Appendix

Saddle Node Bifurcation

Consider the following equation,

$$\frac{du}{dt} = \alpha - u^2 \tag{A.1}$$

When a parameter value is changed certain advancements take place, such as two fixed points emerging from nothing. Sometimes they move closer, collide then mutually disappear [10, 13]. This process is called saddle node bifurcation. It is also known as limit point bifurcation, fold bifurcation, and blue-sky bifurcation. Equation (A.1) is a standard example of saddle node bifurcation. First, we look for the fixed points by setting (A.1) to zero.

$$\alpha - u^2 = 0$$

$$\Rightarrow u = \pm\sqrt{\alpha} \tag{A.2}$$

In the figures below from 15, we have varied the parameter α and looked for equilibrium points. For $\alpha < 0$ there are no fixed points. Increasing α from -2 to 0 we see one fixed point at 0 has been created. Since $\frac{du}{dt}$ is always less than zero, the flow will be to the left. Thus here we have half-stable fixed point.

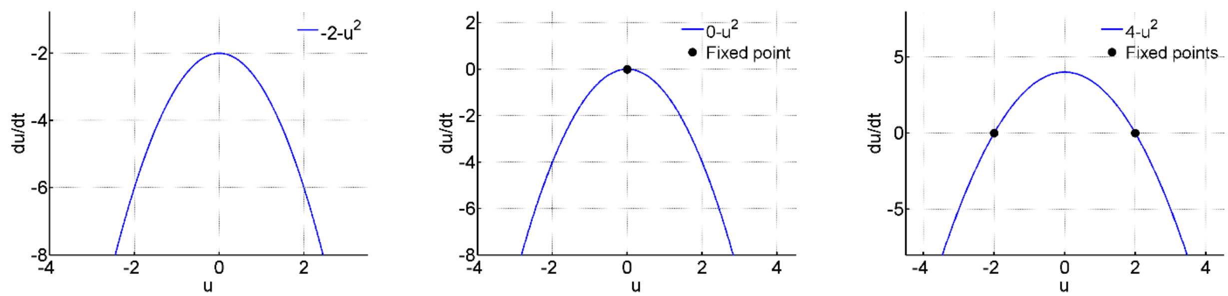

Figure 15: Saddle node bifurcation.

Now when $\alpha = 4$ there are two fixed points, at -2 and 2. To the left of-2, $\frac{du}{dt} < 0$ and to the right $\frac{du}{dt} > 0$. So the flow is outwards and the point is unstable. But at the fixed point 2 the flow is inwards, thus it is stable.

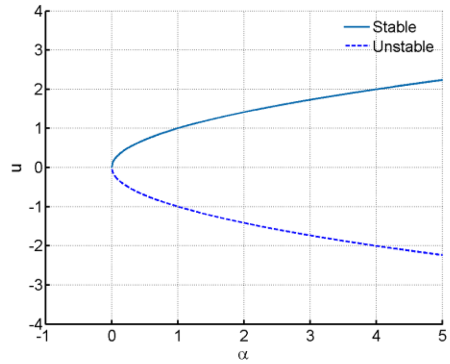

Figure 16: Bifurcation diagram for Saddle node bifurcation.

Another way to depict a saddle node bifurcation is plotting α verses u, known as the bifurcation diagram.[10,14] Here in Figure 16 the solid line $u = \sqrt{\alpha}$ represents stable equilibrium and the dotted line where $u = -\sqrt{\alpha}$ is for unstable equilibrium.

Transcritical Bifurcation

Consider the following equation for the classic form of the transcritical bifurcation in one dimensional form:

$$\frac{du}{dt} = \alpha u - u^2 \tag{A.3}$$

The mechanism where an equilibrium point exists for all parameter values but changes its stability is known as transcritical bifurcation.[13,14] Here from Figure 17 we see the fixed point $u = 0$ indicated as a red dot, exists for every values of α. For $\alpha < 0$ there is one more fixed point at $u = 3$ which is unstable, and the other one is stable, but when $\alpha = 0$ the fixed point is half stable. Now when α is greater than zero there are two fixed points, but the one at $u = 0$ is stable. Thus we see how the fixed point $u = 0$ changes stability when the parameter value is varied.

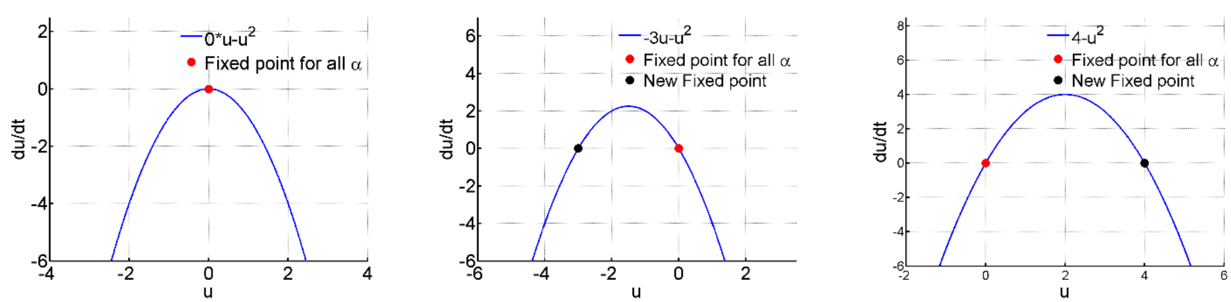

Figure 17: Transcritical bifurcation.

The Figure 18, is the bifurcation diagram where α is the independent variable. We substitute equation (A.3) to 0 and get,

$$\alpha u - u^2 = 0$$

$$\Rightarrow u(\alpha - u) = 0 \tag{A.4}$$

Then we get, $u = 0$ and $u = \alpha$ The stable points are indicated using a solid line and the unstable points are shown using dotted line.

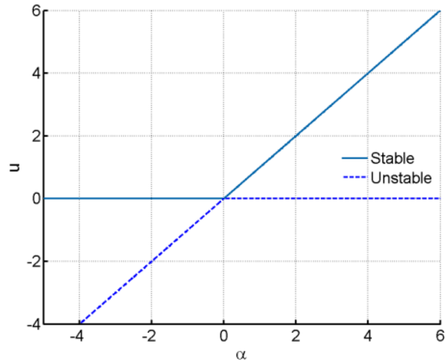

Figure 18: Bifurcation diagram for Transcritical bifurcation.