Literature Review

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A review on recent generalizations of exponential distribution

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Lishamol Tomy,[1] Manju Jose,[2] Veena G[3]
[1]Department of Statistics, Deva Matha College, Kuravilangadu, India
[2]Department of Statistics, Alphonsa College, Palai, India
[3]Department of Statistics, St.Thomas College, Palai, India

Correspondence: Lishamol Tomy, Assistant Professor, Department of Statistics, Deva Matha College, Kuravilangadu, Kerala, India, Email lishatomy@gmail.com

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Introduction

The Exponential distribution is the probability distribution of the time between two consecutive events in a Poisson point process, that is, it is a process in which the events occur continuously and independently at a constant average rate. The distribution is a limit of the scaled Beta distribution, and it is the only continuous probability distribution that has a constant failure rate. It has the maximum entropy probability distribution for a random variate X which is greater than or equal to zero, for which the expectation of X is fixed. The distribution is a particular case of the Gamma distribution, and it can also be seen as the continuous analogue of the Geometric distribution, it also possesses the lack of memory property, which states that the "*process does not remember what has happened until now and the distribution of the waiting time, given that it has already exceeded some amount of time, has the same Exponential distribution form*". The distribution was referred to by Kondo[1] as Pearson's type X distribution when the sampling of standard deviation was discussed.

These distributions have been widely used to analyze lifetime data, on account of its analytical tractability, also used in physics, queueing theory, and hydrology, often used to model the reliability of electronic systems, which do not typically experience wear-out type failures. Exponential variables can also be used to model situations where certain events occur with a constant probability per unit length, such as the distance between mutations on a DNA strand, or between road kills on a given road.

In Physics, if you observe a gas at a fixed temperature and pressure in a uniform gravitational field, the heights of the various molecules also follow an approximate exponential distribution, known as the Barometric formula. In Queuing theory, the service times of agents in a system, for example how long it takes for a bank teller etc. to serve a customer are often modeled as exponentially distributed variables. The length of a process that can be thought of as a sequence of several independent tasks follows the Erlang distribution which is the distribution of the sum of several independent exponentially distributed variables. Reliability theory and reliability engineering also makes an extensive use of the exponential distribution, because of the memory less property of this distribution, it is well-suited to model the constant hazard rate portion of the bathtub curve used in reliability theory. It is also very convenient because it is so easy to add failure rates in a reliability model. The exponential distribution is however not appropriate to model the overall lifetime of organisms or technical devices, because the "failure rates" here are not constant: more failures occur for very young and for very old systems. In hydrology, the exponential distribution is used to analyze extreme values of such variables as monthly and annual maximum values of daily rainfall and river discharge volumes. The applications of Exponential distribution have been widespread, which include models to determine bout criteria for analysis of animal behavior seen in Yeates et al.,[2] design rainfall estimation in the Coast of Chiapas as seen in Escalante-Sandoval,[3] analysis of Los Angeles rainfall data as shown in Madi and Raqab,[4] software reliability growth models for vital quality metrics, seen in Subburaj et al.,[5] models for episode peak, duration for eco-hydroclimatic applications shown by Biondi et al.[6] and estimating mean life of power system equipment with limited end-of-life failure data as seen in Cota-Felix et al.[7]

Exponential distribution is widely used in statistical literature as it possesses several important statistical properties, yet exhibits great mathematical tractability.

Marshall and Olkin[8] introduced a generalization of bivariate exponential distribution through investigating the distribution of joint waiting times in a bivariate Poisson process.

This article is an attempt to cover the most recent generalizations of exponential distribution. Section 2 presents such generalizations existing in recent statistical literature. Section 3 discusses the areas of application and Section 4 concludes the article.

Review on Exponential generalizations

Pogany[9] employed, E-Bayesian, for estimating the parametric functions of the generalized inverted Exponential (G-IE) distribution. Relations are derived under a squared error loss function, type-II censoring and a conjugate prior. E-Bayesian estimations are obtained based on different priors of the hyperparameters to investigate the influence of different prior distributions on these estimations. The asymptotic behaviors of E-Bayesian estimations and relations among them have been investigated. Finally, a comparison among the E-Bayesian, maximum likelihood and Bayes estimations in different sample sizes are made, using a real data and the Monte-Carlo (MC) simulation. These simulations showed that the newly employed method is more efficient than previous methods and is also easy to operate and some comparisons among the results of G-IE distribution, Exponential distribution and Generalized Exponential distribution are provided.

The maximum likelihood estimators (MLE) of the unknown parameters and reliability characteristics of G-IE distribution was derived by Dube et al.[10] by making use of a generalization of progressive censoring, which is the progressive first-failure censoring.

The non-Bayesian and Bayesian estimators for the unknown parameters based on records from the G-IE distribution was derived by Dey et al.[11] Under symmetric and asymmetric loss functions, using gamma priors on both the shape and the scale parameters, Bayes' estimators of the unknown parameters are obtained. Markov Chain Monte Carlo (MCMC) techniques are used to generate samples from the posterior distributions, as the Bayes estimators does not yield explicit forms and in turn computed the Bayes estimators. Bayes interval is generated and both frequentist and the Bayesian prediction intervals of the future record values based on the observed record values are discussed. MC simulations were performed to compare the performances of the proposed methods, for illustrative purposes, a data set has been analyzed.

Under random censoring, the G-IE distribution was studied by Garg et al.[12] They derived MLE's of the parameters and expected Fisher information under random censoring model. Using Lindley's approximation, Bayes estimators of the parameters under squared error loss function are obtained. Based on importance sampling procedure, the highest posterior density credible intervals of the parameters are constructed. A MC simulation is conducted to evaluate the performance of different estimators developed. The estimation method developed was illustrated by a randomly censored real dataset.

The problem of estimation of the stress–strength parameter $P(Y<X)$ based on progressively first-failure-censored samples was considered by Krishna et al.,[13] where X and Y both follow two-parameter G-IE distribution with different and unknown shape and scale parameters. MC simulation study was conducted to compare the proposed methods of estimation and the methods developed were illustrated with a couple of real data examples.

Okagbue et al.[14] proposed an alternative to the approximations used in properties of statistical distributions. It was done by using the differential calculus to obtain some classes of ordinary differential equations for the probability density function, quantile function, survival function, inverse survival function, hazard function and reversed hazard function of the exponentiated generalized exponential distribution. They established that the stated necessary conditions required for the existence of such equations are consistent with the various parameters that defined the distribution.

Oguntunde et al.[15] extended the inverse Exponential distribution using the Weibull generalized family of distributions and investigated studied its properties.

Khan et al.[16] investigated the potential practicability of the three-parameter transmuted generalized exponential distribution for analyzing lifetime data and compared it with various generalizations of the two-parameter exponential distribution using MLE. Mathematical properties of the extended model were also derived.

Kumar and Dey[17] examined the generalized order statistics from the extended exponential distribution by Nadarajah and Haghighi.[18] Exact explicit expressions as well as recurrence relations for the single, product, and conditional moments of generalized order statistics from the extended exponential distribution were obtained. Using the derived results they computed the means, variances, and covariances of order statistics and record values for samples of varied sizes for different values of the shape and scale parameters.

Valiollahi et al.[19] examined the prediction of a future observation based on either Type-I or Type-II hybrid censored samples when the lifetime distribution of the experimental units is assumed to be a generalized exponential random variable, different point and interval predictors using classical and Bayesian approaches were obtained. MC simulations were performed to analyze the performances of the different methods, and analysis of one data set has been conferred for illustrative purposes.

Genc[20] discussed the truncated inverted generalized exponential distribution and derived closed-form expressions for the related moments. They evaluated the modeling performance of the distribution based on two real datasets.

Using the quantile functions of well-known distributions, Zubair et al.[21] introduced a new generalized classes of exponential distribution, called the T-exponential class. Properties of this class including explicit expressions for the quantile function, Shannon entropy, moments and mean deviations were derived. Some generalized exponential families are reviewed and two real data sets are used to exhibit the applicability of the new models.

Torabi et al.[22] derived some properties of the Marshall–Olkin generalized exponential distribution and revealed that this distribution is more flexible than the exponentiated exponential distribution.

By applying the Quadratic rank transmutation map, Abdullahi et al.[23] proposed the transmuted odd generalized exponential-exponential distribution as an extension of the popular odd generalized exponential-exponential distribution.

Hassan et al.[24] dealt with estimation of $R = P(Y < X)$, where X and Y are distributed as two independent G-IE with common scale parameter and different shape parameters. On the basis of upper record values and upper record ranked set samples, the maximum likelihood and Bayesian estimators of R are obtained. Bayes estimator was obtained using Lindley approximation. Simulation study is performed to compare the reliability estimators in each record sampling scheme with respect to mean squared error.

To construct prediction intervals for future two-parameter exponential lifetimes based on a random number of generalized order statistics under a general set-up including progressive type II censored order statistics with general scheme, Barakat et al.[25] developed two pivotal quantities. They derived a maximum likelihood predictor for future exponential lifetimes when the sample size is deterministic or random.

Naseer and Hashmi[26] proposed a new class of continuous distributions with four parameters named the generalized odd Gumbel type-2 exponential distribution. The associated reliability analysis including hazard rate function, survival function, reverse hazard rate function and cumulative hazard rate function were discussed. Statistical properties such as quantile function, moments, moment generating function, and order statistic were discussed.

Estimation problems for the G-IE distribution based on progressively type-II censored order statistics and record values were studied by Kinaci et al.[27] They established some theorems to construct the exact confidence intervals and regions for the parameters. MC simulation studies are used to assess the performance of our proposed methods and it showed that the coverage probabilities of the exact confidence interval and the exact confidence region are all close to the desired level.

A new single acceptance sampling plan (ASP) is proposed by Al-Nasser and Obeidat[28] from truncated life test assuming that the quality characteristics follow the Tsallis q-exponential distribution. The operation characteristics function is derived and calculated the optimal sample size and producer's risk for some given parameter values to measure the performance of this plan. A comparative study is also done with other sampling plans to show the importance of the proposed plan.

Various techniques for estimations from the generalized linear exponential distribution that can be used for modeling bathtub, increasing and decreasing hazard rate behavior is dealt by Mahmoud et al.[29] They concluded with a simulation study to compare the different methods of estimation based on the mean square error and the average absolute bias.

Reliability analysis of electronic devices under varying voltages assuming modified generalized exponential distribution and beta generalized exponential distribution using the inverse power law rule was conducted by Ali et al.[30] They estimated the parameters of the modified distribution assuming Bayesian inference to incorporate prior information and also studied sensitivity of hyper-parameters and selection of an appropriate probability model.

Fallah and Kazemi[31] addressed the inferential aspects of the generalized weighted Exponential (G-WE) distribution, such as the maximum likelihood estimators of the unknown parameters and their corresponding asymptotic confidence intervals. They also developed some new distributional results about the G-WE distribution and provided more interesting closed form expressions for previously presented results. A simulation study and a real world application are also worked out to assess the maximum likelihood estimators and to illustrate the theory.

Garcia et al.[32] presented a three-parameter family of distributions which includes the common exponential and the Marshall–Olkin exponential as special cases. This distribution shows a monotone failure rate, which makes it appealing for practitioners interested in reliability, and means it can be included in the catalogue of appropriate non-symmetric distributions to model these disadvantages, such as the Weibull and gamma three-parameter families. Numerical examples based on real data reflect the suitable behavior of this distribution for modelling purposes.

A three-parameter exponential distribution called the extended odd Weibull exponential distribution was proposed by Afify and Mohamed,[33] which can have constant, increasing, decreasing, upside-down bathtub, bathtub and reversed-J shaped hazard rates, and left-skewed, symmetrical, right-skewed and reversed-J shaped densities. Some properties of the proposed distribution are derived with the model parameters estimated via eight frequentist estimation methods. Simulations are conducted and four practical data sets from the fields of medicine, engineering, and reliability are analyzed, proving the usefulness and flexibility of the proposed distribution.

Al-saiary et al.,[34] introduced the five-parameter beta Kumaraswamy exponential distribution (BKw-E), and some characterizations of this distribution are obtained and the properties are studied. Furthermore, important measures such as Rényi entropy and order statistics are obtained. An illustration of a real data set is discussed.

Louzada et al.[35] derived the different frequentist estimation procedures for the parameters of the exponential-Poisson distribution and is illustrated using two real data sets (rainfall and aircraft data) with the occurrence of zero values.

Hassan et al.[24] characterized and investigated a random variable ξ having the three parameter exponentiated exponential Poisson distribution by giving explicit closed-form formula for its characteristic function $\varphi(t)$, moment generating function $M(t)$, and the Fisher information matrix $I(\theta)$.

Tripathi et al.[36] considered the problem of estimating the quantile of a two-parameter exponential distribution with respect to an arbitrary strictly convex loss function under progressive type II censoring.

Ali et al.[30] developed ten frequentist methods of estimation, namely, maximum likelihood, least square, weighted least square, percentiles, maximum and minimum spacing distance, and variant of the method of the minimum distances for the logistic exponential (LE) distribution parameters as it is the only two parameter distribution that exhibits five hazard rate shapes such as constant, increasing, decreasing, bathtub, and upside-down bathtub.

Using the partial duration series, Won et al.[37] studied the univariate frequency analysis. A bivariate frequency study is conducted using a relatively simple bivariate exponential distribution to give a practical return level to major drought events in the past while reflecting the correlation between durations and drought severities. The drought severity vs duration frequency curves using each of the two frequency analyses are plotted, and these curves are used to inspect how the progress of the occurrence of the drought in the future and illustrations are also considered.

Afify et al.[38] introduced a new heavy-tailed exponential distribution that acquires upside-down bathtub, bathtub, decreasing-constant, decreasing, and increasing hazard rates and studied its actuarial properties with illustration.

Based on shock models, Mohtashami-Borzadaran et al.[39] proposed a new generalization of the bivariate Marshall–Olkin exponential distribution. The proposed model allows wider range tail dependence which is appealing in modeling risky events and effectively analyzes discrete–continuous data and the multivariate extension was also proposed.

Cinardi et al.[39] presented a generalized d-dimensional model for asymptotically-scale-free geographical networks with a wider spectrum for the fitness of the nodes.

Applications of generalized Exponential distribution

In this section, the main areas where generalizations of Exponential distribution find application are presented:

- **Medicine**

Garg et al.[12] illustrated the estimation procedures in the randomly censored data on remission times (in weeks) for a group of 30 patients with leukemia who received a similar treatment. Dube et al.[10] applied the G-IE distribution to the survival times (in days) of guinea pigs

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injected with different doses of tubercle bacilli. Khan[16] illustrated the value of the models on survival remission times (in months) of a random sample of 128 bladder cancer patients and also studied the intensive care unit (ICU) patients' agitation-sedation (A-S) status by using wavelets and Bayesian threshold limits. Genc[20] studied about the iron plasma concentrations in measures taken from 202 elite athletes training in the Australian Institute of Sport. Abdullahi et al.[23] studied the remission times (in months) of a random sample of 128 bladder cancer patients.

- **Reliability**

Dey et al.[11] applied the G-IE distribution based on upper records to study the breakdown of electrical insulating fluid subjected to 30 kilovolts. Krishna et al.[13] studied the strength data of single carbon fibres of 10 and 20 mm in gauge length. Valiollahi et al.[19] predicted the future failures of the accelerator life test of 59 conductors using the generalized exponential distribution under Type-I or Type-II hybrid censoring. Ali et al.[30] studied the lifetimes of electronic devices under different voltage levels by modified the generalized exponential distribution assuming the inverse power law rule.

- **Environmental studies**

Genc[20] studied about the losses due to wind-related catastrophes in 1977. Won et al.[37] studied the behavior of the future extreme drought, keeping in mind the change in climate scenarios in the future at the Daeghwanryeong site using two quantification methods of drought frequency analysis.

Conclusions

In this article, we have reviewed the recent generalizations which can be generated from the widely accepted statistical distribution, namely the Exponential distribution. The review finds that various Exponential generalizations can be used to manage real data sets with complex structure and these models are useful in designing probability models. We hope this review work can help in developing new classes from the Exponential distribution.

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