Research Article

Open Access CrossMark

Shock models leading to $\mathcal{G}^*$ class of lifetime distributions

Abstract

In this paper we study a stochastic ordering namely alternate probability generating function ($a.p.g.f.$) ordering and its properties. The life distribution $H(t)$ of a device subject to shocks governed by a Poisson process is considered as a function of the probabilities $\overline{P}_k$ of surviving the first k shocks. Various properties of the discrete failure distribution P_k are shown to be reflected in corresponding properties of the continuous life distribution $H(t)$. A certain cumulative damage model and various applications of these models in reliability modeling are also considered.

Keywords: lifetime distributions, probability and statistics

Volume 9 Issue 2 - 2020

K.V. Jayamol,[1] K. K. Jose[2]

[1]Department of Statistics, Maharaja's College, India
[2]Department of Biostatistics, St. Thomas College, India

Correspondence: K.V. Jayamol, Department of Statistics, Maharaja's College, Ernakulam, India, Tel 9447036746, Email jkvalavil@gmail.com

Received: April 10, 2020 | **Published:** April 29, 2020

Introduction

Stochastic orders and inequalities are being used at an accelerated rate in many diverse areas of probability and Statistics. For example, in statistical reliability theory, several concepts of partial orderings have been successfully used to develop various notions of ageing of non negative random variables. Ageing concept for discrete distributions were studied by various authors. See for example, Barlow and Proschan,[1] Cai and Kalashnikov,[2] Cai and Willmot,[3] Lai and Xie,[4] Shaked and Shanthikumar,[5] Shaked et al.,[6] Willmot and Cai,[7] Willmot and Lin,[8] Willmot et al.,[9] and references therein. Using Laplace transform, various reliability classes have been characterized by different researches. For details, see Bryson and Siddiqui,[10] Klefsjö,[11,12] Shaked and Wong[13] and the references there in. As a discrete analogue of Laplace transform ordering, Jayamol and Jose[14] introduced $a.p.g.f.$ ordering and $\mathcal{G}^*$ class of lifetime distributions based on this ordering as follows.

Definition 1.1 Let f denotes the probability mass function ($p.m.f.$) of a non-negative integer-valued random variable X, then the $a.p.g.f.$, $G(\cdot)$ of f is defined as

$$G(s) = E(1-s)^X, 0 < s \le 1. \qquad (1.1)$$

Definition 1.2 A non-negative integer-valued life distribution with mean μ belongs to the $\mathcal{G}^*\left(\overline{\mathcal{G}}^*\right)$ class of lifetime distributions if and only if

$$G(s) \le (\ge) \frac{1}{1+s\mu}, \ \mu \ge 0, 0 < s \le 1. \qquad (1.2)$$

It may be noted that the R.H.S. of the inequality (1.2) is the $a.p.g.f.$ of a geometric distribution with p.m.f. $f(x) = pq^x, x = 0,1,2,\cdots$ and with mean μ as that of f. For properties of $\mathcal{G}^*\left(\overline{\mathcal{G}}^*\right)$ classes one may refer to Jayamol and Jose,[15] Jayamol and Jose.[14]

For many equipments, useful life is often measured in discrete integer units, for example the number of copies a plain paper copier makes before a breakdown, the number of completed production runs in an automated assembly line before a malfunction occurs, etc. Even in situations where the time to failure is conceptually a continuous variable, one is often interested in measuring the life in suitably discretized work units successfully completed. For example, the number of days one needs to replace the batteries in an appliance under specified normal pattern of use is discrete. So as a discrete analogue of Laplace transform ordering, Jayamol and Jose[14] introduced a new stochastic ordering namely alternate probability generating function ($a.p.g.f.$) ordering. Some properties of this ordering are considered here.

$a.p.g.f.$ ordering and its properties

As a discrete analogue of Laplace transform ordering introduced by Klefsjö,[12] Jayamol and Jose[14] defined $a.p.g.f.$ ordering as follows.

Definition 2.1 Suppose that X a and Y are two non-negative integer-valued random variables with $p.m.f.$s f_1 and f_2 and $a.p.g.f.$s $G_1(s)$ and $G_2(s)$ respectively. Then X is said to be smaller than Y (or equivalently, f_1 is smaller than f_2) in $a.p.g.f.$ ordering if $G_1(s) \le G_2(s)$, for $0 \le s \le 1$. It is denoted by $X \le_G Y$ (or equivalently, we write $f_1 \le_G f_2$).

In this context, we have the following theorems.

Theorem 2.1 Suppose that X and Y be two non-negative integer-valued random variables with respective $p.m.f.$s f_1 and f_2. Then $X \le_G Y$ implies $E(X) \ge E(Y)$, provided the expectations exist.

Proof

If $X \le_G Y$ then $\sum_{x=0}^{\infty}(1-s)^x f_1(x) \le \sum_{y=0}^{\infty}(1-s)^y f_2(y)$

Differentiating once with respect to s and letting $s \to 0$, we get $E(X) \ge E(Y)$.

Theorem 2.2 Let X and Y be two non-negative integer-valued random variables. If $X \le_G Y$ then $X+K \le_G Y+K$ for every $K \in N_+$.

Proof

If $X \le_G Y, E(1-s)^X \le E(1-s)^Y$. Then we have,

$$E(1-s)^{X+K} = E(1-s)^K (1-s)^X = (1-s)^K E(1-s)^X$$
$$\le (1-s)^K E(1-s)^Y = E(1-s)^{Y+K}.$$

Theorem 2.3 Let $X_1, X_2, \ldots, X_m$ be a set of independently distributed non-negative integer-valued random variables. Let $Y_1, Y_2, \ldots, Y_m$ be another set of independently distributed non-negative integer-valued random variables. If $X_i \le_G Y_i$ for i=1,2… , m. Then

$$\sum_{i=1}^{m} X_i \le_G \sum_{i=1}^{m} Y_i$$

Proof

If $X_i \le_G Y_i$ then $E(1-s)^{X_i} \le E(1-s)^{Y_i}$ for i=1,2... , m.

Let $X = \sum_{i=1}^{m} X_i$ and $Y = \sum_{i=1}^{m} Y_i$

$$E(1-s)^X = E(1-s)^{X_1} \ldots (1-s)^{X_m} = E(1-s)^{X_1} \ldots E(1-s)^{X_m}$$
$$\le E(1-s)^{Y_1} \ldots E(1-s)^{Y_m} = E(1-s)^Y.$$

Theorem 2.4 Let $X_1, X_2, \ldots$ be independently and identically distributed non-negative integer-valued random variables and let N_1 and N_2 be positive integer-valued random variables which are independent of X_i. Then $N_1 \le_G N_2 \Leftrightarrow \sum_{i=1}^{N_1} X_i \le_G \sum_{i=1}^{N_2} X_i$

Proof

We have the $a.p.g.f.$

$$G_{X_1+X_2+\ldots+X_{N_1}}(S) = \sum_{i=1}^{\infty} P[N_1 = i] G_{X_1+X_2+\ldots+X_i}(S)$$

$$= \sum_{i=1}^{\infty} P[N_1 = i](G_{X_1}(s))^i$$

$$\sum_{i=1}^{\infty} P[N_1 = i]\left(1-\left(1-G_{X_1}(S)\right)\right)^i$$

$$= G_{N_1}(s), \quad where\, 0 < s = \left(1-G_{X_1}(s)\right) \le 1.$$

$$N_1 \le_G N_2 \Leftrightarrow G_{N_1}(s) \le G_{N_2}(s)$$

$$\Leftrightarrow G_{X_1+X_2+\cdots+X_{N_1}}(s) \le G_{X_1+X_2+\cdots+X_{N_2}}(s)$$

$$\Leftrightarrow \sum_{i=1}^{N_1} X_i \le_G \sum_{i=1}^{N_2} X_i$$

Theorem 2.5 Let X and Y be two non-negative integer-valued random variables such that $X \le_G Y$. Let $\overline{P}_k$ and $\overline{Q}_k$ be the survival functions of X and Y respectively. Then

$$\sum_{k=0}^{\infty} \overline{P}_k (1-s)^k \ge \sum_{k=0}^{\infty} \overline{Q}_k (1-s)^k$$

Proof

The stated result follows from the definition of $a.p.g.f.$ ordering and from the equation

$$E(1-s)^X = 1 - s\sum_{k=0}^{\infty} \overline{P}_k (1-s)^k. \qquad (2.1)$$

Shock models leading to $\mathcal{G}^*$ class

In reliability analysis one may calculate the reliability of a complex system starting with the reliability of the components. If all components have life distributions belonging to a certain class, then one would like to conclude that the life distribution of the entire system belongs to the same, or a similar class. Shock models of this kind have been considered by a number of authors under all kinds of assumptions. The results center around proving that, subject to suitable assumptions on the point process $\{N(t)\}$ of shocks, various discrete reliability characteristics of the $\{\overline{P}_k\}$ sequence, which arise naturally out of physical considerations are inherited by the continuous survival probability $\overline{F}(t)$. That is if the shock survival probabilities $\{\overline{P}_k\}$ belong to a discrete version of one of the life distribution classes, then under appropriate assumptions the continuous time survival probability $\overline{F}(t)$ belongs to the continuous version of that class. That is the life distribution $H(t)$ of a device subject to shocks governed by a Poisson process is considered as a function of the probabilities $\overline{P}_k$ of surviving the first k shocks. Various properties of the discrete failure distribution P_k are shown to be reflected in the corresponding properties of the continuous life distribution $H(t)$. In the present paper we study some shock models leading to $\mathcal{G}^*$ class. A certain cumulative damage model is also investigated. For that we consider the following definitions and Theorem.

Klefsjö[12] introduced a class denoted by $\mathcal{L}$, which consists of all distribution functions F, for which $L_F(s) \le \frac{1}{1+s\mu_1(F)}$, where $L_F(s)$ is the Laplace transform of F defined by $L_F(s) = \int_0^{\infty} e^{-sx} dF(x), s \ge 0$ and $\mu_k(F) = \int_0^{\infty} x^k dF(x)$ k=1,2,…, Its dual class $\overline{\mathcal{L}}$ is obtained by reversing the inequality.

Definition 3.1 Suppose that X and Y are two non-negative integer-valued random variables with survival functions $\overline{P}_k$ and $\overline{Q}_k$ respectively. Then X is said to be smaller than Y in $a.p.g.f.$ ordering if $\sum_{k=0}^{\infty} \overline{P}_k (1-s)^k \ge \sum_{k=0}^{\infty} \overline{Q}_k (1-s)^k$ for $0 < s \le 1$.

Theorem 3.1 Let X be a non-negative integer-valued random variable with $a.p.g.f. G(s)$ and survival function $P[X > k] = \overline{P}_k$. Then for $s \in (0,1)$, $X \in \mathcal{G}^* \Leftrightarrow \sum_{k=0}^{\infty} (1-s)^k \overline{P}_k \ge \frac{\mu}{(1+s\mu)}$

A Poisson shock model

Assume a device is subject to shocks occurring randomly in time according to a Poisson process with intensity λ. Suppose if the device has the probability $\overline{P}_k$ of surviving k shocks, where $1 = \overline{P}_0 \ge \overline{P}_2 \ge \ldots$, then the survival function of the device is given by,

$$\overline{H}(t) = \sum_{k=0}^{\infty} \frac{e^{-\lambda t}(\lambda t)^k}{k!} \overline{P}_k \qquad (3.1)$$

Esary et al.[16] have shown that if $\{\overline{P}_k\}_{k=0}^{\infty}$ has the discrete Increasing Failure Rate (IFR), Increasing Failure Rate in Average (IFRA), New Better than Used in Expectation (NBUE) or Decreasing Mean Residual Life (DMRL) property, then this property will be reflected to $\overline{H}(t)$ given by (3.1). Klefsjö[11] has shown that a similar result holds for the Harmonically New Better than Used in Expectation (HNBUE) class. Shock models leading to GHNBUE (GHNWUE) classes are studied by A H N Ahmed.[17] We now show that the same is true for $\mathcal{G}^*$ class also.

Theorem 3.2 The survival function $\overline{H}(t)$ in (3.1) is in $\mathcal{L}$ class if and only if $\{\overline{P}_k\}_{k=0}^{\infty}$ is in $\mathcal{G}^*$ class.

Proof

Let μ be the mean of $H(t)$ and m be the mean of $\{\overline{P}_k\}_{k=0}^{\infty}$. We have,

$$\mu=\int_0^\infty \overline{H}(t)dt=\sum_{k=0}^{\infty}\left\{\int_0^\infty \frac{e^{-\lambda t}(\lambda t)^k}{k!}dt\right\}\overline{P}_k=\sum_{k=0}^{\infty}\frac{1}{\lambda}\overline{P}_k=\frac{m}{\lambda}$$

Laplace transform of $\overline{H}(t)$

$$L_{\overline{H}}(S)=\int_0^\infty e^{-st}\overline{H}(t)dt \tag{3.2}$$

$$\sum_{k=0}^{\infty}\overline{P}_k\frac{\lambda^k}{k!}\int_0^\infty e^{-(s+\lambda)t}t^k dt$$

$$=\frac{1}{s+\lambda}\sum_{k=0}^{\infty}\left(\frac{\lambda}{s+\lambda}\right)^k\overline{P}_k \tag{3.3}$$

$$H\in L\,class\,if\,and\,only\,if\;L_{\overline{H}}(s)\geq\frac{\mu}{1+s\mu} \tag{3.4}$$

$$that\,is\,if\,and\,only\,if\,,=\frac{1}{s+\lambda}\sum_{k=0}^{\infty}\left(\frac{\lambda}{s+\lambda}\right)^k\overline{P}_k\geq\frac{\mu}{1+s\mu}$$

$$\sum_{k=0}^{\infty}\left(\frac{\lambda}{s+\lambda}\right)^k\overline{P}_k\geq\frac{(s+\lambda)\mu}{1+s\mu}$$

$$\sum_{k=0}^{\infty}\left(1-\frac{s}{s+\lambda}\right)^k\overline{P}_k\geq\frac{m\left(\frac{s+\lambda}{\lambda}\right)}{1+s\frac{m}{\lambda}}=\frac{m}{\frac{\lambda}{s+\lambda}+\frac{s}{s+\lambda}m}\geq\frac{m}{1+\frac{s}{s+\lambda}m} \tag{3.5}$$

(3.4) holds if and only if (3.5) holds. Hence from Theorem 3.1, we have the result.

Consider another device which is also subjected to shocks occurring randomly as events in a Poisson process with same constant intensity λ, and the device has probability $\overline{Q}_k$ of surviving the first k shocks, where $1=\overline{Q}_0\geq\overline{Q}_1\geq\ldots$. The survival function of this device is given by

$$\overline{F}(t)=\sum_{k=0}^{\infty}\frac{e^{-\lambda t}(\lambda t)^k}{k!}\overline{Q}_k \tag{3.6}$$

Singh and Jain[18] have shown that some partial orderings, namely likelihood ratio (LR) ordering, failure rate (FR) ordering, stochastic (ST) ordering, variable (V) ordering and mean residual life (MRL) ordering between the two shock survival probabilities $\overline{P}_k\,and\,\overline{Q}_k$ are preserved by the corresponding survival functions $\overline{H}(t)\,and\,\overline{F}(t)$ of the devices. Here we extent this preservation property to $a.p.g.f$ ordering.

Theorem 3.3 If $\overline{P}_k\leq_G\overline{Q}_k$ then $\overline{H}(t)\leq_L\overline{F}(t)$.

Proof

Let $\overline{P}_k\leq_G\overline{Q}_k$. From (3.6), we have

$$\int_0^\infty e^{-st}\overline{F}(t)dt=\int_0^\infty e^{-st}\sum_{k=0}^{\infty}\frac{e^{-\lambda t}(\lambda t)^k}{k!}\overline{Q}_k$$

$$\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}\overline{Q}_k\int_0^\infty e^{-(s+\lambda)t}t^k dt=\frac{1}{(s+\lambda)}\sum_{k=0}^{\infty}\left(\frac{\lambda}{s+\lambda}\right)^k\overline{Q}_k$$

$$=\frac{1}{(s+\lambda)}\sum_{k=0}^{\infty}\left(1-\frac{\lambda}{s+\lambda}\right)^k\overline{Q}_k$$

$$From\,Definition\,3.1\leq\frac{1}{s+\lambda}\sum_{k=0}^{\infty}\left(1-\frac{s}{s+\lambda}\right)^k\overline{P}_k=\int_0^\infty e^{-st}\overline{H}(t)dt$$

$$\Leftrightarrow\overline{H}(t)\leq_L\overline{F}(t).$$

Remark 1 When λ is a random variable, denoted by Λ,, whose distribution is Y, in this case $\overline{H}(t)$ can be written as

$$\overline{H}(t)=\sum_{k=0}^{\infty}E\frac{\left(e^{-\Lambda t}\left(\Lambda t\right)^k\right)}{k!} \tag{3.7}$$

A Nonhomogeneous poisson shock model

Suppose that shocks occur according to a nonhomogeneous Poisson process with mean value function $\Omega(t)$. If a device has the probability $\overline{P}_k$ of surviving the first k shocks, its survival function $\overline{H}(t)$ is given by

$$\overline{H}(t)=\sum_{k=0}^{\infty}\frac{e^{-\Omega(t)}\left[\Omega(t)\right]^k}{k!}\overline{P}_k \tag{3.8}$$

This shock model was studied by Hameed and Proschan.[19] They proved that under suitable conditions on $\Omega(t)$, the survival function $\overline{H}(t)$ is IFR, IFRA, New Better than Used (NBU) NBUE or DMRL if $\left\{\overline{P}_k\right\}_{k=0}^{\infty}$ has the corresponding discrete property. We will now give a theorem for $\mathcal{G}^*$ class.

Lemma 3.1 (Klefsjö)[12]: $\overline{H}(t)=\overline{H}_1\left[\Omega(t)\right]$ belongs to $\mathcal{L}(\overline{\mathcal{L}})\,if\,\overline{H}_1\in\mathcal{L}(\overline{\mathcal{L}})$ class and $\Omega(t)$ is starshaped (antistarshaped).

Theorem 3.4 If $\left\{\overline{P}_k\right\}_{k=0}^{\infty}$ is in $\mathcal{G}^*$ class and $\Omega(t)$ is starshaped then $\overline{H}(t)$ in (3.8) belongs to $\mathcal{L}$ class.

Proof

Let $\overline{H}(t)=\sum_{k=0}^{\infty}\frac{e^{-t}t^k}{k!}\overline{P}_k$ Since $\left\{\overline{P}_k\right\}_{k=0}^{\infty}$ is in $\mathcal{G}^*$ class, by Theorem 3.2 $\overline{H}_1(t)$ belongs to $\mathcal{L}$ class. Hence the result follows from Lemma 3.1.

A cumulative damage model

In this section we study special model for the survival probability $\overline{P}_k$. Suppose that a device is subjected to shocks. Every shock causes a random amount of damage. Suppose damage accumulates additively. The device fails when the accumulated damage exceeds a critical threshold Y which has the distribution F, where $F(0-)=0$. If the damages $X_1, X_2, \ldots,$ from successive shocks are independent and exponentially distributed with mean $\frac{1}{\lambda}$, and are independent of the threshold. Let N be the number of shocks survived by the device. Then the survival probabilities are given by

$$\overline{P}_k(\lambda) = \lambda^k \int_0^\infty \frac{x^{k-1}}{(k-1)!} e^{-\lambda x} \overline{F}(x)\,dx \; for\; k = 1,2,\ldots,$$

$$\overline{P}_0(\lambda) = 1. \tag{3.9}$$

Thus the probability function of N, $P[N=k]$ is

$$p_k = \int_0^\infty \frac{e^{-\lambda x}(\lambda x)^{k-1}}{(k-1)!} dF(x), k \geq 1$$

The above cumulative damage model has been studied by Esary et al.[16] for the NBU, IFR and IFRA cases. They proved that if F is NBUE, then $\{\overline{P}_k\}_{k=0}^\infty$ has the discrete NBUE property. Klefsjö[11] proved that the same is true in the case of discrete HNBUE. We now claim that the result is true when $\{\overline{P}_k\}_{k=0}^\infty$ belongs to $\mathcal{G}^*$ class.

Theorem 3.5 The survival probabilities $\overline{P}_k$ in (3.9) belongs to $\mathcal{G}^*$ class for every $\lambda > 0$ if F belongs to $\mathcal{L}$ class.

Proof

First observe that m be mean of $\overline{P}_k$ is

$$m = \sum_{k=0}^\infty \overline{P}_k = 1 + \sum_{k=1}^\infty \lambda^k \int_0^\infty \frac{x^{k-1}}{(k-1)!} e^{-\lambda x}\overline{F}(x)\,dx$$

$$= 1 + \int_0^\infty \lambda \sum_{k=0}^\infty \frac{(\lambda x)^k}{k!} e^{-\lambda x}\overline{F}(x)\,dx$$

$$= 1 + \int_0^\infty \lambda e^{-\lambda x + \lambda x}\overline{F}(x)\,dx \tag{3.10}$$

$$= 1 + \lambda \int_0^\infty \overline{F}dx == 1 + \lambda\mu, where\, \mu\, is\, the\, mean\, of\, F$$

$$So\; \mu = \frac{m-1}{\lambda}. \tag{3.11}$$

Consider

$$\sum_{k=0}^\infty \overline{P}_k(\lambda)(1-s)^k = 1 + \sum_{k=0}^\infty (1-s)^k \lambda^k \int_0^\infty \frac{x^{k-1}}{(k-1)!} e^{-\lambda x}\overline{F}(x)\,dx$$

$$= 1 + \int_0^\infty e^{-\lambda x}\overline{F}(x)(1-s)\lambda \sum_{k=0}^\infty \frac{((1-s)\lambda x)^k}{k!} dx$$

$$= 1 + \int_0^\infty e^{-\lambda x}\overline{F}(x)(1-s)\lambda e^{\lambda(1-s)x} dx$$

$$= 1 + \lambda(1-s)\int_0^\infty e^{-\lambda s x}\overline{F}(x)\,dx \tag{3.12}$$

Let $F \in \mathcal{L}$ class, hence from (3.12)

$$\sum_{k=0}^\infty \overline{P}_k(\lambda)(1-s)^k \geq 1 + \lambda(1-s)\frac{\mu}{1+s\lambda\mu}$$

$$= 1 + \lambda(1-s)\frac{\frac{m-1}{\lambda}}{1+s\lambda\frac{m-1}{\lambda}} \tag{3.13}$$

$$= 1 + \frac{(1-s)(m-1)}{1+s(m-1)} = \frac{m}{(1-s)+sm} \tag{3.14}$$

$$\geq \frac{m}{1+sm}.$$

Thus from the definition of $\mathcal{G}^*$ class, we have the theorem.

Theorem 3.6 The survival probability $\overline{P}_k$ in (3.9) belongs to $\overline{\mathcal{G}}^*$ class for every $\lambda > 0$ if F belongs to $\overline{\mathcal{L}}$ class.

Proof

We have, from (3.1) and (3.12), for $\overline{P}_k \in \overline{\mathcal{G}}^*$

$$1 + \lambda(1-s)\int_0^\infty e^{-\lambda s x}\overline{F}(x)\,dx \leq \frac{m}{1+sm} \leq \frac{m}{1-s+sm}$$

$$\lambda(1-s)\int_0^\infty e^{-\lambda s x}\overline{F}(x)\,dx \leq \frac{(1-s)(m-1)}{1+s(m-1)}$$

$$\int_0^\infty e^{-\lambda s x}\overline{F}(x)\,dx \leq \frac{(1-s)(m-1)}{\lambda(1-s)[1+s(m-1)]} = \frac{\mu}{1+s\mu}$$

Hence from the definition of $\overline{\mathcal{L}}$ class the result follows.

Applications

Random minima and maxima

Let $X_1, X_2, \ldots$ be a sequence of non-negative integer-valued random variables which are independent and identically distributed. Let N_1 be a positive integer-valued random variable which is independent of X_i's. Denote $X_{(1:N_1)} = min(X_1, X_2, \ldots, X_{N_1})$ and $X_{(N_1:N_1)} = max(X_1, X_2, \ldots, X_{N_1})$ (for details refer Gupta and Gupta,[20] Rohatgi,[21] Shaked and Wong[13] and references there in). Since the X_i s are non-negative, the random variable $X_{(N_1:N_1)}$ arises naturally in reliability theory as the lifetime of a parallel system with a random number N_1 of identical components with lifetimes $X_1, X_2, \ldots, X_{N_1}$. The random variable $X_{(1:N_1)}$ arises naturally in transportation theory as the accident free distance of a shipment of explosives, where N_1 of them are defectives which may explode and cause an accident after $X_1, X_2, \ldots, X_{N_1}$ miles respectively. Let N_2 be another positive integer-valued random variable which is also independent of the X_i and let

$$X_{(1:N_2)} = min(X_1, X_2 \ldots X_{N_2})\, and\, X_{(N_2:N_2)} = max(X_1, X_2, \ldots X_{N_2}).$$

Theorem 4.1 Let $X_1, X_2, \ldots$ be a sequence of non-negative integer-valued random variable which are independent and identically distributed. Let N_1 and N_2 be two positive integer-valued random variables which are independent of the $X_i^{'}s$ .. Then the following results are true.

1. If $N_1 \leq_G N_2$, then $X_{(1:N_1)} \leq_{st} X_{(1:N_2)}$.
2. If $N_1 \leq_G N_2$, then $X_{(N_1:N_1)} \geq_{st} X_{(N_2:N_2)}$.

Proof

Let P_k be the common distribution function of X_i 's, that is, $P_k = P[X_i \leq k]\, for\, i = 1,2,\ldots$ and $P_{k(N_1:N_1)}$ denotes the distribution function of $X_{(N_1:N_1)}$. Then we have

$$P_{k(N_1:N_1)} = \sum_{n=1}^{\infty} (P_k)^n P[N_1 = n]$$

$$\sum_{n=1}^{\infty} \left(1 - \overline{P}_k\right)^n P[N_1 = n]$$

$$= G_{N_1}\left(\overline{P}_k\right), 0 < \overline{P}_k \leq 1, \overline{P}_k = 1 - P_k$$

Similarly

$$P_{k(N_2:N_2)} = G_{N_2}\left(\overline{P}_k\right).$$

Also the survival function of $X_{(1:N_1)}, \overline{P}_{k(1:N_1)}\, is$

$$\overline{P}_{k(1:N_1)} = \sum_{n=1}^{\infty} \left(\overline{P}_k\right)^n P[N_1 = n]$$

$$\sum_{n=1}^{\infty} (1 - P_K)^n P[N_1 = n]$$

$$= G_{N_1}(P), 0 < P_k \leq 1$$

Similarly, $\overline{P}_{k(1:N_2)} = G_{N_2}(P_k)$

$$N_1 \leq_G N_2 \Leftrightarrow \overline{P}_{k(1:N_1)} \leq \overline{P}_{k(1:N_2)}$$

$$\Leftrightarrow X_{(1:N_1)} \leq_{st} X_{(1:N_2)}$$

$$N_1 \leq_G N_2 \Leftrightarrow \overline{P}_{k(N_1:N_1)} \leq \overline{P}_{k(N_2:N_2)}$$

$$P_{k(N_1:N_1)} \geq P_{k(N_2:N_2)}$$

$$\Leftrightarrow X_{(N_1:N_1)} \geq_{st} X_{(N_2:N_2)}$$

Conclusion

Similar to continuous ageing classes, discrete classes can be classified according to various stochastic oderings. These discrete classes have been extensively used in different fields such as insurance, finance, reliability, survival analysis and others. In this paper, a. p. g. f. ordering, a discrete analogue of Laplace transform ordering and its properties and certain shock models leading to $\mathcal{G}^*$ class are studied. It has been shown that a.p.g.f ordering between two shock survival functions $\overline{P}_k$ and $\overline{Q}_k$ are preserved by survival function of the system. It has also been shown that it is necessary and sufficient for the survival function of the system to belong to L class is that the survival probability of surviving k shocks belongs to $\mathcal{G}^*$ class, under the assumption that the shock occuring randomly in time according to a Poisson process. If the failure of the system is triggered by a sufficient number of shocks, we proved that the survival probability function is in $\mathcal{G}^*\left(\overline{\mathcal{G}}^*\right)$ class only if the critical threshold is in $\mathcal{L}(\overline{\mathcal{L}})$ under the assumption that the damage is accumulated additively and the shocks do not damage the system unless the accumulated shocks exceeds a critical thershold. Finally stochastic ordering of random maxima and minima has studied in relation to a. p. g. f. ordering.

Acknowledgment

The authors thank the referee for pointing out some inadequacies that the earlier version of the manuscript had, and for valuable comments.

Conflicts of interest

There is no conflicts of interest.

Funding

None.

References

1. Barlow RE, Proschan F. Statistical theory of reliability and life testing: probability models. Florida State Univ Tallahassee.1975.
2. Cai J, Kalashnikov V. NWU property of a class of random sums. *Journal of applied probability*. 2000;37(1):283–289.
3. Cai J, Willmot GE. Monotonicity and aging properties of random sums. Statistics & probability letters, 2005;73(4):381–392.
4. Lai CD, Xie M. *Stochastic ageing and dependence for reliability*. Springer Science & Business Media. 2006.
5. Shaked M, Santhikumar JG. *Stochastic Orders and their Applications*. Academic Press, New York. 1994.
6. Shaked M, Santhikumar JG, Valdez–Torres JB. Discrete hazard rate functions. *Computers and Operation Research*. 1995;22:391–402.
7. Willmot GE, Cai J. Aging and other distributional properties of discrete compound eometric distributions. Insurance: Mathematics and Economics. 2001;28(3),361–379.
8. Willmot GE, Lin XS. *Lundberg approximations for compound distributions with insurance applications.* Springer, New York. 2000.
9. Willmot GE, Drekic S, Cai J. Equilibrium compound distributions and stop–loss moments. *Scandinavian Actuarial Journal*. 2005; 2005(1):6–24.
10. Bryson MC, Siddiqui MM. Some criteria for ageing. *J Amer Statist Assoc*. 1969;64(328):1472–1483.
11. Klefsjö B. The HNBUE and HNWUE classes of life distributions. *Naval Res Logist*. Quart. 1982;29(2):331–344.
12. Klefsjö B. A useful ageing property based on the Laplace transforms. *J Appl Prob*. 1983;20(3):685–616.
13. Shaked M, Wong T. Stochastic orders based on ratios of Laplace transforms. *J Appl Prob.* 1997;34(2):404–419.
14. Jayamol KV, Jose KK. Stochastic ordering with respect to alternating probability generating function. STARS, *Int Journal (Sciences)*. 2008;2(1):20–28.
15. Jayamol KV, Jose KK. On G and G(α) –classes of life distributions. *Statistical Methods, special issue*, 2006;105–120.
16. Esary JD, Marshall AW, Proschan F. Shock Models and wear processes. *Ann Prob*. 1973;1(4):627–650.

17. Ahmed AH. The generalized HNBUE (HNWUE) class of life distributions. *Zeitschrift für Operations Research.* 1990; 34(3):183–194.

18. Singh H, Jain, K. Preservation of some partial orderings under Poisson models. *Adv Appl Prob*. 1989;21(3):713–716.

19. A–Hameed M, Proschan F. Non–stationary shock models. *Stoc Proc Appl.* 1973;1:383–404.

20. Gupta D, Gupta RC. On the distribution of order statistics for a random sample size. *Statistica neerlandica,* 1984;38(1):13–19.

21. Rohatgi VK. Distribution of order statistics with random sample size. *Communications in Statistics–Theory and Methods*, 1987;16(12):3739–3743.