^\sigma(a)\theta^{\beta-2}x_i^{\beta-1}}{k\Gamma(\beta)\theta^{\phi+a-2}x_i^{a-1} + \eta^\sigma\Gamma(a)\theta^{\beta-2}x_i^{\beta-1}}\right] = 0$$

(38)

$$\frac{\partial\mathcal{L}}{\partial\beta} = -n\psi(\beta) + \sum_{i=1}^{n}\left[\frac{k(\beta)\theta^{\phi+a-2}x_i^{a-1} + \eta^\sigma\Gamma(a)\theta^{\beta-2}x_i^{\beta-1}\log\log(\theta x_i)}{k\Gamma(\beta)\theta^{\phi+a-2}x_i^{a-1} + \eta^\sigma\Gamma(a)\theta^{\beta-2}x_i^{\beta-1}}\right] = 0$$

(39)

$$\frac{\partial\mathcal{L}}{\partial k} = -\frac{n\theta^\phi}{\eta^\sigma + \theta^\phi k} + \sum_{i=1}^{n}\left[\frac{\Gamma(\beta)\theta^{\phi+a-2}x_i^{a-1}}{k\Gamma(\beta)\theta^{\phi+a-2}x_i^{a-1} + \eta^\sigma\Gamma(a)\theta^{\beta-2}x_i^{\beta-1}}\right] = 0$$

(40)

$$\frac{\partial\mathcal{L}}{\partial\eta} = -\frac{n\sigma\eta^{\sigma-1}}{\eta^\sigma + \theta^\phi k} + \sum_{i=1}^{n}\left[\frac{\sigma\eta^{\sigma-1}\Gamma(a)\theta^{\beta-2}x_i^{\beta-1}}{k\Gamma(\beta)\theta^{\phi+a-2}x_i^{a-1} + \eta^\sigma\Gamma(a)\theta^{\beta-2}x_i^{\beta-1}}\right] = 0$$

(41)

$$\frac{\partial\mathcal{L}}{\partial\phi} = -\frac{n\theta^\phi k\log\log(\theta)}{\eta^\sigma + \theta^\phi k} + \sum_{i=1}^{n}\left[\frac{k\Gamma(\beta)\theta^{\phi+a-2}x_i^{a-1}\log\log(\theta)}{k\Gamma(\beta)\theta^{\phi+a-2}x_i^{a-1} + \eta^\sigma\Gamma(a)\theta^{\beta-2}x_i^{\beta-1}}\right] = 0$$

(42)

$$\frac{\partial\mathcal{L}}{\partial\sigma} = -\frac{n\eta^\sigma\log\log(\eta)}{\eta^\sigma + \theta^\phi k} + \sum_{i=1}^{n}\left[\frac{\eta^\sigma\Gamma(\alpha)\theta^{\beta-2}x_i^{\beta-1}\log\log(\eta)}{k\Gamma(\beta)\theta^{\phi+a-2}x_i^{a-1} + \eta^\sigma\Gamma(a)\theta^{\beta-2}x_i^{\beta-1}}\right] = 0$$

(43)

Where $\psi(.)$ is the diagamma function, and it is defined as

$$\psi(a) = \frac{d}{da}\ln\ln\Gamma(a) = \frac{\square(a)}{\Gamma(a)}.$$

To solve the equations (37) through (43), it is usually more convenient to use nonlinear optimization algorithms such as quasi-Newton algorithm to numerically maximize the log-likelihood function. In order to compute the standard errors and asymptotic confidence intervals we use the usual large sample approximation, in which the MLEs can be treated as being approximately trivariate normal.

Hence as $\to \infty$, the asymptotic distribution of the MLE is given by, see Zaindin et al.[6]

$$\begin{pmatrix}\hat{\theta}\\\hat{\alpha}\\\hat{\beta}\\\hat{k}\\\hat{\eta}\\\hat{\sigma}\\\hat{\phi}\end{pmatrix} \sim \begin{pmatrix}\hat{\theta}\\\hat{\alpha}\\\hat{\beta}\\\hat{k}\\\hat{\eta}\\\hat{\sigma}\\\hat{\phi}\end{pmatrix}, \begin{pmatrix}\hat{V}_{11}\hat{V}_{12}\hat{V}_{13}\hat{V}_{14}\hat{V}_{15}\hat{V}_{16}\hat{V}_{17}\\\hat{V}_{21}\hat{V}_{22}\hat{V}_{23}\hat{V}_{24}\hat{V}_{25}\hat{V}_{26}\hat{V}_{27}\\\hat{V}_{31}\hat{V}_{32}\hat{V}_{33}\hat{V}_{34}\hat{V}_{35}\hat{V}_{36}\hat{V}_{37}\\\hat{V}_{41}\hat{V}_{42}\hat{V}_{43}\hat{V}_{44}\hat{V}_{45}\hat{V}_{46}\hat{V}_{47}\\\hat{V}_{51}\hat{V}_{52}\hat{V}_{53}\hat{V}_{54}\hat{V}_{55}\hat{V}_{56}\hat{V}_{57}\\\hat{V}_{61}\hat{V}_{62}\hat{V}_{63}\hat{V}_{64}\hat{V}_{65}\hat{V}_{66}\hat{V}_{67}\\\hat{V}_{71}\hat{V}_{72}\hat{V}_{73}\hat{V}_{74}\hat{V}_{75}\hat{V}_{76}\hat{V}_{77}\end{pmatrix}$$

Where $(\hat{V}_{ij} = V_{ij}|\theta = \hat{\theta})$, and

$$\begin{pmatrix}\hat{V}_{11}\hat{V}_{12}\hat{V}_{13}\hat{V}_{14}\hat{V}_{15}\hat{V}_{16}\hat{V}_{17}\\\hat{V}_{21}\hat{V}_{22}\hat{V}_{23}\hat{V}_{24}\hat{V}_{25}\hat{V}_{26}\hat{V}_{27}\\\hat{V}_{31}\hat{V}_{32}\hat{V}_{33}\hat{V}_{34}\hat{V}_{35}\hat{V}_{36}\hat{V}_{37}\\\hat{V}_{41}\hat{V}_{42}\hat{V}_{43}\hat{V}_{44}\hat{V}_{45}\hat{V}_{46}\hat{V}_{47}\\\hat{V}_{51}\hat{V}_{52}\hat{V}_{53}\hat{V}_{54}\hat{V}_{55}\hat{V}_{56}\hat{V}_{57}\\\hat{V}_{61}\hat{V}_{62}\hat{V}_{63}\hat{V}_{64}\hat{V}_{65}\hat{V}_{66}\hat{V}_{67}\\\hat{V}_{71}\hat{V}_{72}\hat{V}_{73}\hat{V}_{74}\hat{V}_{75}\hat{V}_{76}\hat{V}_{77}\end{pmatrix} = \begin{pmatrix}A_{11}A_{12}A_{13}A_{14}A_{15}A_{16}A_{17}\\A_{21}A_{22}A_{23}A_{24}A_{25}A_{26}A_{27}\\A_{31}A_{32}A_{33}A_{34}A_{35}A_{36}A_{37}\\A_{41}A_{42}A_{43}A_{44}A_{45}A_{46}A_{47}\\A_{51}A_{52}A_{53}A_{54}A_{55}A_{56}A_{57}\\A_{61}A_{62}A_{63}A_{64}A_{65}A_{66}A_{67}\\A_{71}A_{72}A_{73}A_{74}A_{75}A_{76}A_{77}\end{pmatrix}$$

is the approximate variance-covariance matrix with its elements obtained from

$$A_{11} = \frac{\partial^2\mathcal{L}}{\partial\theta^2} \quad A_{12} = \frac{\partial^2\mathcal{L}}{\partial\theta\partial\alpha} \quad A_{13} = \frac{\partial^2\mathcal{L}}{\partial\theta\partial\beta} \quad A_{14} = \frac{\partial^2\mathcal{L}}{\partial\theta\partial k} \quad A_{15} = \frac{\partial^2\mathcal{L}}{\partial\theta\partial\eta} \quad A_{16} = \frac{\partial^2\mathcal{L}}{\partial\theta\partial\sigma} \quad A_{17} = \frac{\partial^2\mathcal{L}}{\partial\theta\partial\phi}$$

$$A_{22} = \frac{\partial^2\mathcal{L}}{\partial\alpha^2} \quad A_{23} = \frac{\partial^2\mathcal{L}}{\partial\alpha\partial\beta} \quad A_{24} = \frac{\partial^2\mathcal{L}}{\partial\alpha\partial k} \quad A_{25} = \frac{\partial^2\mathcal{L}}{\partial\alpha\partial\eta} \quad A_{26} = \frac{\partial^2\mathcal{L}}{\partial\alpha\partial\sigma} \quad A_{27} = \frac{\partial^2\mathcal{L}}{\partial\alpha\partial\phi}$$

$$A_{33} = \frac{\partial^2\mathcal{L}}{\partial\beta^2} \quad A_{34} = \frac{\partial^2\mathcal{L}}{\partial\beta\partial k} \quad A_{35} = \frac{\partial^2\mathcal{L}}{\partial\beta\partial\eta} \quad A_{36} = \frac{\partial^2\mathcal{L}}{\partial\beta\partial\sigma} \quad A_{37} = \frac{\partial^2\mathcal{L}}{\partial\beta\partial\phi}$$

$$A_{44} = \frac{\partial^2\mathcal{L}}{\partial k^2} \quad A_{45} = \frac{\partial^2\mathcal{L}}{\partial k\partial\eta} \quad A_{46} = \frac{\partial^2\mathcal{L}}{\partial k\partial\sigma} \quad A_{47} = \frac{\partial^2\mathcal{L}}{\partial k\partial\phi}$$

$$A_{55} = \frac{\partial^2\mathcal{L}}{\partial\eta^2} \quad A_{56} = \frac{\partial^2\mathcal{L}}{\partial\eta\partial\sigma} \quad A_{57} = \frac{\partial^2\mathcal{L}}{\partial\eta\partial\phi}$$

$$A_{66} = \frac{\partial^2\mathcal{L}}{\partial\sigma^2} \quad A_{67} = \frac{\partial^2\mathcal{L}}{\partial\sigma\partial\phi}$$

$$A_{77} = \frac{\partial^2\mathcal{L}}{\partial\phi^2}$$

By solving this inverse dispersion matrix, these solutions will yield the asymptotic variances and co- variances of these MLEs for $\theta, \alpha, \beta, k, \phi, \sigma$ and η.

Approximate $100(1-\alpha)\%$ confidence intervals for $\theta, \alpha, \beta, k, \phi, \sigma$ and η can be determinedas

$$\hat{\theta} \pm Z_{\frac{\alpha}{2}} \sqrt{\hat{V}_{11}}, \hat{\alpha} \pm Z_{\frac{\alpha}{2}} \sqrt{\hat{V}_{22}}, \hat{\beta} \pm Z_{\frac{\alpha}{2}} \sqrt{\hat{V}_{33}},$$

$$\hat{k} \pm Z_{\frac{\alpha}{2}} \sqrt{\hat{V}_{44}}, \hat{\eta} \pm Z_{\frac{\alpha}{2}} \sqrt{\hat{V}_{55}}, \hat{\sigma} \pm Z_{\frac{\alpha}{2}} \sqrt{\hat{V}_{66}} \hat{\phi} \pm Z_{\frac{\alpha}{2}} \sqrt{\hat{V}_{77}}$$

Where $Z_{\alpha/2}$ is the upperpercentile of the standard normal distribution.

Application

In this section, we use a real data set to compare the fits of the SPL distribution with three sub-models. In each case, the parameters are estimated by maximum likelihood as described in Section 4.6, using the R software.

The data set consist of uncensored data set from Nichols and Padgett on the breaking stress of carbon fibers (in Gba). The data are given below:

3.70, 2.74, 2.73, 2.50, 3.60, 3.11, 3.27, 2.87, 1.47, 3.11, 3.56,

4.42, 2.41, 3.19, 3.22, 1.69, 3.28, 3.09, 1.87, 3.15, 4.90, 1.57,

2.67, 2.93, 3.22, 3.39, 2.81, 4.20, 3.33, 2.55, 3.31, 3.31, 2.85,

1.25, 4.38, 1.84, 0.39, 3.68, 2.48, 0.85, 1.61, 2.79, 4.70, 2.03,

1.89, 2.88, 2.82, 2.05, 3.65, 3.75, 2.43, 2.95, 2.97, 3.39, 2.96,

2.35, 2.55, 2.59, 2.03, 1.61, 2.12, 3.15, 1.08, 2.56, 1.80, 2.53.

The summary of the above data is given by

Units	Minimum	Ist Qu.	Median	Mean	3rd Qu.	Maximum
66	0,390	2,178	2,835	2,760	3.278	4.900

In order to compare the two distribution models, we consider criteria like KS (Kolmogorov Smirnov), $-2L$, AIC (Akaike information criterion), AICC (corrected Akaike information criterion), and BIC (Bayesian information criterion) for the data set. The better distribution corresponds to smaller KS, $-2L$, AIC and AICC values:

$$AIC = 2k - 2l, AICC = AIC + \frac{2k(k+1)}{n-k-1},$$

$$BIC = 2l + k \times \log\log n \text{ and}$$

$$KS\max_{1 \le i \le n} \left(F(X_i) - \frac{i-1}{n}, \frac{i}{n} - F(X_i), \right)$$

Where l denotes the log-likelihood function evaluated at the maximum likelihood estimates, k is the number of parameters, and is n the sample size.

Also, for calculating the values of KS we use the sample estimates of and b. Table 1&2 shows the parameter estimation based on the maximum likelihood and least square estimation, and gives the values of the criteria AIC, AICC, BIC, and KS test.

Table 1 The special cases of the SPL distribution

Distribution	Parameters							Author
	θ	α	β	k	η	σ	ϕ	
Gamma			I	I	0	I	I	Brown & Flood[11]
ED		I	I	I	0	I	I	Steffensen[12]
LD		I	2	I	I	I	I	Lindley[1]
Erlang		$v, v \in N$	I	I	0	I	I	A. K. Erlang[13]
QLD		I	2		θ	I	I	Rama & Mishra[7]
GLD			$\alpha + 1$	I		I	I	Zakerzadeh&Dolati[6]
Janardan	$\frac{\theta}{\eta}$	I	2	I		I	I	Rama et al.[8]
NGLD				I	I	I	I	Elbatal et al.[9]
Chi-square	1/2	$v/2, v \in N$	I	I	0	I	I	Fisher[14]
4-p L type I				I		I	I	New
4-p L type II					I	I	I	New
2-p L		I	2		I	I	I	New
5-p L	θ	α	β	k	η	I	I	New

The values in Table 2 indicate that the SPL distribution leads to a better fit over all the other models.

	SPL		FPLD		Lindley		Gamma		exponential	
	coef	std.e	coef	std.e	coef	std.e	coef	std.e	coef	std.e
alpha	4.918145682	3.1418	4.918019	2.273544	4.9181	2.2736	7.48803	1.27552	-	-
beta	13.32661771	2.273564	13.32648	3.141746	13.3266	3.1418	2.7135	0.47806	-	-
theta	4.608695814	1.033154	4.608644	1.033136	4.6087	1.0332	-	-	0.362379	0.044606

Table Continued...

	SPL		FPLD		Lindley		Gamma		exponential	
	coef	std.e	coef	std.e	coef	std.e	coef	std.e	coef	std.e
phai	0.033886236	0.452362	-	-	-	-	-	-	-	-
k	1.471738523	0.056224	0.103372	0.080925	-	-	-	-	-	-
eta	3.406750687	0.187052	6.104362	0.001398	59.0538	46.2436	-	-	-	-
sigma	2.438262765	0.236172	-	-	-	-	-	-	-	-
AIC	177.3931		181.3931		179.3931		186.3351		267.9887	
BIC	182.0655		192.3413		188.1517		190.7144		270.1784	
AICC	175.462		182.3931		180.0488		186.5256		268.0512	
HQIC	171.3364		185.7192		182.854		188.0656		268.8539	
K-S	0.07		0.070003		0.0713806		0.13285		0.35811	
P-value	0.9031		0.9028		0.8569		0.1945		8.89E-08	

A density plot compares the fitted densities of the models with the empirical histogram of the observed data (Figures 3-5). The fitted density for the **SPL** model is closer to the empirical histogram than the fits of the other models.[15-22]
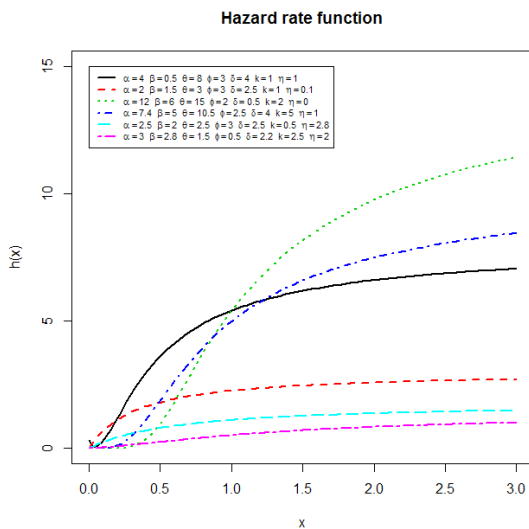

Hazard rate function

Figure 3 Increasing, decreasing, constant, bathtub and upside-down shapes for the hazad rate function of the SPLD.
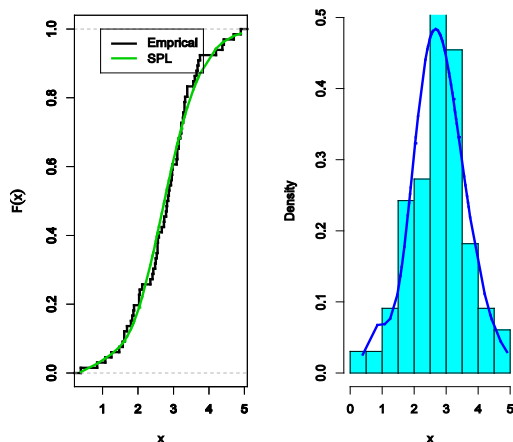

Figure 4 (a) Estimated densities of the SPL distributions for the data. (b) Estimated cdf function from the fitted the fitted the SPL distributions and the empirical cdf for the data.
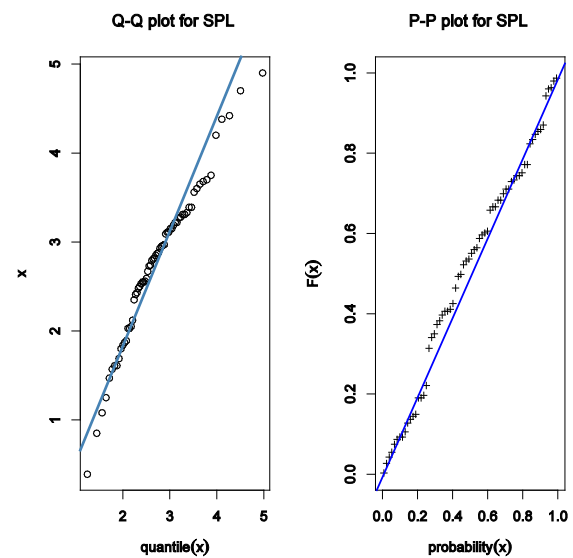

Q-Q plot for SPL   P-P plot for SPL

Figure 5 PP plots for the fitted SPLD distribution and for the data set.

Concluding remarks

There has been a great interest among statisticians and applied researchers in constructing flexible lifetime models to facilitate better modeling of survival data. Consequently, a significant progress has been made towards the generalization of some well-known lifetime models and their successful application to problems in several areas. In this paper, we introduce a new five-parameter distribution obtained using the idea of mixture of distributions. We refer to the new model as the Five Parameter Lindley Distribution (FPLD) and study some of its mathematical and statistical properties. We provide the pdf, the cdf and the hazard rate function of the new model and explicit expressions for the moments. The model parameters are estimated by the method of maximum likelihood. The new model is compared with three lifetime models and provides consistently better fit than them. We hope that the proposed distribution will serve as an alternative model to other models available in the literature for modeling positive real data in many areas such as engineering, survival analysis, hydrology and economics.

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