Research Article

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A two-parameter power Rama distribution with properties and applications

Abstract

In this paper a two-parameter Power Rama distribution, which includes one parameter Rama distribution as a particular case, has been proposed. Its statistical and reliability properties including shapes of density for varying values of parameters, the moments about origin, the mean and variance, survival function, hazard rate function, mean residual life function have been discussed. The maximum likelihood estimation for estimating the parameters has been discussed. Finally, the goodness of fit of the proposed distribution has been discussed with two real lifetime dataset.

Keywords: Rama distribution, Hazard rate function, moments, maximum likelihood estimation, goodness of fit

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Introduction

Shanker[1] proposed a continuous lifetime distribution named Rama distribution defined by its probability density function (pdf) and cumulative distribution function (cdf)

$$f(y;\theta)=\frac{\theta^4}{\theta^3+6}\left(1+y^3\right)e^{-\theta y};\ y\ge 0,\theta\ge 0 \qquad (1.1)$$

$$F(y,\theta)=1-\left[1+\frac{\theta^3y^3+3\theta^2y^2+6\theta y}{\theta^3+6}\right]e^{-\theta y}\ ;x>0,\theta>0 \quad (1.2)$$

The pdf (1.1) is a convex combination of exponential (θ) and gamma $(4,\theta)$ distribution with mixing proportion $\frac{\theta^3}{\theta^3+6}$. We have

$$f(y,\theta)=p\,g_1(y)+(1-p)g_2(y)$$

where $p=\frac{\theta^3}{\theta^3+6}$, $g_1(y)=\theta e^{-\theta y}$, and $g_2(y)=\frac{\theta^4y^3e^{-\theta y}}{6}$.

The statistical properties, estimation of parameter using maximum likelihood estimation and applications for modeling lifetime data of Rama distribution are available in Shanker.[1]

The pdf and the cdf of Power Lindley distribution (PLD) introduced by Ghitany et al.,[2] are given by

$$f(x;\theta,\alpha)=\frac{\alpha\theta^2}{(\theta+1)}x^{\alpha-1}\left(1+x^\alpha\right)e^{-\theta x^\alpha}\ ;x>0,\theta>0,\alpha>0 \quad (1.3)$$

$$F(x;\theta,\alpha)=1-\left[1+\frac{\theta x^\alpha}{\theta^2+1}\right]e^{-\theta x^\alpha}\ ;x>0,\theta>0,\alpha>0 \quad (1.4)$$

Obviously at $\alpha=1$, PLD reduces to Lindley distribution, introduced by Lindley[3] having pdf and cdf

$$f(x;\theta)=\frac{\theta^2}{\theta+1}(1+x)e^{-\theta x};x>0,\theta>0 \qquad (1.5)$$

$$F(x;\theta)=1-\left[1+\frac{\theta x}{\theta+1}\right]e^{-\theta x};x>0,\theta>0 \qquad (1.6)$$

The statistical properties, estimation of parameter and application of Lindley distribution are discussed in Ghitany et al.,[4] Shanker et al.,5 have detailed study on applications of Lindley distribution and exponential distribution to model real lifetime datasets from engineering and biomedical sciences. Since Rama distribution has only one parameter, it has less flexibility to model data of varying natures. In the present paper an attempt has been made to derive two-parameter power Rama distribution which includes one parameter Rama distribution as particular cases as power transformation of Rama distribution. The shapes of the density, moments, hazard rate function, and mean residual life function of the distribution have been discussed. The maximum likelihood estimation has been explained. The goodness of fit of the proposed distribution has been discussed with two real lifetime dataset and fit shows quite satisfactory fit over other one parameter and two-parameter lifetime distributions.

A two-parameter power Rama distribution

Taking the power transformation $X=Y^{1/\alpha}$ in (1.1), the pdf of the random variable X can be obtained as

$$f_1(x;\theta,\alpha)=\frac{\alpha\theta^4}{\theta^3+6}\left(1+x^{3\alpha}\right)x^{\alpha-1}e^{-\theta x^\alpha};x\ge 0,\ \theta\ge 0,\ \alpha\ge 0 \quad (2.1)$$

$$=p\,g_1(x;\theta,\alpha)+(1-p)g_2(x;\theta,\alpha) \qquad (2.2)$$

where $p=\frac{\theta^3}{\theta^3+6}$, $g_1(x;\theta,\alpha)=\alpha\theta e^{-\theta x^\alpha}x^{\alpha-1};\quad x>0,\theta>0,\alpha>0$

and $g_2(x;\theta,\alpha)=\frac{\alpha\theta^4e^{-\theta x^\alpha}x^{4\alpha-1}}{6};\quad x>0,\theta>0,\alpha>0$.

We would call the pdf in (2.1) power Rama distribution (PRD). At $\alpha=1$, (2.1) reduces to one Rama distribution. Like Rama distribution, the PRD is also a convex combination of Weibull distribution (with shape parameter α and scale parameter θ) and a generalized gamma distribution (with shape parameter 4 and α and scale parameter θ) with mixing proportion $p=\frac{\theta^3}{\theta^3+6}$.

The corresponding cdf of PRD can be obtained as

$$F_1(x;\theta,\alpha)=1-\frac{\theta^3e^{-\theta x^\alpha}+\Gamma\left(4,\theta x^\alpha\right)}{\theta^3+6}\quad x>0,\theta>0\ ,\alpha>0 \quad (2.3)$$

Various graphs of the pdf of PRD for varying values of parameters have been drawn and presented in Figure 1. As the value of the shape parameter alpha increases the graph of PRD approaches normal distribution. Various graphs of the cdf of PRD for varying values of its parameters have been drawn and shown in Figure 2.

Statistical constants

Using (2.2), the r^th^ moment about the origin, μ_r' of PRD can be obtained as

$$\mu_r' = E\left(X^r\right) = p\int_0^\infty x^r g_1\left(x;\theta,\alpha\right)dx + \left(1-p\right)\int_0^\infty x^r g_2\left(x;\theta,\alpha\right)dx$$

$$= \frac{\theta^3\Gamma\left(\frac{r}{\alpha}+1\right)+\Gamma\left(\frac{r}{\alpha}+4\right)}{\theta^{\frac{r}{\alpha}}\left(\theta^3+6\right)};r=1,2,3,... \quad (3.1)$$

At $\alpha=1$, the above expression reduces to the r[th] moment about origin of Rama distribution given by

$$\mu_r' = \frac{\theta^3\Gamma\left(r+1\right)+\Gamma\left(r+4\right)}{\theta^r\left(\theta^3+6\right)};r=1,2,3,...$$

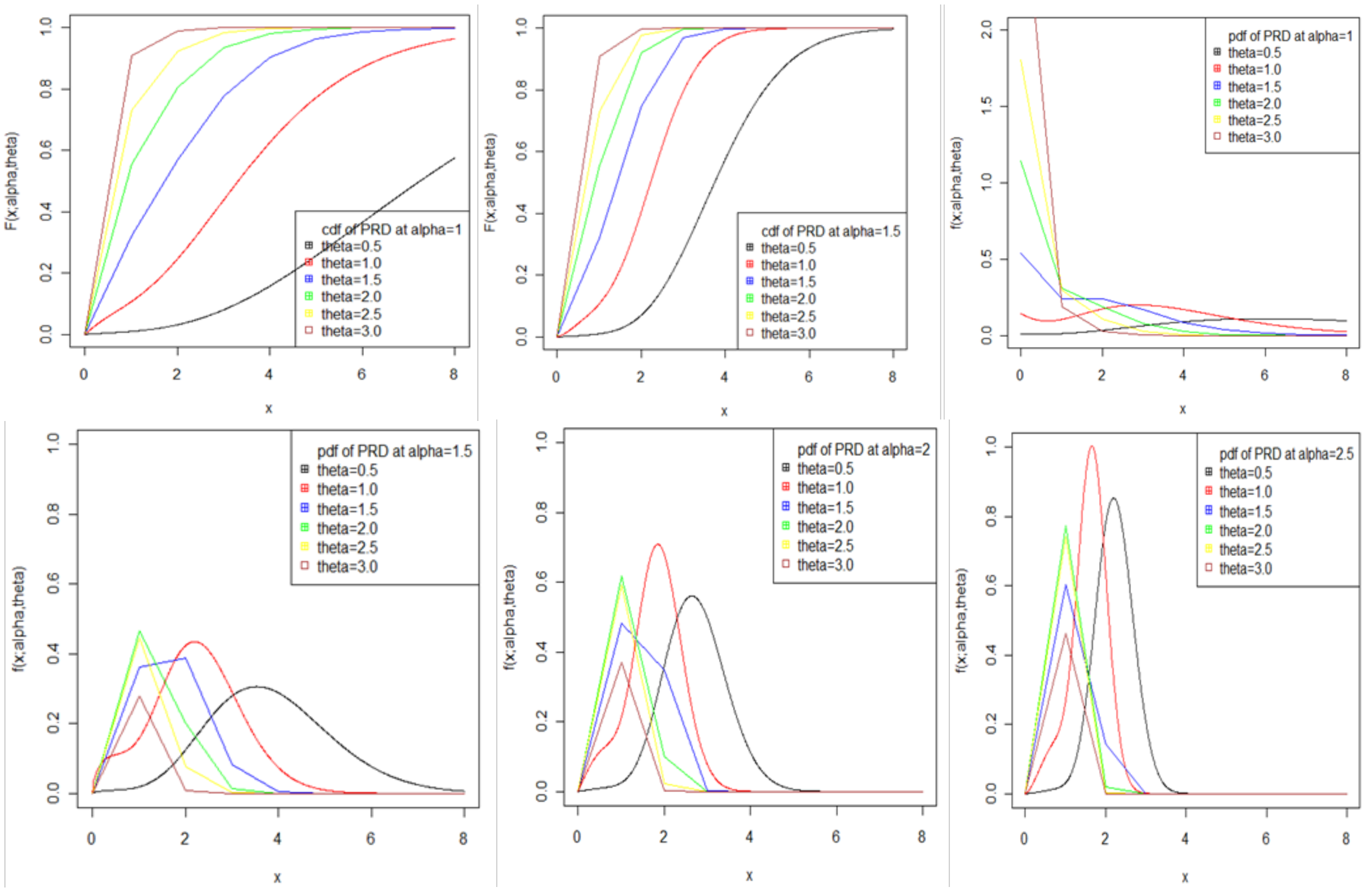

Figure 1 Behavior of the pdf of PRD for varying values of parameters θ and α .

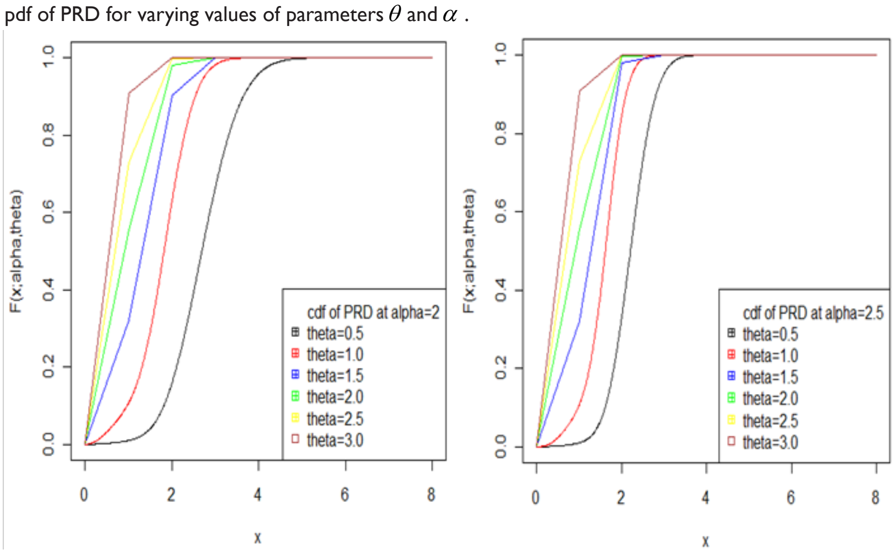

Figure 2 Behavior of the cdf of PRD for varying values of parameters θ and α .

Thus the first four moments about origin of the PRD are obtained as

$$\mu_1' = \frac{\frac{1}{\alpha}\Gamma\left(\frac{1}{\alpha}\right)\left[\theta^3+\left(\frac{1}{\alpha}+3\right)\left(\frac{1}{\alpha}+2\right)\left(\frac{1}{\alpha}+1\right)\right]}{\theta^{\frac{1}{\alpha}}\left(\theta^3+6\right)},$$

$$\mu_2' = \frac{\frac{2}{\alpha}\Gamma\left(\frac{2}{\alpha}\right)\left[\theta^3+\left(\frac{2}{\alpha}+3\right)\left(\frac{2}{\alpha}+2\right)\left(\frac{2}{\alpha}+1\right)\right]}{\theta^{\frac{2}{\alpha}}\left(\theta^3+6\right)}$$

$$\mu_3' = \frac{\frac{3}{\alpha}\Gamma\left(\frac{3}{\alpha}\right)\left[\theta^3+\left(\frac{3}{\alpha}+3\right)\right]\left(\frac{3}{\alpha}+2\right)\left(\frac{3}{\alpha}+1\right)}{\theta^{\frac{3}{\alpha}}\left(\theta^3+6\right)},$$

$$\mu_4' = \frac{\frac{4}{\alpha}\Gamma\left(\frac{4}{\alpha}\right)\left[\theta^3+\left(\frac{4}{\alpha}+3\right)\right]\left(\frac{4}{\alpha}+2\right)\left(\frac{4}{\alpha}+1\right)}{\theta^{\frac{4}{\alpha}}\left(\theta^3+6\right)}$$

The variance of the PRD thus can be expressed as

$$\sigma^2 = \frac{1}{\theta^{\frac{2}{\alpha}}\left(\theta^3+6\right)}\left[\frac{2}{\alpha}\Gamma\left(\frac{2}{\alpha}\right)\left[\theta^3+\left(\frac{2}{\alpha}+3\right)\left(\frac{2}{\alpha}+2\right)\left(\frac{2}{\alpha}+1\right)\right] - \frac{\left\{\frac{1}{\alpha}\Gamma\left(\frac{1}{\alpha}\right)\left[\theta^3+\left(\frac{1}{\alpha}+3\right)\left(\frac{1}{\alpha}+2\right)\left(\frac{1}{\alpha}+1\right)\right]\right\}^2}{\left(\theta^3+3\right)}\right].$$

Reliability properties

The survival function, $S(x;\theta,\alpha)$ of PRD can be obtained as

$$S(x;\theta,\alpha) = P(X>x) = \int_x^\infty f(t;\theta,\alpha)\,dt$$

$$= \frac{\theta^3 e^{-\theta x^\alpha} + \Gamma\left(4,\theta x^\alpha\right)}{\theta^3+6};\quad x>0,\theta>0,\alpha>0 \quad (4.1)$$

The hazard rate function, $h(x;\theta,\alpha)$ and the mean residual function, $m(x;\theta,\alpha)$ of PRD are obtained as

$$h(x;\theta,\alpha) = \frac{f(x;\theta,\alpha)}{S(x;\theta,\alpha)} = \frac{\alpha\theta^4\left(1+x^{3\alpha}\right)x^{\alpha-1}e^{-\theta x^\alpha}}{\theta^3 e^{-\theta x^\alpha}+\Gamma\left(4,\theta x^\alpha\right)}$$

and

$$m(x;\theta,\alpha) = \frac{1}{S(x;\theta,\alpha)}\int_x^\infty t\,f(t;\theta,\alpha)\,dt - x$$

$$= \frac{\frac{1}{\alpha}\Gamma\left(\frac{1}{\alpha},\theta x^\alpha\right)\left[\theta^3+\left(\frac{1}{\alpha}+1\right)\left(\frac{1}{\alpha}+2\right)\right]\left(\frac{1}{\alpha}+3\right) + e^{-\theta x^\alpha}\left(\theta x^\alpha\right)\left[\theta^3+\left(\theta x^\alpha\right)^3\left(\frac{1}{\alpha}+3\right)\left(\theta x^\alpha\right)^2+\left(\frac{1}{\alpha}+3\right)\left(\frac{1}{\alpha}+2\right)\left(\theta x^\alpha\right)\right]}{\theta^{\frac{1}{\alpha}}\left[\Gamma\left(4,\theta x^\alpha\right)+\theta^3 e^{-\theta x^\alpha}\right]} - x.$$

Obviously

$$m(x;\theta,\alpha) = \frac{\frac{1}{\alpha}\Gamma\left(\frac{1}{\alpha}\right)\left[\theta^3+\left(\frac{1}{\alpha}+3\right)\left(\frac{1}{\alpha}+2\right)\left(\frac{1}{\alpha}+1\right)\right]}{\theta^{\frac{1}{\alpha}}\left(\theta^3+6\right)} = \mu_1' \quad \text{for} \quad x=0,$$

the mean of PRD.

The behaviors of hazard rate function of PRD for varying values of parameters have been shown graphically in Figure 3. The graphs of hazard rate function of PRD are monotonically increasing for varying values of parameters. The behaviors of mean residual life function of PRD for varying values of parameters are shown in Figure 4. The graphs of mean residual life function of PRD are monotonically deceasing for varying values of parameters.

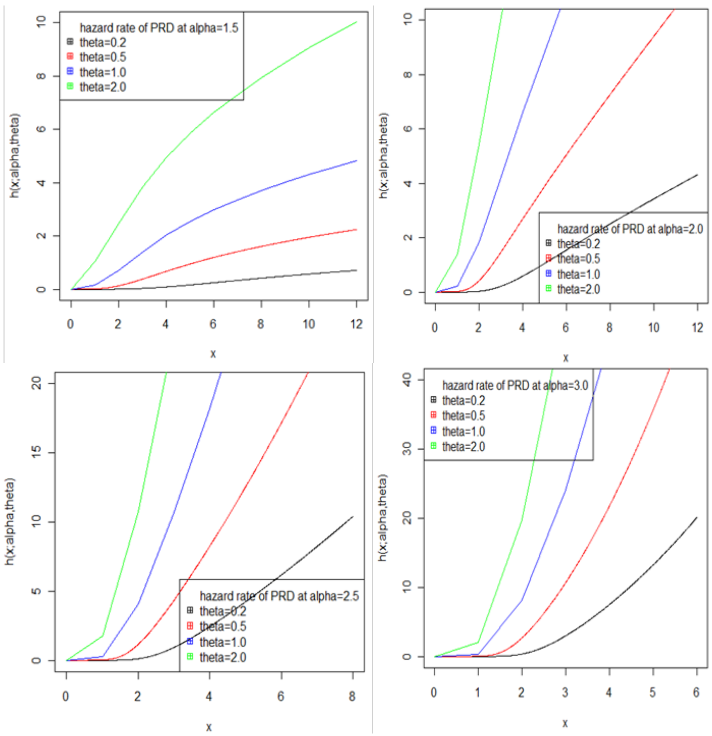

Figure 3 Behavior of the hazard rate function of PRD for varying values of paramete θ and α .

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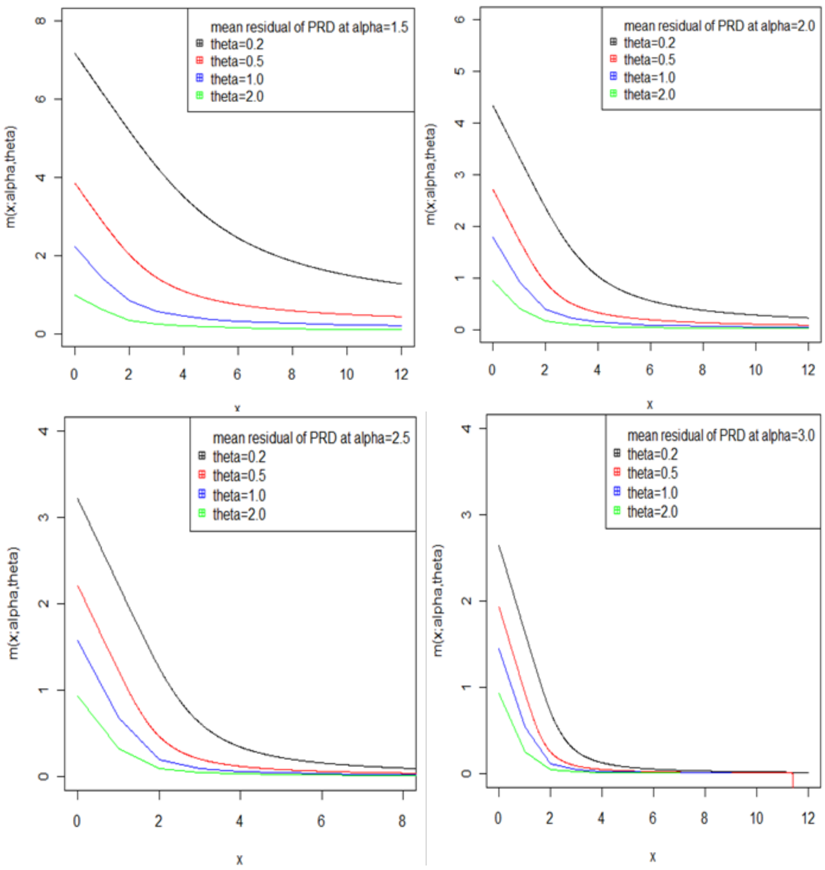

Figure 4 Behavior of the mean residual function of PRD for varying values of parameters θ and α .

Estimation of parameters

The natural log likelihood function based on random samples $(x_1, x_2, ..., x_n)$ from PRD (θ, α) can be expressed as

$$\ln L = \sum_{i=1}^{n} \ln f(x_i; \theta, \alpha)$$

$$= n\left[4\ln\theta + \ln\alpha - \ln\left(\theta^3 + 6\right)\right] + (\alpha - 1)\sum_{i=1}^{n} \ln x_i + \sum_{i=1}^{n}\left(1 + x_i^{3\alpha}\right) - \theta\sum_{i=1}^{n} x_i^{\alpha}$$

Thus, the maximum likelihood estimates, $\left(\hat{\theta}, \hat{\alpha}\right)$ of parameters (θ, α) of PRD is the solution of the following log-likelihood equations

$$\frac{\partial \ln L}{\partial \theta} = \frac{4n}{\theta} - \frac{3n\theta^2}{\theta^3 + 6} - \sum_{i=1}^{n} x_i^{\alpha} = 0$$

$$\frac{\partial \ln L}{\partial \alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n} \ln x_i - \theta\sum_{i=1}^{n} x_i^{\alpha} = 0$$

Since these two natural log likelihood equations cannot be expressed in closed form, these two log-likelihood equations are not directly solvable. The $\hat{\theta}, \hat{\alpha}$ of parameters θ, α can be obtained directly from the natural log likelihood equation using Newton-Raphson iteration using R-software till sufficiently close values of $\hat{\theta}$ and $\hat{\alpha}$ are obtained.

Applications

The application and the goodness of fit of PRD based on maximum likelihood estimates of parameters have been explained with two real datasets from engineering and biomedical Sciences. The fit given by one parameter exponential, Lindley and Rama distributions and two-parameter power Lindley distribution (PLD) and Weibull distribution, introduced by Weibull,[6] have also given for ready comparison. The following two real lifetime datasets have been considered for showing the superiority of PRD over other lifetime distributions.

Data set 1: The data set is from Smith & Naylor[7] relating to the strength of 1.5cm glass fibers measured at the National Physical Laboratory, England.

0.55	0.93	1.25	1.36	1.49	1.52	1.58	1.61	1.64	1.68	1.73	1.81
2.00	0.74	1.04	1.27	1.39	1.49	1.53	1.59	1.61	1.66	1.68	1.76
1.82	2.01	0.77	1.11	1.28	1.42	1.50	1.54	1.60	1.62	1.66	1.69
1.76	1.84	2.24	0.81	1.13	1.29	1.48	1.50	1.55	1.61	1.62	1.66
1.70	1.77	1.84	0.84	1.24	1.30	1.48	1.51	1.55	1.61	1.63	1.67
1.70	1.78	1.89									

In order to compare considered distributions, values of MLE$\left(\hat{\theta},\hat{\alpha}\right)$ along with their standard errors, -2ln L, AIC (Akaike Information Criterion), K-S statistic and p-value for the dataset have been computed and presented in Table 1. The best distribution is the distribution corresponding to lower values of -2ln L, AIC and K-S statistic.

The Variance-Covariance matrix and 95% confidence interval (CI's) for the parameters $\hat{\theta}$ and $\hat{\alpha}$ of PRD for the dataset 1 has been presented in Table 2.

In order to see the closeness of the fit given by one parameter exponential, Lindley and Rama distributions and two-parameter PLD and Weibull distribution, the fitted plot of pdfs of these distributions for the dataset 1 have been shown in Figure 5. Obviously the fitted plots of the distributions and the histogram of the original dataset shows that PRD gives much closer fit over the considered distributions.

Data set 2: The following dataset represents waiting time (in minutes) of 154 patients, waiting before OPD (Out Patient Diagnosis) from the 25th - 30th December, 2017(in the 4th week of December) at Halibet Hospital., available in the master thesis of Berhane Abebe,[8] Department of Statistics, College of Science, Eritrea Institute of Technology, Eritrea.

2(13)	3(29)	4(32)	5(29)	6(18)	7(15)	8(6)	9(6)
10(2)	11(2)	12	17				

Table 1 MLE's, -2ln L, se$\left(\hat{\theta},\hat{\alpha}\right)$, AIC, K-S statistic, and p-value of the fitted distribution of dataset 1

Model	ML estimate	se$\left(\hat{\theta},\hat{\alpha}\right)$	-2ln L	AIC	K-S	p-value
PRD	$\hat{\theta}=0.8436$ $\hat{\alpha}=3.2697$	0.0942 0.2241	26.19	30.19	0.1297	0.2398
PLD	$\hat{\theta}=0.2224$ $\hat{\alpha}=4.4583$	0.0466 0.3871	29.38	33.38	0.1442	0.1457
Weibull	$\hat{\theta}=0.0598$ $\hat{\alpha}=5.7800$	0.0205 0.5752	30.41	34.31	0.1523	0.1075
Rama	$\hat{\theta}=1.7313$	0.0991	169.72	171.72	0.3580	< 0.0001
Lindley	$\hat{\theta}=0.9961$	0.0948	162.56	164.56	0.3864	< 0.0001
Exponential	$\hat{\theta}=0.6636$	0.0836	177.66	179.66	0.4180	< 0.0001

Table 2 Variance-Covariance matrix and 95% confidence interval (CI's) for the parameters $\hat{\theta}$ and $\hat{\alpha}$ of PRD

Parameters	Variance-covariance matrix		95% CI	
	$\hat{\theta}$	$\hat{\alpha}$	Lower	Upper
$\hat{\theta}$	0.0089	-0.0179	0.0668	1.0367
$\hat{\alpha}$	-0.0179	0.0504	2.8450	3.7278

The maximum likelihood estimates of parameters, standard error of estimates of parameters, AIC, K-S and p-values of the fitted distributions for dataset 2 are shown in Table 3. The Variance-Covariance matrix and 95% confidence interval (CI's) for the parameters $\hat{\theta}$ and $\hat{\alpha}$ of PRD for the given dataset 2 has been presented in Table 4.

In order to see the closeness of the fit given by one parameter exponential, Lindley and Rama distributions and two-parameter PLD and Weibull distribution the fitted plot of pdfs of these distributions for the dataset 2 have been shown in Figure 6. Obviously the fitted plots of the distributions and the histogram of the dataset 2 shows that PRD gives much closer fit over the considered distributions.

Table 3 MLE's, -2ln L, se$\left(\hat{\theta},\hat{\alpha}\right)$, AIC, K-S statistic, and P-value of the fitted distribution of dataset 2

Model	ML Estimate	se$\left(\hat{\theta},\hat{\alpha}\right)$	-2ln L	AIC	K-S	P-value
PRD	$\hat{\theta}=0.5378$ $\hat{\alpha}=1.2173$	0.0555 0.0576	659.57	663.57	0.1329	0.0087
PLD	$\hat{\theta}=0.1054$ $\hat{\alpha}=1.7223$	0.0173 0.0873	665.69	669.69	0.1368	0.0062
Weibull	$\hat{\theta}=0.0189$ $\hat{\alpha}=2.2892$	0.0049 0.1271	673.14	677.14	0.1402	0.0047
Rama	$\hat{\theta}=0.7575$	0.0292	675.87	677.87	0.1581	0.0009
Lindley	$\hat{\theta}=0.3475$	0.0201	746.74	748.74	0.2903	< 0.0001
Exponential	$\hat{\theta}=0.1995$	0.0161	804.51	806.51	0.3659	< 0.0001

Table 4 Variance-Covariance matrix and 95% confidence interval (CI's) for the parameters $\hat{\theta}$ and $\hat{\alpha}$ of PRD for dataset 2

Parameters	Variance-covariance matrix		95% CI	
	$\hat{\theta}$	$\hat{\alpha}$	Lower	Upper
$\hat{\theta}$	0.0031	-0.0030	0.4355	0.6526
$\hat{\alpha}$	-0.0030	0.0033	1.1080	1.3339

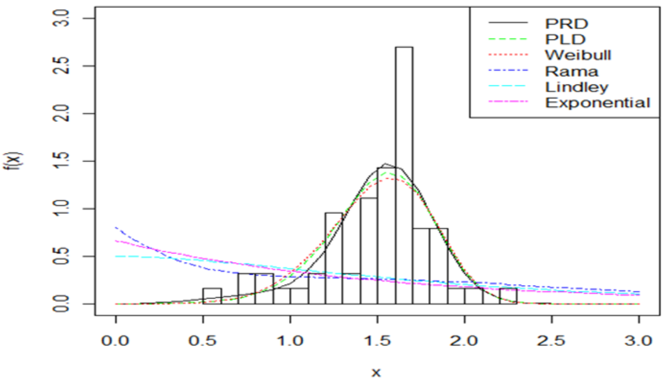

Figure 5 Fitted pdf plots of the considered distributions for the given dataset 1.

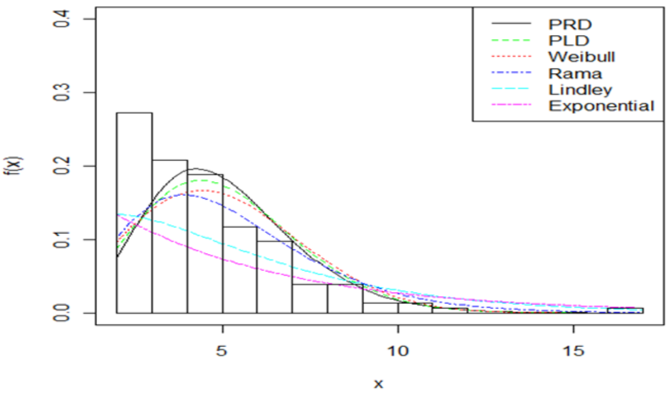

Figure 6 Fitted pdf plots of the considered distributions for the given dataset 2.

Concluding remarks

A two-parameter Power Rama distribution (PRD) has been proposed which includes Rama distribution, introduced by Shanker (2017), as a particular case. The statistical and reliability properties including moments, survival function, hazard rate function, mean residual function of PRD have been discussed. Maximum likelihood estimation has been explained for estimating the parameters. The goodness of fit of PRD has been discussed with two real lifetime datasets and the fit has been found quite satisfactory over one parameter Rama, Lindley and Exponential distributions and two-parameter Power Lindley distribution (PLD) and Weibull distribution. Hence, PRD can be considered an important two-parameter lifetime distribution.

Acknowledgments

None.

Conflicts of interest

The author declares there is no conflicts of interest.

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