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Second-order efficiency of fully sequential designs for estimating the product of two means with application in reliability estimation

Abstract

For estimating the product of two means from the general one parameter exponential family, we consider a fully Bayesian approach with conjugate priors. We derive a sharp lower bound for the Bayes Risk. We also propose a fully sequential design with an incurred Bayes Risk near the second order lower bound. An application to reliability estimation is performed analytically and through Monte Carlo simulation.

Keywords: Bayesian estimation, sequential designs, second-order optimality, one-parameter exponential family, reliability estimation

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Introduction

Solving estimation problems using a fully Bayesian approach for the one-parameter exponential family with conjugate priors for sequential designs was first investigated by Shapiro and Wardrop and developed for linear combinations of means thoroughly by Rekab et al. (2013) and many others.[1,2]

For estimating the product of means of two independent Bernoulli populations with Beta priors which has an application in estimating the reliability of a series system, Rekab & Song[3] have derived the first order lower bound for the Bayes Risk and proposed a sequential design that achieves it.

The first-order efficiency for any sequential procedure, ρ,[3] is

$$R(p) \geq \frac{E\left[\sum_{i=1}^{k}\sqrt{\psi(\eta_i)}\left|\prod_{j\neq i}^{k}\psi(\eta_j)\right|\right]^2}{T} + o\left(\frac{1}{T}\right).$$

In this article, we will derive a sharper lower bound for the Bayes Risk and propose a sequential design that will achieve it at least asymptotically. Such a design will be referred to as second order efficient design.

The problem of estimating system reliability is the same as estimating the product of means of independent Bernoulli populations. We use independent Beta priors for the means and propose a sequential design that is second order efficient, it converges faster to the optimal ratio than the first order designs. Second order sequential designs are sought and show the optimality of the fully sequential design through an application of reliability estimation using Monte Carlo simulation.

Second-order efficiency

The density function of one-parameter exponential family

$$f_{\eta_i}(x) = \exp\{\eta_i x - \varphi(\eta_i)\}\, d\wedge(x), -\infty < x < \infty, \eta_i \in \Omega, \quad (2.1)$$

where $\eta_i = \eta(\theta_i)$ is real-valued function, φ is a continuously differentiable, real-valued function, $\wedge$ is a non-degenerate sigma-finite measure and $\Omega = \left(\underline{\eta}, \overline{\eta}\right)$, $-\infty < \underline{\eta} < \overline{\eta} < \infty$, is a non-empty open interval, $i = 1,2$. Using the square error loss, and by adopting the Bayesian approach, the conjugate prior $\pi(\theta_i)$ and the posterior density $\pi(\theta_i | X_1, X_2)$ are both the one-parameter exponential family. The prior density as derived by Diaconis & Ylvisaker.[4]

$$\pi(\theta_i) = \frac{\exp\{r_{i,0}\mu_{i,0}\eta_i - r_{i,0}\psi(\eta_i)\}}{C(r_{i,0}, \mu_{i,0})}, \mu_i \in \Omega, \quad (2.2)$$

Where

$$C(r_{i,0}, \mu_{i,0}) = \int \exp\{r_{i,0}\mu_{i,0}\eta_i - r_{i,0}\psi(\eta_i)\}\, d\theta_i \quad (2.3)$$

Consider the problem of allocating a fixed total number of observations from the independent populations, where $T = M_1 + M_2$, M_i is the sample size of component $i, i = 1,2$.

The Bayes Risk for estimating the product of two means

$$R(p) = E\left[\frac{E\left[\psi''(\eta_1)|\mathcal{F}_T\right]E^2\left[\psi'(\eta_2)|\mathcal{F}_T\right]}{M_1} + \frac{E\left[\psi''(\eta_2)|\mathcal{F}_T\right]E^2\left[\psi'(\eta_1)|\mathcal{F}_T\right]}{M_2} + \frac{\left[\psi''(\eta_1)|\mathcal{F}_T\right]E\left[\psi''(\eta_2)|\mathcal{F}_T\right]}{M_1 M_2}\right]. \quad (2.4)$$

where $\mathcal{F}_T$ is the sigma algebra generated by a total of T observations.

To ease the notation, let $D_i = E\left[\psi''(\eta_i)\right]; C_i = E^2\left[\psi'(\eta_i)\right]$, $i = 1,2.$

Next, we derive the second order lower bound for the Bayes Risk.

Theorem 2.1: For any sequential procedure, p_s that satisfies the following conditions:

$M_i \to \infty$ in probability, as $T \to \infty$,

$$\frac{M_i}{T} \to \frac{\sqrt{\psi''(\eta_i)\psi'(\eta_i)}}{\sum_{i=1}^{2}\sqrt{\psi''(\eta_i)\psi'(\eta_i)}} \text{ in probability, as } T \to \infty,$$

$$R(p_s) \geq \frac{E\left[\left(\sqrt{D_1C_2}+\sqrt{D_2C_1}\right)^2\right]}{T}\left(1+\frac{E\left[\sqrt{\frac{D_1D_2}{C_1C_2}}\right]}{T}\right)+O\left(\frac{1}{T^2}\right) \quad (2.5)$$

Proof: The first two terms establish the first order lower bound for the Bayes Risk, whereas the third term establishes the second order lower bound for the Bayes Risk.

$$E\left[\frac{E\left[\psi''(\eta_1)|\mathcal{F}\right]E\left[\psi''(\eta_2)|\mathcal{F}\right]}{M_1M_2}\right]=\frac{1}{T^2}E\left[\frac{T}{M_1}\frac{T}{M_2}\left[\psi''(\eta_1)|\mathcal{F}\right]E\left[\psi''(\eta_2)|\mathcal{F}\right]\right].$$

Reliability estimation of series system with two independent components

Consider a series system with two independent components with unknown system reliability. The problem is to determine the optimal number units to be tested from each component. Each tested unit can be considered a Bernoulli trial. That is, suppose $X_i=\{X_{i1},\ldots,X_{iM_i}\}\sim Ber(\theta_i)$ and $\theta_i\sim Beta(a_i,b_i)$, where $0\leq\theta_i\leq 1, i=1,2.$

Then the Bayes Risk incurred after T units have been tested is:

$$R(p_s)=\frac{E\left[\left(\sqrt{\theta_1(1-\theta_1)\theta_2}+\sqrt{\theta_2(1-\theta_2)\theta_1}\right)^2\right]}{T}\left(1+\frac{E\left[\frac{\sqrt{\theta_1(1-\theta_1)}\sqrt{\theta_1(1-\theta_1)}}{\theta_1\theta_2}\right]}{T}\right) \quad (3.1)$$

as the second-order lower bound of Bayes Risk.

We also need to rely on the optimal ratio indicated in Theorem 2.1, that is

$$C=\frac{\sqrt{\theta_1(1-\theta_1)\theta_2}}{\sqrt{\theta_1(1-\theta_1)\theta_2}+\sqrt{\theta_2(1-\theta_2)\theta_1}} \quad (3.2)$$

We proceed with the test as follows:

Step 1: Collect one sample case ($m_1=n_1=1$) from each component, $\mathcal{P}_1$ and $\mathcal{P}_2$.

Step 2: We collect m_ℓ sample cases from component, $\mathcal{P}_1$ where $\ell\geq 2$,

If

$$\frac{m_{\ell+1}+r_{1,o}}{T}<\hat{C}(m_{\ell+1},n_{\ell+1}) \quad (3.3)$$

where $\hat{C}=E[C|\mathcal{F}]$. Then $m_{\ell+1}=m_\ell+1$, where $m_{\ell+1}$ are the cumulative sample cases from component $\mathcal{P}_1$;

Otherwise, $n_{\ell+1}=n_\ell+1$, where $n_{\ell+1}$ is the cumulative sample cases from component $\mathcal{P}_2$.

Step 3: We stop the iteration in step 2 when $T=m_{\ell+1}+n_{\ell+1}$, which is the fixed total sample size.

It should be noted that $C=\frac{1}{2}$ when both components have equal prior. The following example through Monte Carlo simulation with 5000 iterations shows that the expected number of units to be tested from each component is approximately $\frac{T}{2}$ which agrees with (3.2) (Table 1).

Next, we establish the rate of convergence of $\Delta=R(p)-R(p_s)$. Let the speed be defined as $T^\beta\cdot\Delta, 0\leq\beta<2$. It is clear that $T\cdot\Delta\to 0$. See Table 2. It is also clear that $T^\beta\cdot\Delta$ is bounded as $T\to\infty$, where $\theta_i\sim Beta(a_i,b_i)$ (Figure 1).[6–17]

Table 1 Fully sequential sampling design with uniform priors

T	50	70	100	200	400	600	800	1000	3000
$E[m_1]$	26	36	51	101	200	301	400	498	1504
$E[m_2]$	24	34	49	99	200	299	400	502	1496

Table 2 $T^\beta.\Delta<C\ as\ t\to\infty$, where $\Delta=R(p)-R(p_s), 0\leq\beta<2, C\in\mathbb{R}$

T	50	70	100	200	400	600	800	1000	3000
$T\ \Delta$	0.02	0.019	0.026	0.029	0.033	0.025	0.034	0.031	0.032
$T^{1.5}.\Delta$	0.138	0.159	0.262	0.415	0.662	0.619	0.955	0.969	1.776
$T^2.\Delta$	0.976	1.329	2.62	5.863	13.231	15.169	27.001	30.627	97.281

Conclusion

The proposed sequential design enables the experimenter to determine the optimal allocation of units to be tested from each component when component is functional independently. Rekab[5] used a classical approach to estimate the reliability of a series system, and proposed a sequential design that was shown to be first-order optimal, that is $T\cdot\Delta\to 0$. The fully Bayesian approach enables us to investigate a design with a converge rate of the order of $T^{-\beta}$, where $0\leq\beta<2$.

Data availability

The data used to support the findings of this study have been produced by Monte Carlo simulation from Bernoulli trials with 5000 replications.

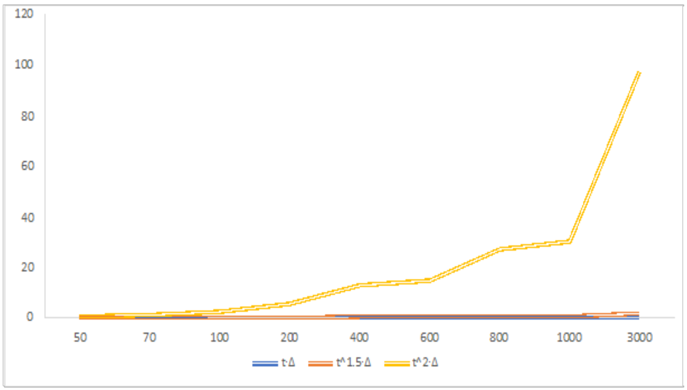

Figure 1 $\Delta = R(p) - R(p_s)$ is bounded by $T^{-\beta}$ as $T \to \infty$, where $0 \le \beta < 2$.

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Conflicts of interest

Authors declare that there are no conflicts of interest.

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