Research Article

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A new three-parameter size-biased poisson-lindley distribution with properties and applications

Abstract

A new three-parameter size-biased Poisson-Lindley distribution which includes several one parameter and two-parameter size-biased distributions including size-biased geometric distribution (SBGD), size-biased negative binomial distribution (SBNBD), size-biased Poisson-Lindley distribution (SBPLD), size-biased Poisson-Shanker distribution (SBPSD), size-biased two-parameter Poisson-Lindley distribution-1 (SBTPPLD-1), size-biased two-parameter Poisson-Lindley distribution-2(SBTPPLD-2), size-biased quasi Poisson-Lindley distribution (SBQPLD) and size-biased new quasi Poisson-Lindley distribution (SBNQPLD) for particular cases of parameters has been proposed. Its various statistical properties based on moments including coefficient of variation, skewness, kurtosis and index of dispersion have been studied. Maximum likelihood estimation has been discussed for estimating the parameters of the distribution. Goodness of fit of the proposed distribution has been discussed.

Keywords: three-parameter Lindley distribution, new three-parameter Poisson-Lindley distribution, size-biased distributions, maximum likelihood estimation, goodness of fit

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Abbreviations: SBGD, size-biased geometric distribution; SBNBD, size-biased negative binomial distribution; SBPLD, size-biased Poisson-Lindley distribution; SBPSD, size-biased Poisson-Shanker distribution; SBTPPLD-1, size-biased two-parameter Poisson-Lindley distribution-1; SBTPPLD-2, size-biased two-parameter Poisson-Lindley distribution-2; SBQPLD, size-biased quasi Poisson-Lindley distribution; SBNQPLD, size-biased new quasi Poisson-Lindley distribution; ATPLD, A three- parameter Lindley distribution

Introduction

A three- parameter Lindley distribution (ATPLD) introduced by Shanker et al.,[1] is defined by its probability density function (pdf) and cumulative distribution function (cdf)

$$f(x;\theta,\alpha,\beta)=\frac{\theta^2}{\theta\alpha+\beta}(\alpha+\beta x)e^{-\theta x}\ ;x>0,\theta>0,\beta>0,\theta\alpha+\beta>0 \quad (1.1)$$

$$F(x;\theta,\alpha,\beta)=1-\left[1+\frac{\theta\beta x}{\theta\alpha+\beta}\right]e^{-\theta x}\ ;x>0,\theta>,\beta>0,\theta\alpha+\beta>0 \quad (1.2)$$

It has been observed that ATPLD is a convex combination of exponential (θ) and gamma $(2,\theta)$ distributions with mixing proportion $p=\frac{\theta\alpha}{\theta\alpha+\beta}$. Shanker et al.,[1] discussed its statistical properties, estimation of parameters using maximum likelihood estimation and applications to model lifetime data. Further, ATPLD includes several one parameter and two-parameter lifetime distributions for particular values of parameters θ,α and β. The particular distributions of (1.2) are summarized in table 1 along with their pdf and introducers.

Although Lindley distribution was proposed by Lindley,[2] but various statistical properties of Lindley distribution was studied by Ghitany et al.[3] Statistical properties, estimation of parameters and applications of the particular distributions of ATPLD given in table 1 are available in the respective papers.

Recently, Das et al.[4] proposed a new three-parameter Poisson-Lindley distribution (NTPPLD) by mixing Poisson distribution with ATPLD introduced by Shanker et al.[1] given in (1.1). The probability mass function of NTPPLD proposed by Das et al.[4] is given by

$$P_0(x;\theta,\alpha,\beta)=\frac{\theta^2}{\theta\alpha+\beta}\frac{\beta x+(\theta\alpha+\alpha+\beta)}{(\theta+1)^{x+2}};x=0,1,2,...,\theta>0,\alpha>0,\theta\alpha+\beta>0 \quad (1.3)$$

Table 1 Particular continuous distributions for specific values of parameters of ATPLD with probability density function and its introducers (year)

Parameter Values	Probability density function	Name of the distribution	Introducers (years)
$\alpha=1,\beta=0$	$f(x;\theta)=\theta e^{-\theta x};x>0$	Exponential distribution	
$\alpha=\beta=1$	$f(x;\theta)=\frac{\theta^2}{\theta+1}(1+x)e^{-\theta x};x>0$	Lindley distribution	Lindley[2]
$\alpha=\theta,\beta=1$	$f(x;\theta)=\frac{\theta^2}{\theta^2+1}(\theta+x)e^{-\theta x};x>0$	Shanker distribution	Shanker[11]
$\beta=1$	$f(x;\theta,\alpha)=\frac{\theta^2}{\theta\alpha+1}(\alpha+x)e^{-\theta x};x>0$	Two-parameter Lindley distribution-1 (TPLD-1)	Shanker and Mishra[12]

Table continue

Parameter Values	Probability density function	Name of the distribution	Introducers (years)
$\alpha=1$	$f(x;\theta,\beta)=\frac{\theta^2}{\theta+\beta}(1+\beta x)e^{-\theta x};x>0$	Two-parameter Lindley distribution-2 (TPLD-2)	Shanker *et al.*[13]
$\beta=\theta$	$f(x;\theta,\alpha)=\frac{\theta}{\alpha+1}(\alpha+\theta x)e^{-\theta x};x>0$	Quasi Lindley distribution (QLD)	Shanker and Mishra[14]
$\alpha=\theta$	$f(x;\theta,\beta)=\frac{\theta^2}{\theta^2+\beta}(\theta+\beta x)e^{-\theta x};x>0$	New Quasi Lindley distribution (NQLD)	Shanker and Amanuel[15]

Statistical properties including moments based measures, generating functions, estimation of parameters and applications of the distribution have been discussed by Das et al.[4]

It has been observed that NTPPLD includes several one parameter and two-parameter discrete distributions based on Poisson mixture of lifetime distributions given in table 1. The particular discrete distributions of (1.3) for particular values of parameters θ,α and β are summarized in table 2 along with their probability mass function (pmf) and introducers (year).

The first four moments about origin and the variance of NTPPLD, obtained by Das et al.,[4] are given by

$$\mu_1'=\frac{\theta\alpha+2\beta}{\theta(\theta\alpha+\beta)} \qquad \mu_2'=\frac{\theta^2\alpha+2(\alpha+\beta)\theta+6\beta}{\theta^2(\theta\alpha+\beta)}$$

$$\mu_3'=\frac{\theta^3\alpha+6(\alpha+\beta)\theta^2+6(\alpha+3\beta)\theta+24\beta}{\theta^3(\theta\alpha+\beta)}$$

$$\mu_4'=\frac{\theta^4\alpha+2(7\alpha+\beta)\theta^3+6(6\alpha+7\beta)\theta^2+24(\alpha+6\beta)\theta+120\beta}{\theta^4(\theta\alpha+\beta)}.$$

$$\mu_2=\sigma^2=\frac{\theta^3\alpha^2+(\alpha+3\beta)\theta^2\alpha+2(2\alpha+\beta)\theta\beta+2\beta^2}{\theta^2(\theta\alpha+\beta)^2}.$$

Table 2 Particular discrete distributions for specific values of parameters of NTPPLD with pmf and its introducers (year)

Parameter Values	Probability mass function	Name of the distribution	Introducers (years)
$\beta=0,\alpha=1$	$P(X=x)=\frac{\theta}{\theta+1}\left(\frac{1}{\theta+1}\right)^x;x=0,1,2,...$	Geometric distribution	
$\alpha=0,\beta=1$	$P(X=x)=(x+1)\left(\frac{\theta}{\theta+1}\right)^2\left(\frac{1}{\theta+1}\right)^x;x=0,1,2,...$	Negative Binomial distribution	Greenwood and Yule[16]
$\alpha=\beta=1$	$P(X=x)=\frac{\theta^2(x+\theta+2)}{(\theta+1)^{x+3}};x=0,1,2,...$	Poisson-Lindley distribution (PLD)	Sankaran[17]
$\alpha=\theta,\beta=1$	$P(X=x)=\frac{\theta^2}{\theta^2+1}\cdot\frac{x+(\theta^2+\theta+1)}{(\theta+1)^{x+2}};x=0,1,2,...$	Poisson-Shanker distribution (PSD)	Shanker[6]
$\alpha=1$	$P(X=x)=\frac{\theta^2}{\theta^2+\beta}\cdot\frac{\beta x+\theta^2+\theta+\beta}{(\theta+1)^{x+2}};x=0,1,2,...$	Two-parameter Poisson-Lindley distribution-1 (TPPLD-1)	Shanker et al.[18]
$\beta=1$	$P(X=x)=\frac{\theta^2}{\theta\alpha+1}\cdot\frac{x+\theta\alpha+\alpha+1}{(\theta+1)^{x+2}};x=0,1,2,...$	Two-parameter Poisson-Lindley distribution-2 (TPPLD-2)	Shanker and Mishra[18]
$\beta=\theta$	$P(X=x)=\frac{\theta}{\alpha+1}\cdot\frac{\theta x+\theta\alpha+\alpha+\theta}{(\theta+1)^{x+2}};x=0,1,2,...$	Quasi Poisson-Lindley distribution (QPLD)	Shanker and Mishra[19]
$\alpha=\theta$	$P(X=x)=\frac{\theta^2}{\theta^2+\beta}\cdot\frac{\beta x+\theta^2+\theta+\beta}{(\theta+1)^{x+2}};x=0,1,2,...$	New Quasi Poisson-Lindley distribution (NQPLD)	Shanker and Tekie[20]

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The main purpose of this paper is to propose a new three-parameter size-biased Poisson-Lindley distribution which includes several one parameter and two-parameter size-biased distributions for particular cases of parameters. Its moments have been derived and various statistical properties based on moments have been studied. Maximum likelihood estimation has been discussed. Goodness of fit of the distribution has been discussed with several count datasets.

A new three-parameter size-biased poisson-lindley distribution

Using the pmf (1.3) and the mean of NTPPLD, a new three-parameter size-biased Poisson-Lindley distribution (NTPSBPLD) can be obtained as

$$P_1(x;\theta,\alpha,\beta)=\frac{x\cdot P_0(x;\theta,\alpha,\beta)}{\mu_1'}=\frac{\theta^3}{\theta\alpha+2\beta}\frac{x(\beta x+\theta\alpha+\alpha+\beta)}{(\theta+1)^{x+2}} \quad (2.1)$$

$$;x=1,2,3,..,(\theta,\alpha,\beta)>0$$

It can be easily verified that NTPSBPLD contains several one-parameter and two-parameter size-biased distributions including size-biased geometric distribution (SBGD), size-biased negative binomial distribution (SBNBD), size-biased Poisson-Lindley distribution (SBPLD) proposed by Ghitany and Mutairi,[5] size-biased Poisson-Shanker distribution(SBPSD) proposed by Shanker,[6] size-biased two-parameter Poisson-Lindley distribution-1 (SBTPPLD-1) introduced by Shanker,[7] size-biased two-parameter Poisson-Lindley distribution-2 (SBTPPLD-2) suggested by Shanker and Mishra,[8] size-biased quasi Poisson-Lindley distribution (SBQPLD) proposed by Shanker and Mishra[9] and size-biased new quasi Poisson-Lindley distribution (SBNQPLD) introduced by Shanker et al.,[10] respectively for $(\beta=0,\alpha=1)$, $(\alpha=0,\beta=1)$, $(\alpha=\beta=1)$, $(\alpha=\theta,\beta=1)$, $(\alpha=1)$, $(\beta=1)$, $(\beta=\theta)$, $(\alpha=\theta)$ respectively.

Various characteristics of a distribution are based on their moments and it not easy to derive the moments of NTPSBPLD directly. Therefore, to derive the moments of NTPSBPLD, the pmf of NTPSBPLD can also be obtained as follows:

Let the random variable X follows the size-biased Poisson distribution (SBPD) with parameter λ and pmf

$$g(x\mid\lambda)=\frac{e^{-\lambda}\lambda^{x-1}}{(x-1)!}\;;x=1,2,3,...,;\lambda>0 \quad (2.2)$$

Suppose the parameter λ of SBPD follows the size-biased three-parameter Lindley distribution with pdf

$$h(\lambda;\theta)=\frac{\theta^3}{\theta\alpha+2\beta}\lambda(\alpha+\lambda\beta)e^{-\theta\lambda}\;;\lambda>0,(\theta,\alpha,\beta)>0 \quad (2.3)$$

Thus the pmf of NTPSBPLD can be obtained as

$$P(X=x)=\int_0^\infty g(x\mid\lambda)\cdot h(\lambda;\theta)\,d\lambda$$

$$=\int_0^\infty \frac{e^{-\lambda}\lambda^{x-1}}{(x-1)!}\frac{\theta^3}{\theta\alpha+2\beta}\lambda(\alpha+\lambda\beta)e^{-\theta\lambda}\,d\lambda \quad (2.4)$$

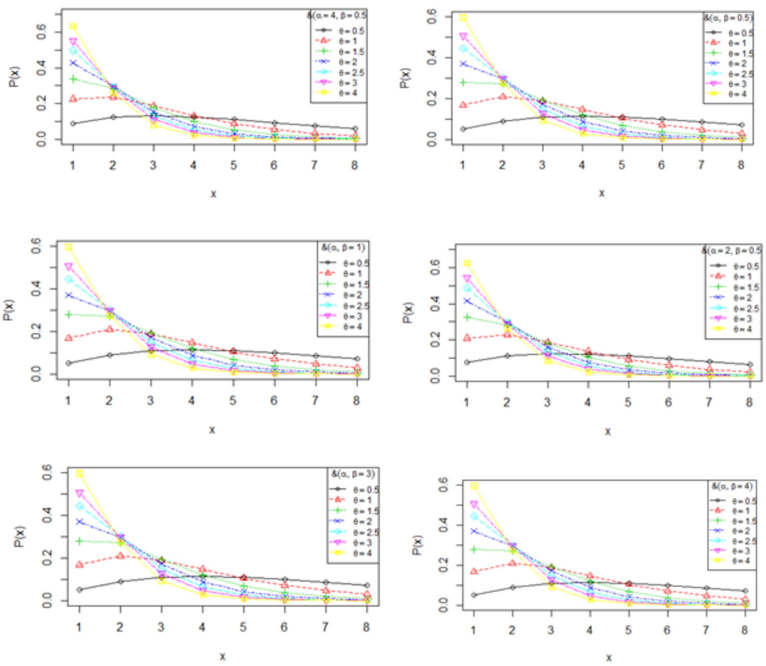
$$=\frac{\theta^3}{(\theta\alpha+2\beta)(x-1)!}\int_0^\infty e^{-(\theta+1)\lambda}\lambda^x(\alpha+\lambda\beta)\,d\lambda$$

$$=\frac{\theta^3}{(\theta\alpha+2\beta)(x-1)!}\left[\frac{\alpha\Gamma(x+1)}{(\theta+1)^{x+1}}+\frac{\beta\Gamma(x+2)}{(\theta+1)^{x+2}}\right]$$

$$=\frac{\theta^3}{\theta\alpha+2\beta}\frac{\beta x^2+(\theta\alpha+\alpha+\beta)x}{(\theta+1)^{x+2}}\;;x=1,2,3,..,(\theta,\alpha,\beta)>0$$

which is the pmf of NTPSBPLD obtained in (2.1).

The behavior of the pmf of NTPSBPLD for varying values of parameters (θ,α,β) has been shown in Figure 1.



Citation: Shanker R, Shukla KK. A new three-parameter size-biased poisson-lindley distribution with properties and applications. *Biom Biostat Int J.* 2020;9(1):1‒14. DOI: 10.15406/bbij.2020.09.00294

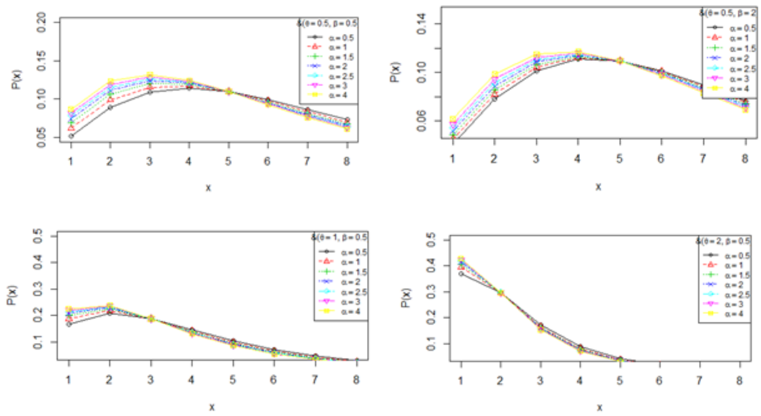


Figure 1 Behavior of NTPSBPLD for (θ,α,β).

Moments

Using (2.4), the r th factorial moment about origin $\mu_{(r)}'$ of the NTPSBPLD (2.1) can be obtained as

$$\mu_{(r)}' = E\left[E\left(X^{(r)}\,|\,\lambda\right)\right], \text{ where } X^{(r)} = X(X-1)(X-2)\dots(X-r+1)$$

$$= \int_0^\infty \left[\sum_{x=1}^\infty x^{(r)} \frac{e^{-\lambda}\lambda^{x-1}}{(x-1)!}\right]\cdot\frac{\theta^3}{\theta\alpha+2\beta}\lambda(\alpha+\lambda\beta)e^{-\theta\lambda}d\lambda$$

$$= \int_0^\infty \left[\lambda^{r-1}\left\{\sum_{x=r}^\infty x\frac{e^{-\lambda}\lambda^{x-r}}{(x-r)!}\right\}\right]\cdot\frac{\theta^3}{\theta\alpha+2\beta}\lambda(\alpha+\lambda\beta)e^{-\theta\lambda}d\lambda$$

Taking $y = x-r$, we get

$$\mu_{(r)}' = \int_0^\infty \left[\lambda^{r-1}\left\{\sum_{y=0}^\infty (y+r)\frac{e^{-\lambda}\lambda^{y}}{y!}\right\}\right]\cdot\frac{\theta^3}{\theta\alpha+2\beta}\lambda(\alpha+\lambda\beta)e^{-\theta\lambda}d\lambda$$

$$= \frac{\theta^3}{\theta\alpha+2\beta}\int_0^\infty \lambda^r(\lambda+r)(\alpha+\lambda\beta)e^{-\theta\lambda}d\lambda$$

$$= \frac{\theta^3}{\theta\alpha+2\beta}\int_0^\infty \left\{\beta\lambda^{r+2}+(\alpha+\beta r)\lambda^{r+1}+r\alpha\lambda^r\right\}e^{-\theta\lambda}d\lambda$$

After a little tedious algebraic simplification, the r th factorial moment about origin of NTPSBPLD (2.1) can be expressed as

$$\mu_{(r)}' = \frac{r!\left\{r\alpha\theta^2+(r+1)(\alpha+r\beta)\theta+(r+1)(r+2)\beta\right\}}{\theta^r(\theta\alpha+2\beta)};r=1,2,3,\dots \quad (3.1)$$

The first four factorial moments about origin can be obtained by taking $r=1,2,3,$ and 4 in (3.1). The first four moments about origin of the NTPSBPLD, using the relationship between moments about origin and factorial moments about origin, are obtained as

$$\mu_1' = \frac{\alpha\theta^2+2(\alpha+\beta)\theta+6\beta}{\theta(\theta\alpha+2\beta)}$$

$$\mu_2' = \frac{\alpha\theta^3+(6\alpha+2\beta)\theta^2+(6\alpha+18\beta)\theta+24\beta}{\theta^2(\theta\alpha+2\beta)}$$

$$\mu_3' = \frac{\alpha\theta^4+(14\alpha+2\beta)\theta^3+(36\alpha+42\beta)\theta^2+(24\alpha+144\beta)\theta+120\beta}{\theta^3(\theta\alpha+2\beta)}$$

$$\mu_4' = \frac{\alpha\theta^5+(30\alpha+2\beta)\theta^4+(150\alpha+90\beta)\theta^3+(240\alpha+600\beta)\theta^2+(120\alpha+1200\beta)+720\beta}{\theta^4(\theta\alpha+2\beta)}$$

Now, using the relationship between moments about mean and the moments about origin, the moments about mean of the NTPSBPLD (2.1) can be obtained as

$$\mu_2 = \frac{2\left\{\alpha^2\theta^3+(\alpha^2+5\alpha\beta)\theta^2+(6\beta^2+6\alpha\beta)\theta+6\beta^2\right\}}{\theta^2(\theta\alpha+2\beta)^2}$$

$$\mu_3 = \frac{2\left\{\begin{array}{l}\alpha^3\theta^5+(7\alpha^2\beta+3\alpha^3)\theta^4+(16\alpha\beta^2+24\alpha^2\beta+2\alpha^3)\theta^3\\+(54\alpha\beta^2+12\beta^3+18\alpha^2\beta)\theta^2+(36\alpha\beta^2+36\beta^3)\theta+24\beta^3\end{array}\right\}}{\theta^3(\theta\alpha+2\beta)^3}$$

$$\mu_4 = \frac{2\left\{\begin{array}{l}\alpha^4\theta^7+(13\alpha^4+9\alpha^3\beta)\theta^6+(30\alpha^2\beta^2+130\alpha^3\beta+24\alpha^2)\theta^5\\+(460\alpha^2\beta^2+44\alpha\beta^3+264\alpha^3\beta+12\alpha^4)\theta^4+(936\alpha^2\beta^2+24\beta^4+144\alpha^3\beta+696\alpha\beta^3)\theta^3\\+(384\beta^4+1368\alpha\beta^3+504\alpha^2\beta^2)\theta^2+(720\beta^4+720\alpha\beta^3)\theta+360\beta^4\end{array}\right\}}{\theta^4(\theta\alpha+2\beta)^4}$$

The coefficient of variation $(C.V)$, coefficient of Skewness $\left(\sqrt{\beta_1}\right)$, coefficient of Kurtosis (β_2) and index of dispersion (γ) of the NTPSBPLD (2.1)) are thus obtained as

$$C.V = \frac{\sigma}{\mu_1'} = \frac{\sqrt{2\left\{\alpha^2\theta^3+(\alpha^2+5\alpha\beta)\theta^2+(6\beta^2+6\alpha\beta)\theta+6\beta^2\right\}}}{\left\{\alpha\theta^2+2(\alpha+\beta)\theta+6\beta\right\}}$$

$$\sqrt{\beta_1} = \frac{\mu_3}{\mu_2^{3/2}} = \frac{\left\{\begin{array}{l}\alpha^3\theta^5+(7\alpha^2\beta+3\alpha^3)\theta^4+(16\alpha\beta^2+24\alpha^2\beta+2\alpha^3)\theta^3\\+(54\alpha\beta^2+12\beta^3+18\alpha^2\beta)\theta^2+(36\alpha\beta^2+36\beta^3)\theta+24\beta^3\end{array}\right\}}{\sqrt{2}\left\{\alpha^2\theta^3+(\alpha^2+5\alpha\beta)\theta^2+(6\beta^2+6\alpha\beta)\theta+6\beta^2\right\}^{3/2}}$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{\left\{\begin{array}{l}\alpha^4\theta^7+(13\alpha^4+9\alpha^3\beta)\theta^6+(30\alpha^2\beta^2+130\alpha^3\beta+24\alpha^2)\theta^5\\+(460\alpha^2\beta^2+44\alpha\beta^3+264\alpha^3\beta+12\alpha^4)\theta^4+(936\alpha^2\beta^2+24\beta^4+144\alpha^3\beta+696\alpha\beta^3)\theta^3\\+(384\beta^4+1368\alpha\beta^3+504\alpha^2\beta^2)\theta^2+(720\beta^4+720\alpha\beta^3)\theta+360\beta^4\end{array}\right\}}{2\left\{\alpha^2\theta^3+(\alpha^2+5\alpha\beta)\theta^2+(6\beta^2+6\alpha\beta)\theta+6\beta^2\right\}^2}$$

$$\gamma = \frac{\sigma^2}{\mu_1'} = \frac{2\left\{\alpha^2\theta^3 + \left(\alpha^2 + 5\alpha\beta\right)\theta^2 + \left(6\beta^2 + 6\alpha\beta\right)\theta + 6\beta^2\right\}}{\theta\left(\theta\alpha + 2\beta\right)\left\{\left(\alpha\theta^2 + 2\left(\alpha + \beta\right)\theta + 6\beta\right)\right\}}$$

The graphs of coefficient of variation $(C.V)$, coefficient of Skewness $(\sqrt{\beta_1})$, coefficient of Kurtosis (β_2) and index of dispersion (γ) of the NTPSBPLD are shown in figures 2,3,4 and 5 respectively.

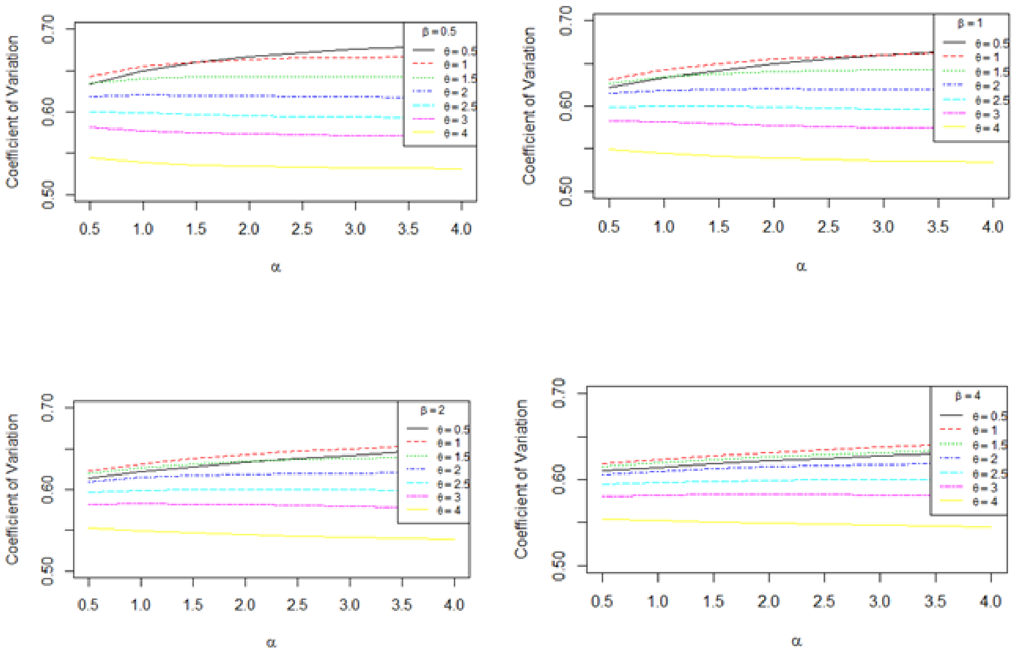


Figure 2 Graphs of coefficient of Variation of the NTPSBPLD for varying values of the parameters (θ, α, β).

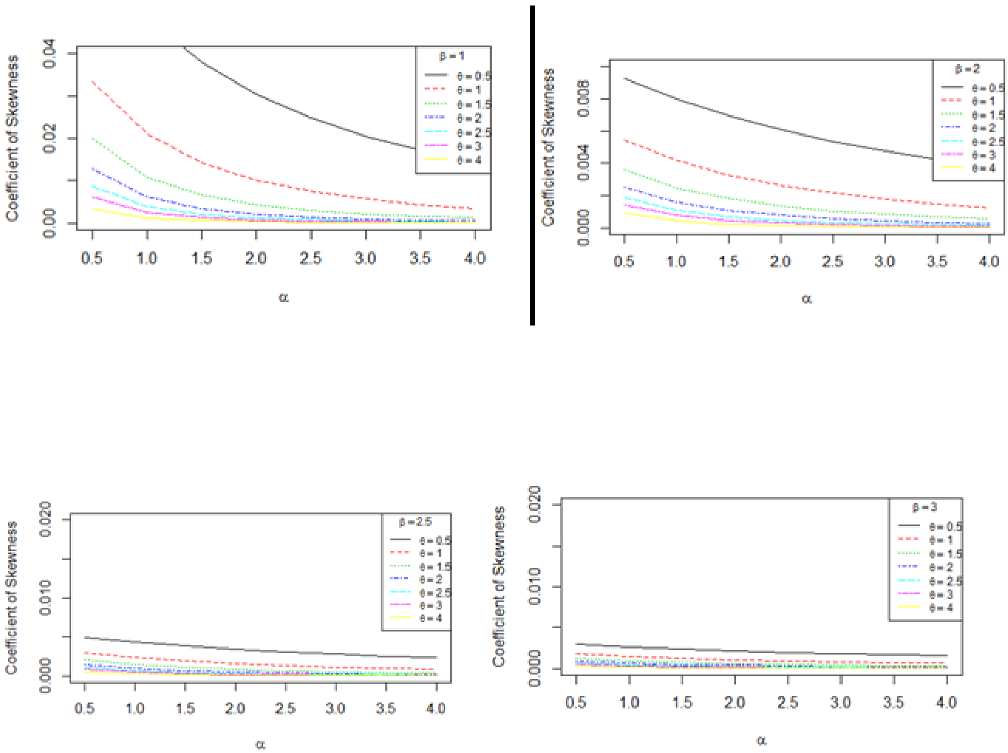


Figure 3 Graphs of coefficient of Skewness of the NTPSBPLD for varying values of the parameters (θ, α, β).

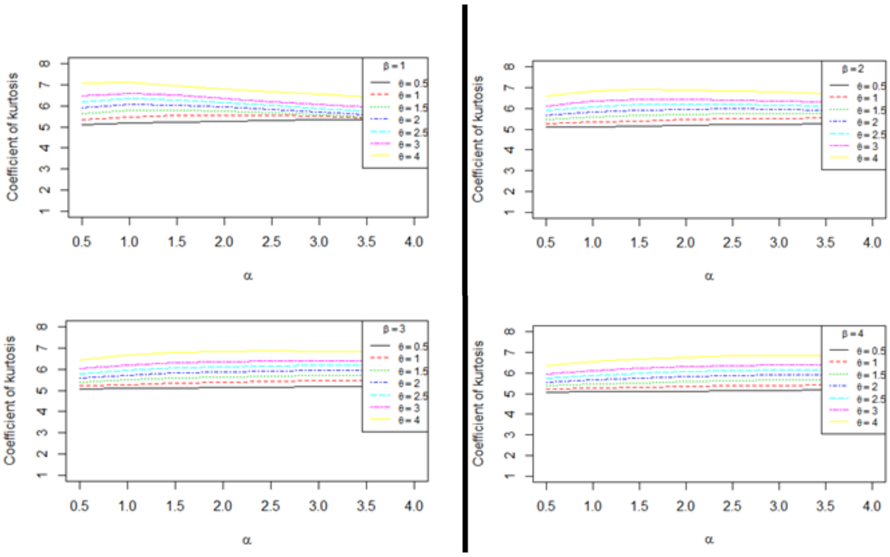


Figure 4 Graphs of Coefficient of Kurtosis of the NTPSBPLD for varying values of the parameter (θ,α,β).

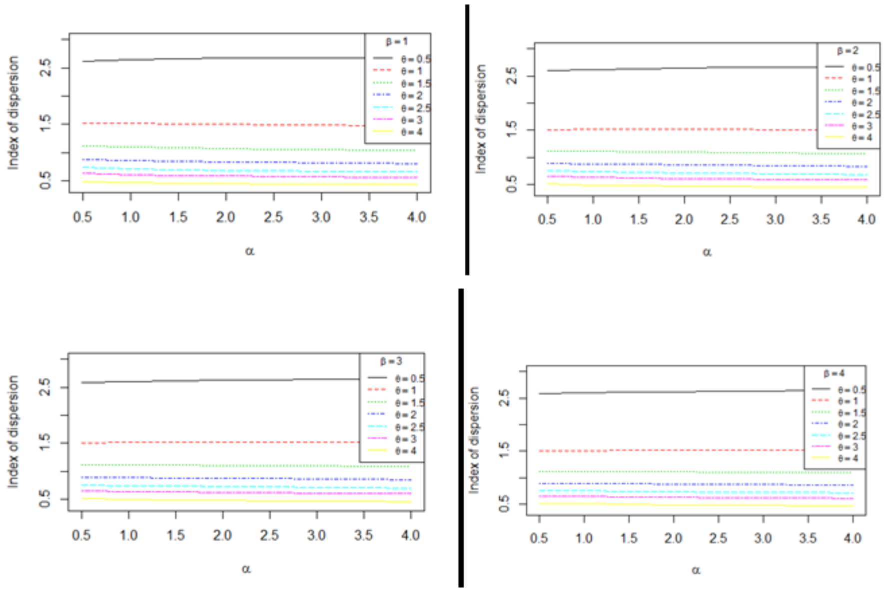


Figure 5 Index of dispersion of the NTPSBPLD for varying values of the parameter (θ,α,β).

Maximum likelihood estimation

Let us consider $(x_1,x_2,x_3,...,x_n)$ as random sample from NTPSBPLD (θ,α,β). Suppose f_x be the observed frequency in the sample corresponding to $X=x\ (x=1,2,3,...,k)$ such that $\sum_{x=1}^{k} f_x = n$, where k is the largest observed value having non-zero frequency. The likelihood function L of NTPSBPLD (θ,α,β) can be expressed as

$$L=\left(\frac{\theta^3}{\theta\alpha+2\beta}\right)^n \frac{1}{(\theta+1)^{\sum_{x=1}^{k} f_x(x+2)}} \prod_{x=1}^{k}\left[\beta x^2+(\theta\alpha+\alpha+\beta)x\right]^{f_x}.$$

The log likelihood function of NTPSBPLD (θ,α,β) is

$$\log L = n\left\{3\log\theta-\log(\theta\alpha+2\beta)\right\}-\sum_{x=1}^{k} f_x(x+2)\log(\theta+1)+\sum_{x=1}^{k} f_x\log\left\{\beta x^2+(\theta\alpha+\alpha+\beta)x\right\}$$

The maximum likelihood estimates, MLE's $(\hat{\theta},\hat{\alpha},\hat{\beta})$, of parameters (θ,α,β) of NTPSBPLD (θ,α,β) is the solutions of the following log likelihood equations

$$\frac{\partial\log L}{\partial\theta}=\frac{3n}{\theta}-\frac{3n\alpha}{\theta\alpha+2\beta}-\frac{n(\bar{x}+2)}{\theta+1}+\sum_{x=1}^{k}\frac{\alpha x f_x}{\beta x^2+(\theta\alpha+\alpha+\beta)x}=0$$

$$\frac{\partial\log L}{\partial\alpha}=-\frac{n\theta}{\theta\alpha+2\beta}+\sum_{x=1}^{k}\frac{(\theta+1)x f_x}{\beta x^2+(\theta\alpha+\alpha+\beta)x}=0$$

$$\frac{\partial\log L}{\partial\beta}=-\frac{2n}{\theta\alpha+2\beta}+\sum_{x=1}^{k}\frac{(x^2+x)f_x}{\beta x^2+(\theta\alpha+\alpha+\beta)x}=0,$$

where $\bar{x}$ is the sample mean.

Since these log likelihood equations cannot be expressed in closed forms and hence do not seem to be solved directly, the (MLE's) $\left(\hat{\theta},\hat{\alpha},\hat{\beta}\right)$ of parameters $\left(\theta,\alpha,\beta\right)$ can be computed directly by solving the log likelihood equation using R-software till sufficiently close estimates of $\left(\hat{\theta},\hat{\alpha},\hat{\beta}\right)$ are attained.

Goodness of fit

The goodness of fit of NTPSBPLD has been discussed with several count data from various fields of knowledge. The expected frequencies according to the SBPLD, SBQPLD and SBNQPLD using maximum likelihood estimates of parameters have also been given in these tables for ready comparison with those obtained by the NTPSBPLD. Clearly the goodness of fit of NTPSBPLD provides better fit over SBPLD and competing well with SBQPLD and SBNQPLD in majority of datasets. In some of the tables the degree of freedom is zero, and hence p-values have not been given and thus in such tables comparisons can be done on the basis of values of $-2\log L$ and AIC (Akaike information criterion). The datasets considered for testing the goodness of fit of SBPLD, SBQPLD, SBNQPLD and NTPSBPLD as follows: (Tables i-x).

Table i Pedestrians-Eugene, Spring, Morning, available in Coleman and James[21]

Group Size	Observed frequency	Expected frequency			
		SBPLD	**SBQPLD**	**SBNQPLD**	**NTPSBPLD**
1	1486	1532.5	1485.4	1505.5	1485.4
2	694	630.6	697.2	656.8	697.2
3	195	191.9	189.7	202.5	189.7
4	37	51.3	41.1	49.2	41.1
5	10	12.8	7.8	9.0	7.8
6	1	3.9	1.8	0.0	
Total	2423	2423.0	2423		
ML Estimate		$\hat{\theta}=4.5082$	$\hat{\theta}=7.14063$ $\hat{\alpha}=-0.79104$	$\hat{\theta}=2.69606$ $\hat{\alpha}=-1.39128$	$\hat{\theta}=7.1386$ $\hat{\alpha}=-0.9318$ $\hat{\beta}=8.4164$
χ^2		13.760	0.776	6.1	0.77
d.f.		3	2	2	1
p-value		0.003	0.6804	0.04735	0.3802
$-2\log L$		4622.36	4607.8	4610.0	4607.8
AIC		4624.36	4611.8	4614.0	4613.8

Table ii Play Groups-Eugene, spring, Public Playground A, available in Coleman and James[21]

1	316	322.9	315.7	313.5	315.7
2	141	132.5	142.7	141.4	142.7
3	44	40.2	40.1	44.1	40.1
4	5	10.7	9.1	10.4	9.1
5	4	3.7	2.4	0.6	2.4
Total	510	510.0	510.0		

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Table continue

Group Size	Observed frequency	Expected frequency			
		SBPLD	SBQPLD	SBNQPLD	NTPSBPLD
ML Estimate		$\hat{\theta}=4.5211$	$\hat{\theta}=6.5501$ $\hat{\alpha}=-0.5069$	$\hat{\theta}=2.4693$ $\hat{\alpha}=-1.2977$	$\hat{\theta}=6.5560$ $\hat{\alpha}=-0.6029$ $\hat{\beta}=7.7499$
χ^2		3.07	0.94	0.38	0.94
d.f.		2	1	1	0
p-value		0.2154	0.3322	0.5376	
$-2\log L$		972.78	971.07	970.24	971.07
AIC		974.78	975.07	974.24	977.07

Table iii Play Groups-Eugene, spring, Public Playground A, available in Coleman and James[21]

Group Size	Observed frequency	Expected frequency			
		SBPLD	SBQPLD	SBNQPLD	NTPSBPLD
1	306	309.4	304.4	306.4	304.4
2	132	131.2	137.9	134.4	137.9
3	47	41.1	41.3	41.6	41.3
4	10	11.3	10.3	11.0	10.3
5	2	4.0	3.1	3.6	3.1
Total	497	497.0			
ML Estimate		$\hat{\theta}=4.3548$	$\hat{\theta}=5.71547$ $\hat{\alpha}=-0.06947$	$\hat{\theta}=4.9998$ $\hat{\alpha}=25.6948$	$\hat{\theta}=5.7156$ $\hat{\alpha}=-0.0708$ $\hat{\beta}=5.8180$
χ^2		0.932	1.19	1.2	1.19
d.f.		2	1	1	0
p-value		0.6281	0.2753	0.2733	
$-2\log L$		971.86	970.96	971.25	970.9
AIC		973.86	974.96	975.25	976.9

Table iv Play Groups-Eugene, Spring, Public Playground D, available in Coleman and James[21]

Group Size	Observed frequency	Expected frequency			
		SBPLD	SBQPLD	SBNQPLD	NTPSBPLD
1	305	314.4	304.3	310.1	304.3
2	144	134.4	148.2	138.8	148.2
3	50	42.5	42.3	43.1	42.3
4	5	11.8	9.6	11.3	9.6
5	2	3.1	1.9	2.7	1.9
6	1	0.8	0.7	1.0	0.7
Total	507	507.0	507.0	507.0	
ML Estimate		$\hat{\theta} = 4.3179$	$\hat{\theta} = 6.70804$ $\hat{\alpha} = -0.74907$	$\hat{\theta} = 5.1516$ $\hat{\alpha} = 48.6067$	$\hat{\theta} = 6.7082$ $\hat{\alpha} = -0.8290$ $\hat{\beta} = 7.4234$
χ^2		6.415	2.96	4.64	2.96
d.f.		2	1	1	0
p-value		0.040	0.0853	0.0312	
$-2\log L$		993.10	990.02	991.51	990.02
AIC		995.1	994.02	995.51	996.02

Table v Play Groups-Eugene, Spring, Public Playground D, available in Coleman and James[21]

Group Size	Observed frequency	Expected frequency			
		SBPLD	SBQPLD	SBNQPLD	NTPSBPLD
1	276	319.6	276.0	313.7	276.0
2	229	166.5	228.3	173.1	228.3
3	61	63.8	61.9	65.2	61.9
4	12	21.4	12.2	20.7	12.2
5	3	9.7	2.6	8.3	2.6
Total	581	581.0	581.0	581.0	581.0

Table continue

Group Size	Observed frequency	Expected frequency			
		SBPLD	SBQPLD	SBNQPLD	NTPSBPLD
ML Estimate		$\hat{\theta} = 3.4359$	$\hat{\theta} = 8.6724$ $\hat{\alpha} = -1.4944$	$\hat{\theta} = 4.1645$ $\hat{\alpha} = 61.0287$	$\hat{\theta} = 8.6726$ $\hat{\alpha} = -2.5854$ $\hat{\beta} = 15.0041$
χ^2		37.86	0.017	29.6	0.017
d.f.		2	1	1	0
p-value		0.00	0.8962	0.000	0.0000
$-2\log L$		1277.42	1238.11	1268.77	1238.11
AIC		1279.42	1242.11	1272.77	1244.11

Table vi Distribution of number of counts of sites with particles from Immunogold data, available in Mathews and Appleton[22]

No. of sites with particles	Observed Frequency	Expected Frequency			
		SBPLD	SBQPLD	SBNQPLD	NTPSBPLD
1	122	119.0	119.2	119.3	119.3
2	50	53.8	53.5	53.3	53.3
3	18	18.0	17.9	17.8	17.8
4	4	5.3	5.3	5.3	5.3
5	4	1.9	2.1	2.3	2.3
Total	198	198.0	198.0	198.0	198.0
ML estimate		$\hat{\theta} = 4.050987$	$\hat{\theta} = 3.7564$ $\hat{\alpha} = 10.1281$	$\hat{\theta} = 3.4795$ $\hat{\alpha} = 0.0216$	$\hat{\theta} = 3.4737$ $\hat{\alpha} = 1.3965$ $\hat{\beta} = 0.0001$
χ^2		0.43	0.34	0.28	0.28
d.f.		2	1	1	0
p-value		0.8065	0.5598	0.5967	
$-2\log L$		409.28	409.17	409.13	409.13
AIC		411.28	413.17	413.13	415.13

Table vii Distribution of snowshoe hares captured over 7 days, available in Keith and Meslow[23]

No. times hares caught	Observed Frequency	Expected Frequency			
		SBPLD	SBQPLD	SBNQPLD	NTPSBPLD
1	184	177.3	177.4	177.5	177.5
2	55	62.5	62.3	62.2	62.2
3	14	16.4 }	16.3	16.3	16.3
4	4	3.8 }	3.8	3.8	3.8
5	4	1.0 }	1.2	1.2	1.2
Total	261	261.0	261	261.0	
ML estimate		$\hat{\theta}=5.351256$	$\hat{\theta}=4.9800$ $\hat{\alpha}=14.9193$	$\hat{\theta}=4.6959$ $\hat{\alpha}=-0.0302$	$\hat{\theta}=4.6994$ $\hat{\alpha}=12.0044$ $\hat{\beta}=-0.0390$
χ^2		1.18	3.2	3.19	3.19
d.f.		1	1	1	0
p-value		0.2773	0.0736	0.07409	
$-2\log L$		457.10	456.86	456.80	456.80
AIC		459.10	460.86	460.80	462.80

Table viii Number of counts of pairs of running shoes owned by 60 members of an athletics club, reported by Simonoff[24]

Number of pairs of running shoes	Observed frequency	Expected Frequency			
		SBPLD	SBQPLD	SBNQPLD	NTPSBPLD
1	18	20.3	17.4	19.5	17.4
2	18	17.4	19.6	18.0	19.6
3	12	10.9	12.3	11.3	12.3
4	7	5.9	6.1	6.0	6.1
5	5	5.5	4.6	5.2	4.6
Total	60	60.0	60.0	60	60
ML Estimate		$\hat{\theta}=1.818978$	$\hat{\theta}=2.5858$ $\hat{\alpha}=-0.7318$	$\hat{\theta}=2.08739$ $\hat{\alpha}=17.3228$	$\hat{\theta}=2.5870$ $\hat{\alpha}=-0.4739$ $\hat{\beta}=1.6732$

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Table continue

Number of pairs of running shoes	Observed frequency	Expected Frequency			
		SBPLD	SBQPLD	SBNQPLD	NTPSBPLD
χ^2		0.64	0.31	0.33	0.31
d.f.		3	1	2	0
P-value		0.8872	0.5777	0.8478	
$-2\log L$		187.08	185.55	186.33	185.55
AIC		189.08	189.55	190.33	191.55

Table ix The numbers of counts of flower heads as per the number of fly eggs reported by Finney and Varley[25]

Number of fly eggs	Observed Frequency	Expected Frequency			
		SBPLD	SBQPLD	SBNQPLD	NTPSBPLD
1	22	20.3	19.8	19.8	19.8
2	18	22.0	22.1	22.1	22.1
3	18	17.2	17.5	17.5	17.5
4	11	11.6	11.8	11.8	11.8
5	9	7.2	7.3	7.3	7.3
6	6	4.2	4.2	4.2	4.2
7	3	2.4	2.3	2.3	2.3
8	0	1.3	1.3	1.3	1.3
9	1	1.8	1.7	1.7	1.7
Total	88		88.0	88.0	88.0
ML estimate		$\hat{\theta} = 1.2822$	$\hat{\theta} = 1.3483$ $\hat{\alpha} = 0.6925$	$\hat{\theta} = 1.3465$ $\hat{\alpha} = 2.5654$	$\hat{\theta} = 1.4477$ $\hat{\alpha} = 0.4315$ $\hat{\beta} = 1.3594$
χ^2		1.39	1.49	1.49	1.49
d.f.		4	3	3	3
p-value		0.8459	0.6845	0.6845	0.6845
$-2\log L$		329.92	329.86	329.86	329.86
AIC		331.92	333.86	333.86	335.86

Table x Number of households having at least one migrant (X) according to the number of observed migrants, reported by Singh and Yadav[26]

X	Observed frequency	Expected frequency			
		SBPLD	SBQPLD	SBNQPLD	NTPSBPLD
1	375	262.8	363.3	363.6	363.6
2	143	157.4	156.5	156.3	156.3
3	49	50.4	50.4	50.4	50.4
4	17	14.2	14.4	14.4	14.4
5	2	3.7	3.9	3.8	3.8
6	2	0.9	1.0	1.0	1.0
7	1	0.2	0.2	0.2	0.2
8	1	0.3	0.4	0.3	0.3
Total	590	590.0	590.0	590.0	590.0
ML Estimate		$\hat{\theta} = 4.24$	$\hat{\theta} = 3.8386$ $\hat{\alpha} = 17.2968$	$\hat{\theta} = 3.6534$ $\hat{\alpha} = 0.00067$	$\hat{\theta} = 3.6504$ $\hat{\alpha} = 12.9869$ $\hat{\beta} = -0.0377$
χ^2		2.48	2.11	2.08	2.08
d.f.		3	2	2	1
P-value		0.4789	0.3481	0.3534	0.1492
$-2\log L$		1190.4	1189.67	1189.57	1189.57
AIC		1192.4	1193.67	1193.57	1193.57

Conclusions

A new three-parameter size-biased Poisson-Lindley distribution which includes several size-biased distributions including size-biased geometric distribution (SBGD), size-biased negative binomial distribution (SBNBD), size-biased Poisson-Lindley distribution (SBPLD), size-biased Poisson-Shanker distribution (SBPSD), size-biased two-parameter Poisson-Lindley distribution-1 (SBTPPLD-1), size-biased two-parameter Poisson-Lindley distribution-2 (SBTPPLD-2), size-biased quasi Poisson-Lindley distribution (SBQPLD) and size-biased new quasi Poisson-Lindley distribution (SBNQPLD) for particular values of parameters has been proposed. Its coefficient of variation, skewness, kurtosis and index of dispersion has been studied. Estimation of parameters has been discussed using maximum likelihood. Goodness of fit of the proposed distribution has been discussed with several count datasets.

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Conflicts of interest

Authors declare that there is no conflict of interests.

References

1. Shanker R, Shukla KK, Shanker R, et al. A three–parameter lindley distribution. *American Journal of Mathematics and Statistics.* 2017;7(1):15–26.
2. Lindley DV. Fiducial distributions and Bayes' theorem. *Journal of the Royal Statistical Society, Series B.* 1958;20:102–107.
3. Ghitany ME, Atieh B, Nadarajah S. Lindley distribution and its application. *Mathematics Computing and Simulation*. 2008;78:493–506.
4. Das KK, Ahmad J, Bhattacharjee S. A new three–parameter poisson–lindley distribution for modeling over dispersed count data, *International journal of Applied Engineering Research.* 2018;13(25):16468–16477.

Citation: Shanker R, Shukla KK. A new three-parameter size-biased poisson-lindley distribution with properties and applications. *Biom Biostat Int J.* 2020;9(1):1–14. DOI: 10.15406/bbij.2020.09.00294

5. Ghitany ME, Al–Mutairi DK. Size–biased poisson–lindley distribution and its applications, *Metron.* 2008; LXVI(3):299–311.
6. Shanker R. A Size–biased poisson–shanker distribution and its applications. *International Journal of Probability and Statistics, 2017;*6(3):33–44.
7. Shanker R. Size–biased discrete two parameters poisson–lindley distribution and its applications, *Global Journal of Pure and Applied Mathematics*. 2013;9(4):325–334.
8. Shanker R, Mishra A. On size–biased two–parameter poisson–lindley distribution and its applications. *American Journal of Mathematics and Statistics*. 2017;7(3):99–107.
9. Shanker R, Mishra A. on a size–biased quasi poisson lindley distribution, *International Journal of Probability and Statistics.* 2013b;2(2):28–34.
10. Shanker R, Shanker R, Tekie AL, et al. Size–biased new quasi poisson lindley distribution and its applications, *International Journal of Statistics and Systems*. 2014; 9(2):109–118.
11. Shanker R. Shanker Distribution and Its Applications. *International Journal of Statistics and Applications.* 2015;5(6):338–348.
12. Shanker R, Mishra A. A two–parameter Lindley distribution, Statistics in Transition–new series. 2013a;14(1):45–56.
13. Shanker R, Sharma S, Shanker R. A two–parameter Lindley distribution for modeling waiting and survival times data. *Applied Mathematics.* 2013;4(2):363–368.
14. Shanker R, Mishra A. A quasi lindley distribution. *African Journal of Mathematics and Computer Science Research.* 6(4):64–71.
15. Shanker R, Amanuel AG. A new quasi lindley distribution, International Journal of Statistics and Systems. 2013;8(2):143–156.
16. Greenwood M, Yule Udny G. An Inquiry into the nature of frequency distributions representative of multiple happenings with particular reference to the occurrence of multiple attacks of disease or of repeated accidents. *Journal of the Royal Statistical Society*. 1920;83(2):255–279.
17. Sankaran M. The discrete Poisson–Lindley distribution, *Biometrics.* 1970;26(1):145–149.
18. Shanker R, Mishra A. A two–parameter poisson–lindley distribution, *International Journal of Statistics and Systems*. 2014;9(1):79–85.
19. Shanker R, Mishra A. A quasi Poisson– Lindley distribution. Journal of 2014. Indian Statistical Association. 2016;54(1&2):113–125.
20. Shanker R, Tekie AL. A new quasi Poisson–Lindley distribution, International Journal of Statistics and Systems. 2014;9(1):79 –85.
21. Coleman JS, James J. The equilibrium size distribution of freely forming groups. Sociometry, 1961;24:36–45.
22. Mathews JNS, Appleton DR. An application of the truncated Poisson distribution to Immunogold assay. *Biometrics*. 1993;49(2):617–621.
23. Keith LB, Meslow EC. Trap response by snowshoe hares. *Journal of Wildlife Management*. 1968;32(4):795–801.
24. Simonoff JS. Analyzing Categorical Data, New York Springer. 2003.
25. Finney DJ, Varley GC. An example of the truncated Poisson distribution, Biometrics, 1955;11(3):387–394.
26. Singh SN, Yadav RC. Trends in rural out–migration at household level. *Rural Demogr.* 1971;8(1):53–61.