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A generalized Aradhana distribution with properties and applications

Abstract

A two-parameter generalized Aradhana distribution which includes one parameter exponential and Aradhana distributions as special cases has been proposed. Its statistical properties including shapes of probability density function for varying values of parameters, coefficient of variation, skeweness, kurtosis, index of dispersion, hazard rate function, mean residual life function, stochastic ordering, mean deviations, Bonferroni and Lorenz curves and stress-strength reliability have been discussed. Maximum likelihood estimation has been discussed for estimating the parameters of the distribution. Applications of the distribution have been explained with two real life time data.

Keywords: Aradhana distribution, statistical properties, maximum likelihood estimation, applications

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Introduction

In almost every fields of knowledge including engineering, biomedical science, social science, insurance, finance, etc, the statistical analysis and modeling of real life time data are crucial for researchers and policy makers. The classical one parameter life time distributions, namely exponential and Lindley, introduced by Lindley,[1] are not always suitable due to theoretical or applied point of view for real lifetime data. To overcome the shortcomings of these classical one parameter distributions and have a better lifetime distribution, a number of one parameter lifetime distributions have been introduced in statistics literature and the statistics literature is flooded with a number of one parameter life time distributions. Shanker[2] has introduced a one parameter lifetime distribution named Aradhana distribution having scale parameter θ and defined by its probability density function (pdf) and cumulative distribution function (cdf)

$$f_1(x;\theta)=\frac{\theta^3}{\theta^2+2\theta+2}(1+x)^2e^{-\theta x};x>0,\theta>0 \qquad (1.1)$$

$$F_1(x,\theta)=1-\left[1+\frac{\theta x(\theta x+2\theta+2)}{\theta^2+2\theta+2}\right]e^{-\theta x};x>0,\theta>0 \qquad (1.2)$$

The rth raw moments (moments about origin), μ_r' of Aradhana distribution obtained by Shanker[2] is given by

$$\mu_r'=\frac{r!\left\{\theta^2+2(r+1)\theta+(r+1)(r+2)\right\}}{\theta^r\left(\theta^2+2\theta+2\right)};r=1,2,3,...$$

Thus, the first four raw moments of Aradhana distribution are obtained as

$$\mu_1'=\frac{\left(\theta^2+4\theta+6\right)}{\theta\left(\theta^2+2\theta+2\right)} \qquad \mu_3'=\frac{6\left(\theta^2+8\theta+20\right)}{\theta^3\left(\theta^2+2\theta+2\right)}$$

$$\mu_3'=\frac{6\left(\theta^2+8\theta+20\right)}{\theta^3\left(\theta^2+2\theta+2\right)} \qquad \mu_4'=\frac{24\left(\theta^2+10\theta+30\right)}{\theta^4\left(\theta^2+2\theta+2\right)}$$

Using the relationship between central moments (moments about mean) and the raw moments, the central moments of Aradhana distribution are given by

$$\mu_2=\frac{\theta^4+8\theta^3+24\theta^2+24\theta+12}{\theta^2\left(\theta^2+2\theta+2\right)^2}$$

$$\mu_3=\frac{2\left(\theta^6+12\theta^5+54\theta^4+100\theta^3+108\theta^2+72\theta+24\right)}{\theta^3\left(\theta^2+2\theta+2\right)^3}$$

$$\mu_4=\frac{3\begin{pmatrix}3\theta^8+48\theta^7\alpha+304\theta^6\alpha^2+944\theta^5\alpha^3+1816\theta^4\alpha^4+2304\alpha^5\theta^3\\+1920\alpha^6\theta^2+960\alpha^7\theta+240\alpha^8\end{pmatrix}}{\theta^4\left(\theta^2+2\theta+2\right)^4}$$

Shanker[2] has discussed various statistical properties based on moments including coefficient of variation, skewness, kurtosis, index of dispersion, hazard rate function, mean residual life function, stochastic ordering, mean deviation, Bonferoni and Lorenz curves, stress-strength reliability along with estimation of parameter and applications of Aradhana distribution for modeling real lifetime data from engineering and biomedical sciences. A discrete Poisson-Aradhana distribution, a Poisson mixture of Aradhana distribution, has been obtained by Shanker[3] and its statistical properties, estimation of parameter along with applications to model count data are available in Shanker.[3] The Lindley distribution is defined by its pdf and cdf

$$f_2(x;\theta)=\frac{\theta^2}{\theta+1}(1+x)e^{-\theta x};x>0,\theta>0 \qquad (1.3)$$

$$F_2(x,\theta)=1-\left(1+\frac{\theta x}{\theta+1}\right);x>0,\theta>0 \qquad (1.4)$$

Ghitany et al.,[4] have detailed study on statistical properties, estimation of parameter and application of Lindley distribution for modeling waiting time data in a bank. Shanker et al.,[5] have critical and comparative study on modeling of real lifetime data from biomedical sciences and engineering and observed that there are several lifetime data where exponential distribution gives much better fit than Lindley distribution. Recently, Berhane & Shanker[6] proposed a discrete Lindley distribution using infinite series approach of discretization and studied its various statistical properties, estimation of parameter and applications.

In this paper a two -parameter generalized Aradhana distribution (GAD) has been proposed which includes exponential and Aradhana distributions. GAD has been found to be more general in nature and wider in scope and possess tremendous capacity to fit observed real lifetime data. Its moments and moments based measures have been obtained and discussed. Statistical properties including hazard rate function, mean residual life function, stochastic ordering, mean deviation, Bonferroni and Lorenz curves and stress -strength parameter of GAD have been discussed. Estimation of parameters has been discussed using the method of maximum likelihood. Applications of the distribution have been explained with two real lifetime data and the fit has been compared with one parameter exponential, Lindley and Aradhana distributions and a generalization of Sujatha distribution (AGSD), proposed by Shanker et al.[7]

A generalized Aradhana distribution

A generalized Aradhana distribution (GAD) having parameters θ and α is defined by its pdf and cdf

$$f_3(x;\theta,\alpha)=\frac{\theta^3}{\theta^2+2\alpha\theta+2\alpha^2}(1+\alpha x)^2 e^{-\theta x};x>0,\theta>0,\alpha\ge 0. \tag{2.1}$$

$$F_3(x;\theta,\alpha)=1-\left[1+\frac{\alpha\theta x(2\theta+\alpha\theta x+2\alpha)}{\theta^2+2\theta\alpha+2\alpha^2}\right]e^{-\theta x};x>0,\theta>0 \tag{2.2}$$

Note that exponential distribution and Aradhana distribution are special cases of GAD for $\alpha=0$ and $\alpha=1$, respectively. Further, Like Aradhana distribution, GAD is also a three-component mixture of exponential (θ), gamma $(2,\theta)$ and $(3,\theta)$ distributions. That is

$$f_3(x;\theta,\alpha)=p_1 g_1(x;\theta)+p_2 g_2(x,\theta)+(1-p_1-p_2)g_3(x,\theta) \tag{2.3}$$

where

$$p=\frac{\theta}{\theta+2\theta\alpha+2\alpha}$$

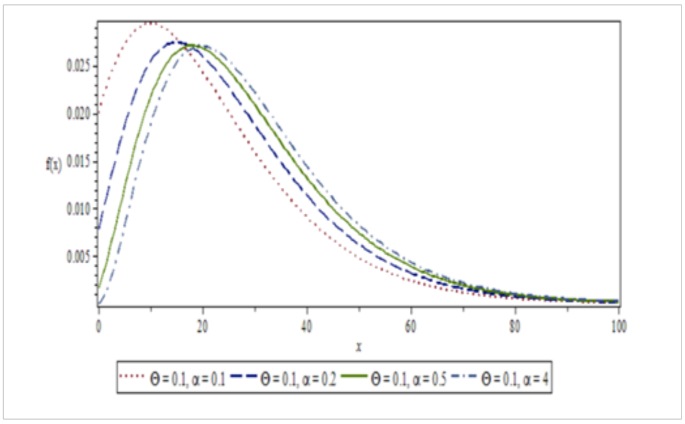
$$p=\frac{2\alpha\theta}{\theta+2\theta\alpha+2\alpha}$$

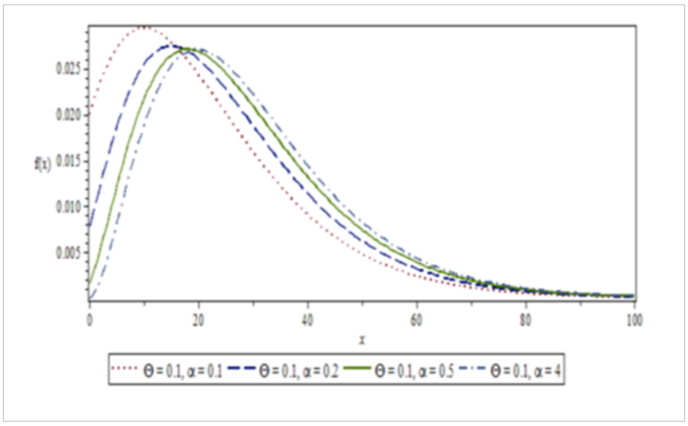
$$g(x;\theta)=\theta e\ ;x>0,\theta>0$$

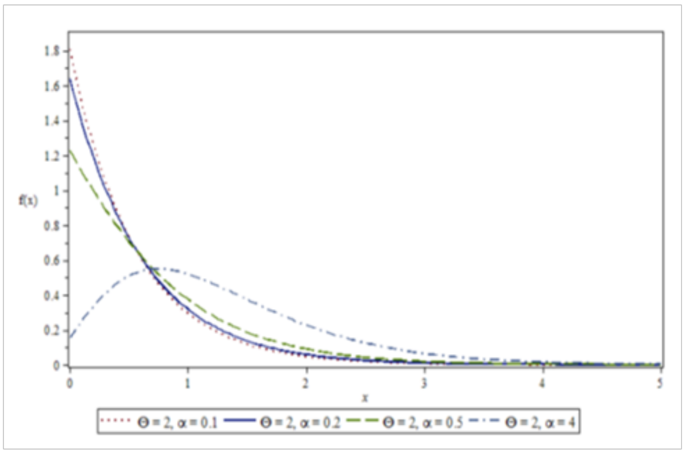
$$g(x;2,\theta)=\frac{\theta}{\Gamma(2)}e\ x\ ;x>0,\theta>0$$

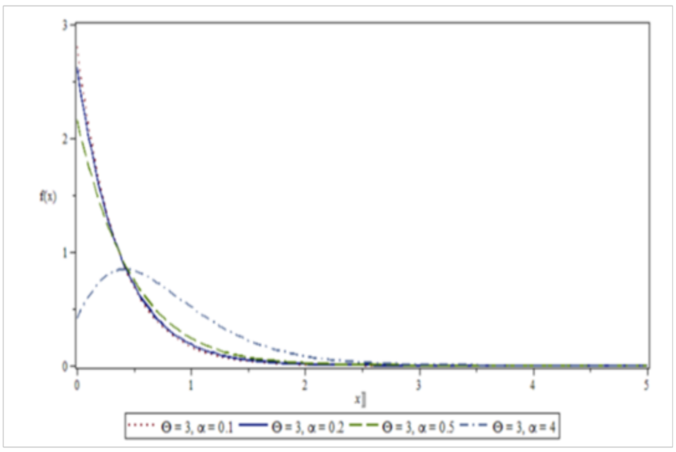
$$g(x;3,\theta)=\frac{\theta}{\Gamma(3)}e\ x\ ;x>0,\theta>0$$

The nature of the pdf and the cdf of GAD for various combinations of the parameters θ and α are presented in Figure 1 & Figure 2, respectively.









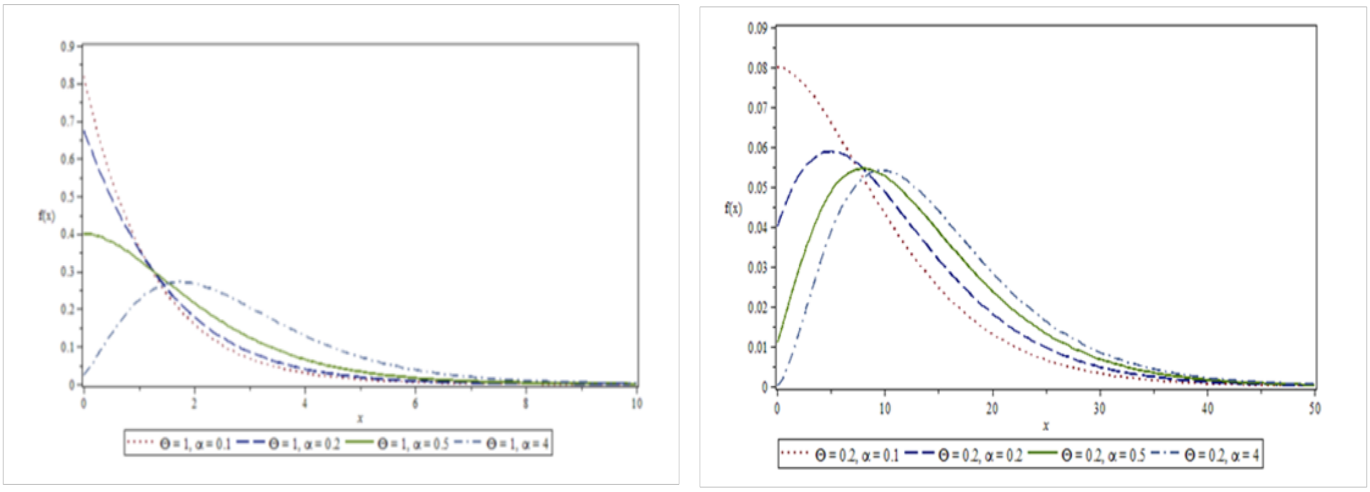
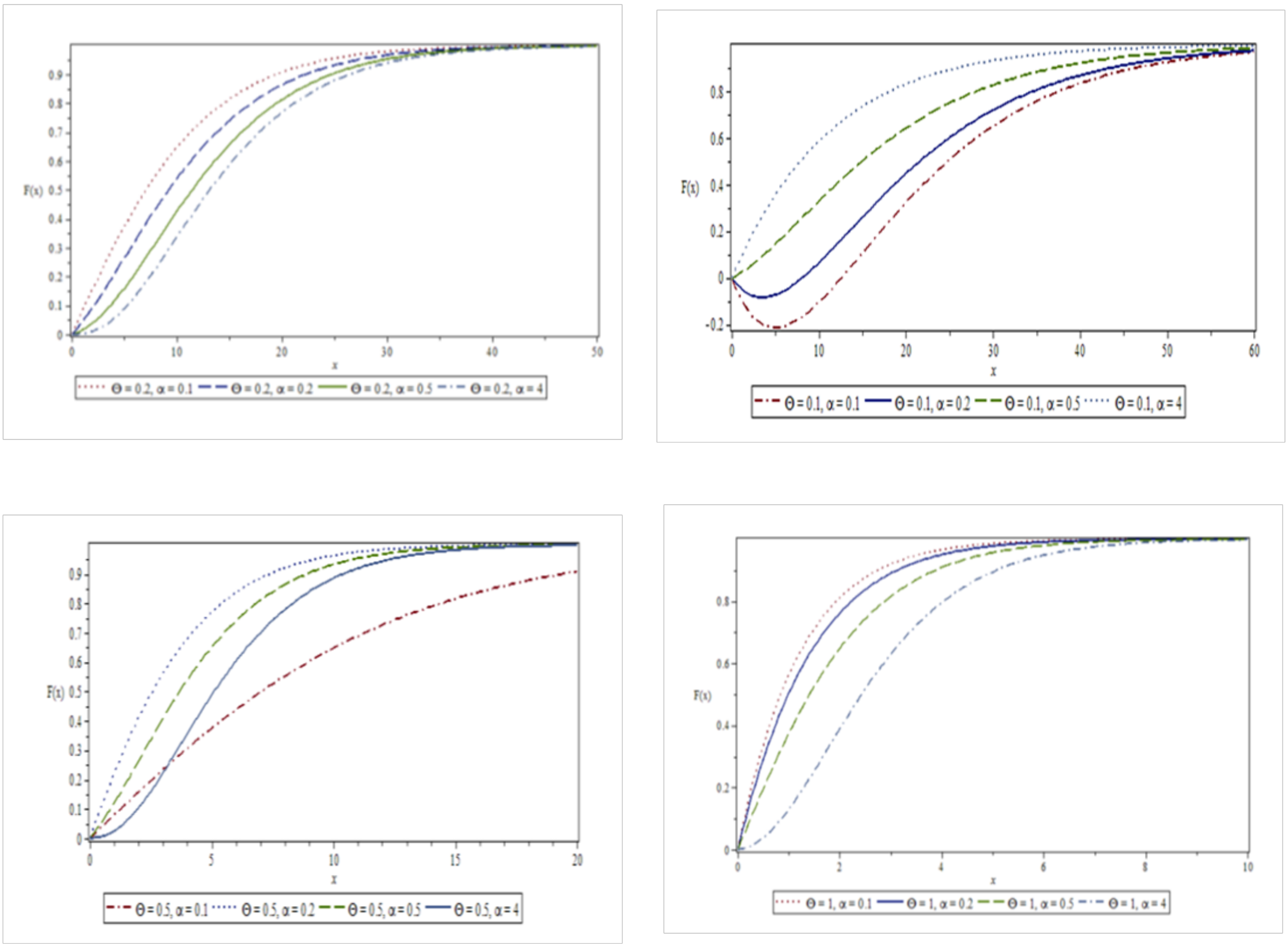


Figure 1 Behavior of the pdf of GAD for varying values of parameters θ and α .



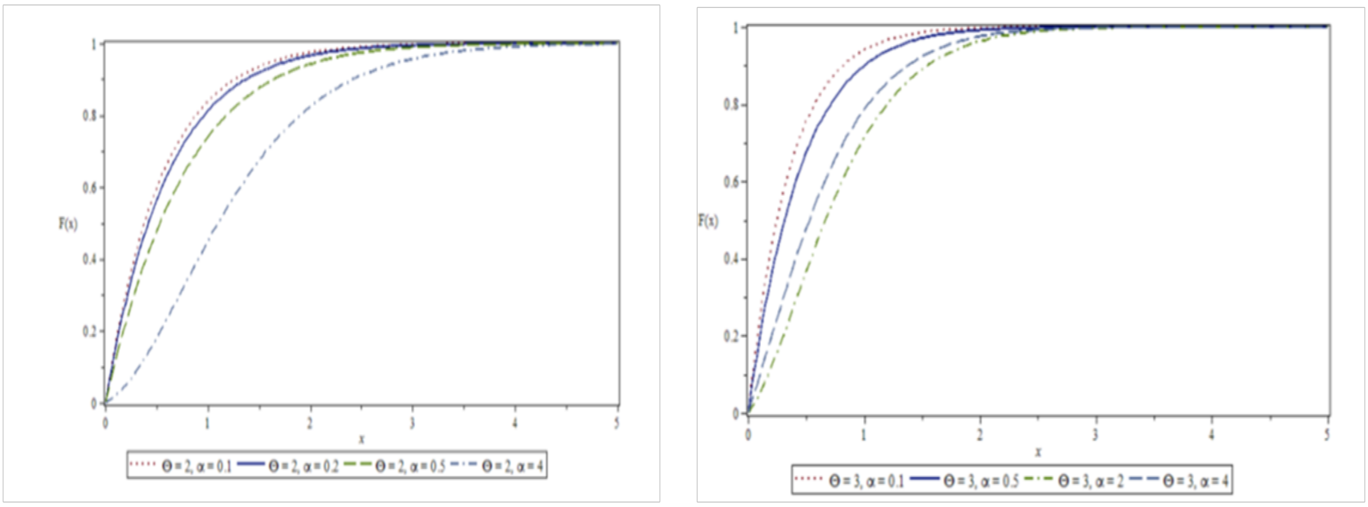


Figure 2 Behavior of the cdf of GAD for varying values of parameters θ and α.

Statistical properties of GAD

Moments and moments based measures

The r[th] raw moment, μ_r', of GAD can be obtained as

$$\mu_r' = \frac{r!\left(\theta^2 + 2\alpha\theta(r+1) + \alpha^2(r+2)(r+1)\right)}{\theta^r\left(\theta^2 + 2\theta\alpha + 2\alpha^2\right)};r = 1,2,3,...$$

The first four raw moments of GAD are given by

$$\mu_1' = \frac{\left(\theta^2 + 4\theta\alpha + 6\alpha^2\right)}{\theta\left(\theta^2 + 2\theta\alpha + 2\alpha^2\right)},\ \mu_2' = \frac{2\left(\theta^2 + 6\theta\alpha + 12\alpha^2\right)}{\theta^2\left(\theta^2 + 2\theta\alpha + 2\alpha^2\right)}$$

$$\mu_3' = \frac{6\left(\theta^2 + 8\theta\alpha + 20\alpha^2\right)}{\theta^3\left(\theta^2 + 2\theta\alpha + 2\alpha^2\right)},\ \mu_4' = \frac{24\left(\theta^2 + 10\theta\alpha + 30\alpha^2\right)}{\theta^4\left(\theta^2 + 2\theta\alpha + 2\alpha^2\right)}.$$

The central moments of GAD (2.1) are thus obtained as

$$\mu_2 = \frac{\theta^4 + 8\alpha\theta^3 + 24\alpha^2\theta^2 + 24\alpha^3\theta + 24\alpha^4}{\theta^2\left(\theta^2 + 2\alpha\theta + 2\alpha^2\right)^2}$$

$$\mu_3 = \frac{2\left(\theta^6 + 12\theta^5\alpha + 54\theta^4\alpha^2 + 100\alpha^3\theta^3 + 108\alpha^4\theta^2 + 72\alpha^5\theta + 24\alpha^6\right)}{\theta^3\left(\theta^2 + 2\alpha\theta + 2\alpha^2\right)^3}$$

$$\mu_4 = \frac{3\left(\begin{array}{l}3\theta^8 + 48\theta^7\alpha + 304\theta^6\alpha^2 + 944\theta^5\alpha^3 + 1816\theta^4\alpha^4 + 2304\alpha^5\theta^3\\ +1920\alpha^6\theta^2 + 960\alpha^7\theta + 240\alpha^8\end{array}\right)}{\theta^4\left(\theta^2 + 2\alpha\theta + 2\alpha^2\right)^4}$$

The expressions for various coefficients including coefficient of variation $(C.V)$, skewness $\left(\sqrt{\beta_1}\right)$, kurtosis (β_2) and index of dispersion (γ) of GAD are obtained as

$$C.V = \frac{\sigma}{\mu_1'} = \frac{\sqrt{\theta^4 + 8\theta^3\alpha + 24\theta^2\alpha^2 + 24\theta\alpha^3 + 12\alpha^4}}{\theta^2 + 4\theta\alpha + 6\alpha^2}$$

$$\sqrt{\beta_1} = \frac{\mu_3}{\mu_2^{3/2}} = \frac{2\left(\theta^6 + 12\theta^5\alpha + 54\theta^4\alpha^2 + 100\alpha^3\theta^3 + 108\alpha^4\theta^2 + 72\alpha^5\theta + 24\alpha^6\right)}{\left(\theta^4 + 8\alpha\theta^3 + 24\alpha^2\theta^2 + 24\alpha^3\theta + 24\alpha^4\right)^{3/2}}$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{3\left(\begin{array}{l}3\theta^8 + 48\theta^7\alpha + 304\theta^6\alpha^2 + 944\theta^5\alpha^3 + 1816\theta^4\alpha^4 + 2304\alpha^5\theta^3\\ +1920\alpha^6\theta^2 + 960\alpha^7\theta + 240\alpha^8\end{array}\right)}{\left(\theta^4 + 8\alpha\theta^3 + 24\alpha^2\theta^2 + 24\alpha^3\theta + 12\alpha^4\right)^2}$$

$$\gamma = \frac{\sigma^2}{\mu_1'} = \frac{\theta^4 + 8\alpha\theta^3 + 24\alpha^2\theta^2 + 24\alpha^3\theta + 12\alpha^4}{\theta\left(\theta^2 + 2\alpha\theta + 2\alpha^2\right)\left(\theta^2 + 4\theta\alpha + 6\alpha^2\right)}$$

The behaviors of C.V, skewness $\left(\sqrt{\beta_1}\right)$, kurtosis (β_2) and index of dispersion (γ) of GAD for various combination of parameters θ and α have been presented in Tables 1-4, respectively.

For a fixed value of α, C.V increases as the value of θ increases. Again for a fixed value of θ, C.V decreases as the value of α increases.

Clearly for any given values of parameters θ and α, coefficient of skewness is always positive and this means that it is always positively skewed.

Since $\beta_2 > 3$, GAD is always leptokurtic, which means that GAD is more peaked than the normal curve

As long as $0 < \theta < 1 \text{and} 0 < \alpha < 5$, GAD is over dispersed $(\sigma^2 > \mu_1')$ and for $\theta > 2 \text{and} \alpha > 0$, GAD is under-dispersed $(\sigma^2 < \mu_1')$. The nature of C.V, $\sqrt{\beta_1}$, β_2 and γ of GAD for various combinations of parameters θ and α have been shown graphically in Figures 3-6, respectively.

Hazard rate function and mean residual life function

The hazard rate function (also known as the failure rate function) $h(x)$ and the mean residual life function $m(x)$ of a continuous random variable X having pdf $f(x)$ and cdf $F(x)$ are respectively defined as

$$h(x) = \lim_{\Delta x \to 0} \frac{P\left(X < x + \Delta x | X > x\right)}{\Delta x} = \frac{f(x)}{1 - F(x)}$$

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and $m(x) = E\left[X - x \mid X > x\right] = \frac{1}{1-F(x)}\int_x^{\infty}\left[1-F(t)\right]dt$

Thus, $h(x)$ and $m(x)$ of GAD are thus obtained as

$$h(x) = \frac{\theta^3(1+\alpha x)^2}{\theta\alpha x(2\theta+\alpha\theta x+2\alpha)+(\theta^2+2\alpha\theta+2\alpha^2)}$$

and

$$m(x) = \frac{\theta^2+2\theta\alpha+2\alpha^2}{\left\{\theta\alpha x(2\theta+2x+2\alpha)+\left(\theta^2+2\theta\alpha+2\alpha^2\right)\right\}e^{-\theta x}}\int_x^{\infty}\left(\frac{\left\{\theta\alpha t(2\theta+2t+2\alpha)+\left(\theta^2+2\theta\alpha+2\alpha^2\right)\right\}}{\theta^2+2\theta\alpha+2\alpha^2}\right)e^{-\theta t}dt.$$

$$= \frac{\left(\theta^2\alpha^2x^2+2\alpha\theta x(2\alpha+\theta)+(\theta^2+4\theta\alpha+6\alpha^2)\right)}{\theta\left(\theta\alpha x(2\theta+2\theta x+2\alpha)+(\theta^2+2\theta\alpha+2\alpha^2)\right)}$$

It can be easily verified that $h(0)=f(0)$ and $m(0)=\mu_1'$. The behaviors of $h(x)$ and $m(x)$ of GAD for various combinations of parameters θ and α have been shown graphically in Figure 7 & Figure 8, respectively. Clearly $h(x)$ is monotonically increasing whereas $m(x)$ is monotonically decreasing.

Table 1 C.V of GAD for varying values of parameters θ and α

α \ θ	**0.1**	**0.2**	**0.5**	**1**	**2**	**3**	**4**	**5**
0.1	0.755	0.8535	0.9519	0.9843	0.9955	0.9979	0.9988	0.9992
0.2	0.6755	0.7552	0.8831	0.9519	0.9843	0.9924	0.9955	0.997
0.5	0.6172	0.6568	0.7551	0.8534	0.9341	0.9635	0.9771	0.984
1	0.597	0.6173	0.6756	0.7551	0.8535	0.9049	0.9341	0.952
2	0.587	0.5971	0.6273	0.6756	0.7552	0.8125	0.8535	0.833
3	0.583	0.5904	0.6105	0.6439	0.7048	0.7551	0.7955	0.8277
4	0.582	0.5871	0.6021	0.6273	0.6756	0.7184	0.7551	0.7863
5	0.581	0.5851	0.5971	0.6105	0.6568	0.6934	0.7262	0.755

Table 2 $\sqrt{\beta_1}$ of GAD for varying values of parameters θ and α

α \ θ	**0.1**	**0.2**	**0.5**	**1**	**2**	**3**	**4**	**5**
0.1	1.2981	1.5314	2.0638	2.5955	3.1058	3.3467	3.4859	3.5763
0.2	1.1932	1.2981	1.6376	2.0639	2.5955	2.9057	3.1058	3.2448
0.5	1.1589	1.1781	1.2981	1.5313	1.9114	2.1974	2.4194	2.5955
1	1.1553	1.1589	1.1932	1.2981	1.5314	1.7358	1.9114	2.0639
2	1.1548	1.1553	1.1622	1.1932	1.2981	1.417	1.5314	1.6376
3	1.1547	1.1549	1.1573	1.17	1.224	1.2981	1.3775	1.456
4	1.1547	1.1548	1.1559	1.1622	1.1932	1.2414	1.2981	1.3576
5	1.1547	1.1547	1.1553	1.1589	1.1781	1.211	1.2523	1.2981

Table 3 β_2 of GAD for varying values of parameters θ and α

α \ θ	**0.1**	**0.2**	**0.5**	**1**	**2**	**3**	**4**	**5**
0.1	5.3806	6.1353	7.163	8.4341	8.8155	8.9103	8.9473	8.9654
0.2	5.0871	5.3806	6.4784	7.613	8.4341	8.7	8.8155	8.8754
0.5	5.0082	5.0507	5.3806	6.1353	7.26	7.8744	8.2225	8.4341
1	5.0012	5.0082	5.0871	5.3806	6.1353	6.779	7.2601	7.613
2	5.0002	5.0012	5.0151	5.087	5.3806	5.7583	6.1353	6.4784
3	5.0001	5.0004	5.005	5.032	5.1673	5.3806	5.6297	5.8864
4	5.00002	5.0002	5.0022	5.015	5.0871	5.2151	5.3806	5.5659
5	5.00001	5.0001	5.0012	5.0082	5.0507	5.1325	5.2459	5.3806

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Table 4 θ of GAD for varying values of parameters θ and α

α \ θ	0.1	0.2	0.5	1	2	3	4	5
0.1	12.5454	6.5556	2.4981	1.1594	0.5449	0.354	0.2619	0.2077
0.2	11.5851	6.2727	2.619	1.2491	0.5797	0.3717	0.2724	0.2147
0.5	10.6807	5.6548	2.509	1.3111	0.6377	0.4085	0.2969	0.2319
1	10.3402	5.3403	2.317	1.2545	0.6556	0.4336	0.3189	0.2498
2	10.1689	5.1701	2.1692	1.1585	0.6273	0.4325	0.3278	0.2619
3	10.1122	5.1129	2.1136	1.111	0.599	0.4182	0.3219	0.261
4	10.084	5.0844	2.0852	1.0846	0.5793	0.4049	0.3136	0.2562
5	10.067	5.0674	2.068	1.0681	0.5655	0.3944	0.306	0.2509

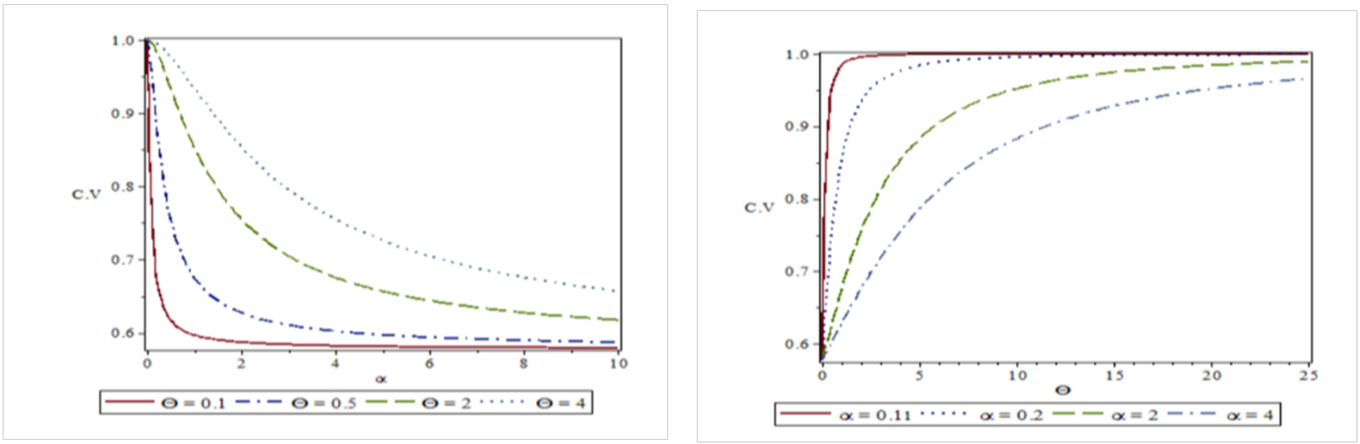


Figure 3 Coefficient of variation of GAD for different values of parameters θ and α.

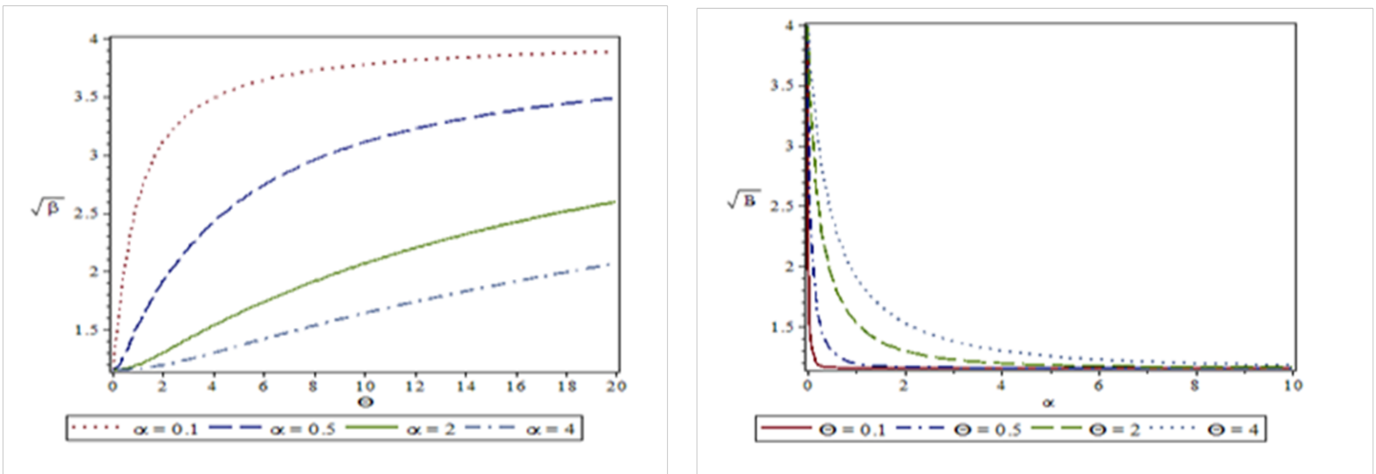


Figure 4 Coefficient of Skewness of GAD for different values of parameters θ and α.

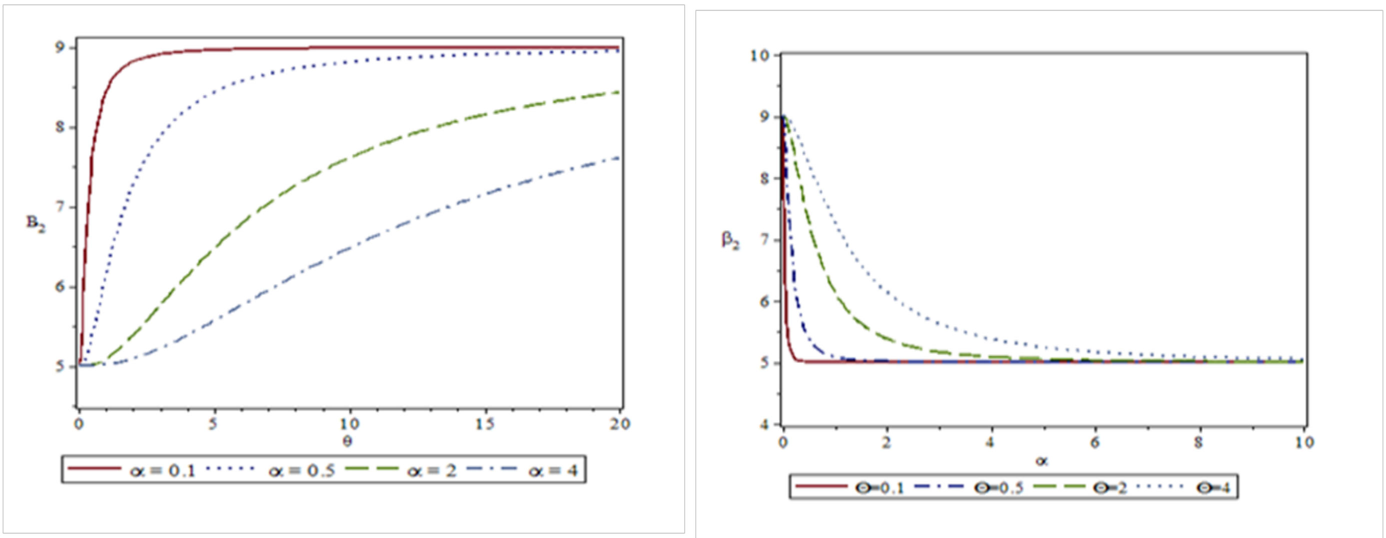


Figure 5 Coefficient of kurtosis of GAD for different values of parameters θ and α .

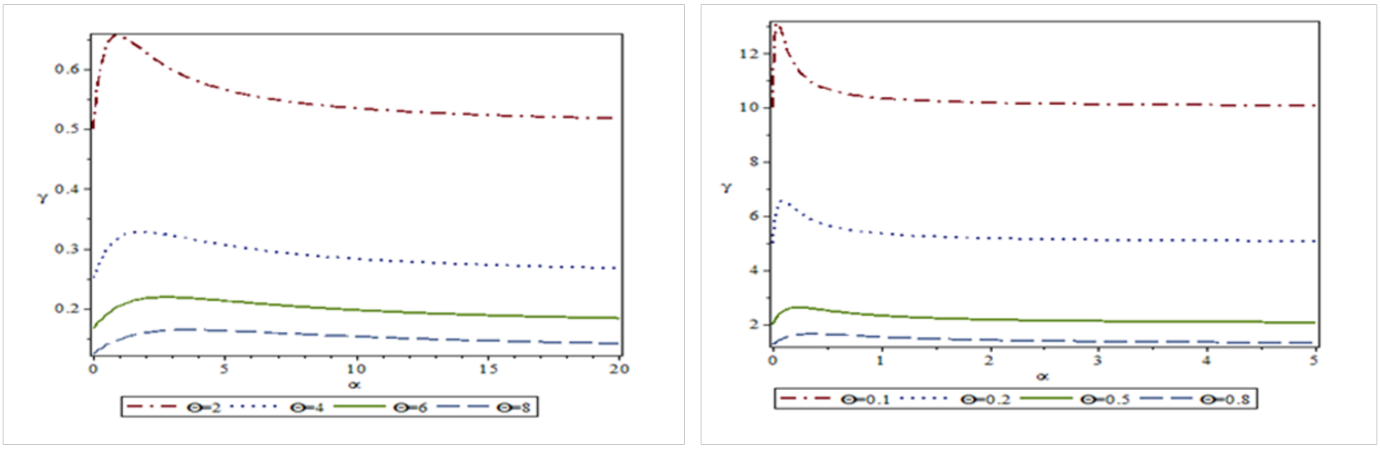
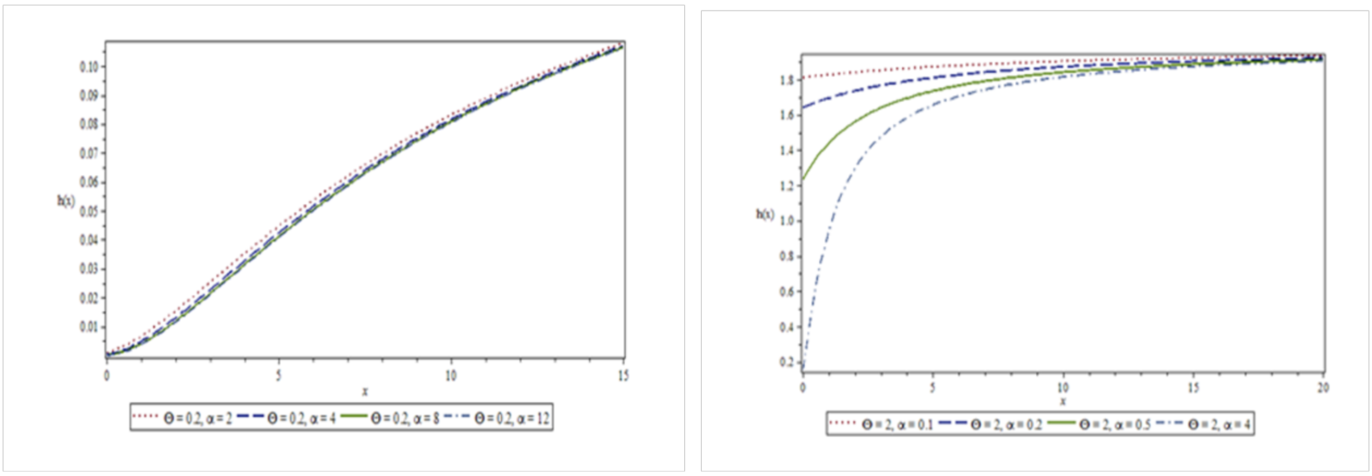


Figure 6 Index of dispersion of GAD for different values of parameters θ and α .



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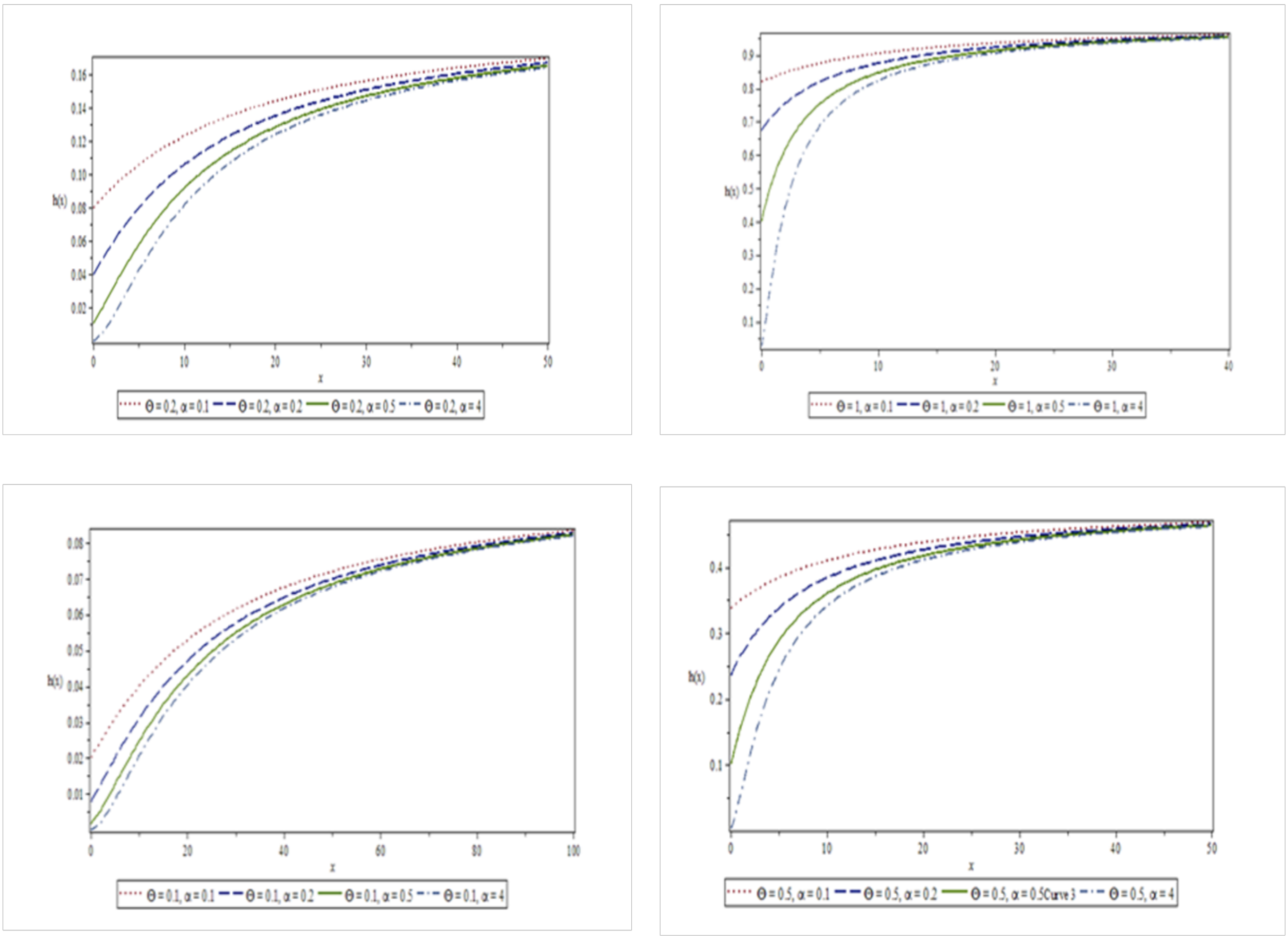
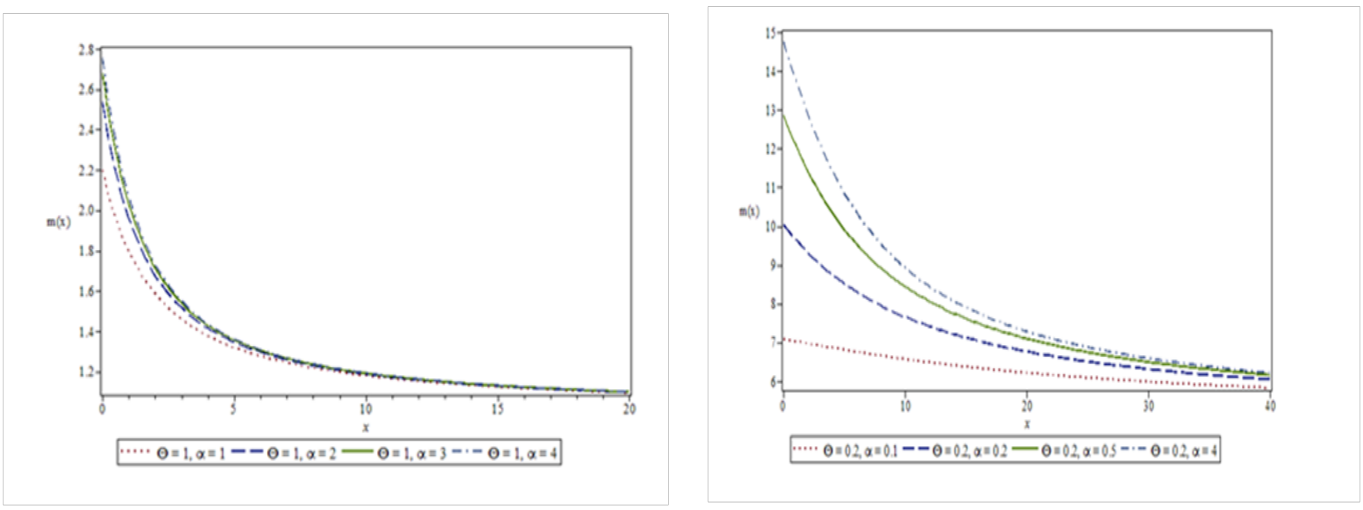


Figure 7 Behaviors of $h(x)$ of GAD for varying values of parameters θ and α .



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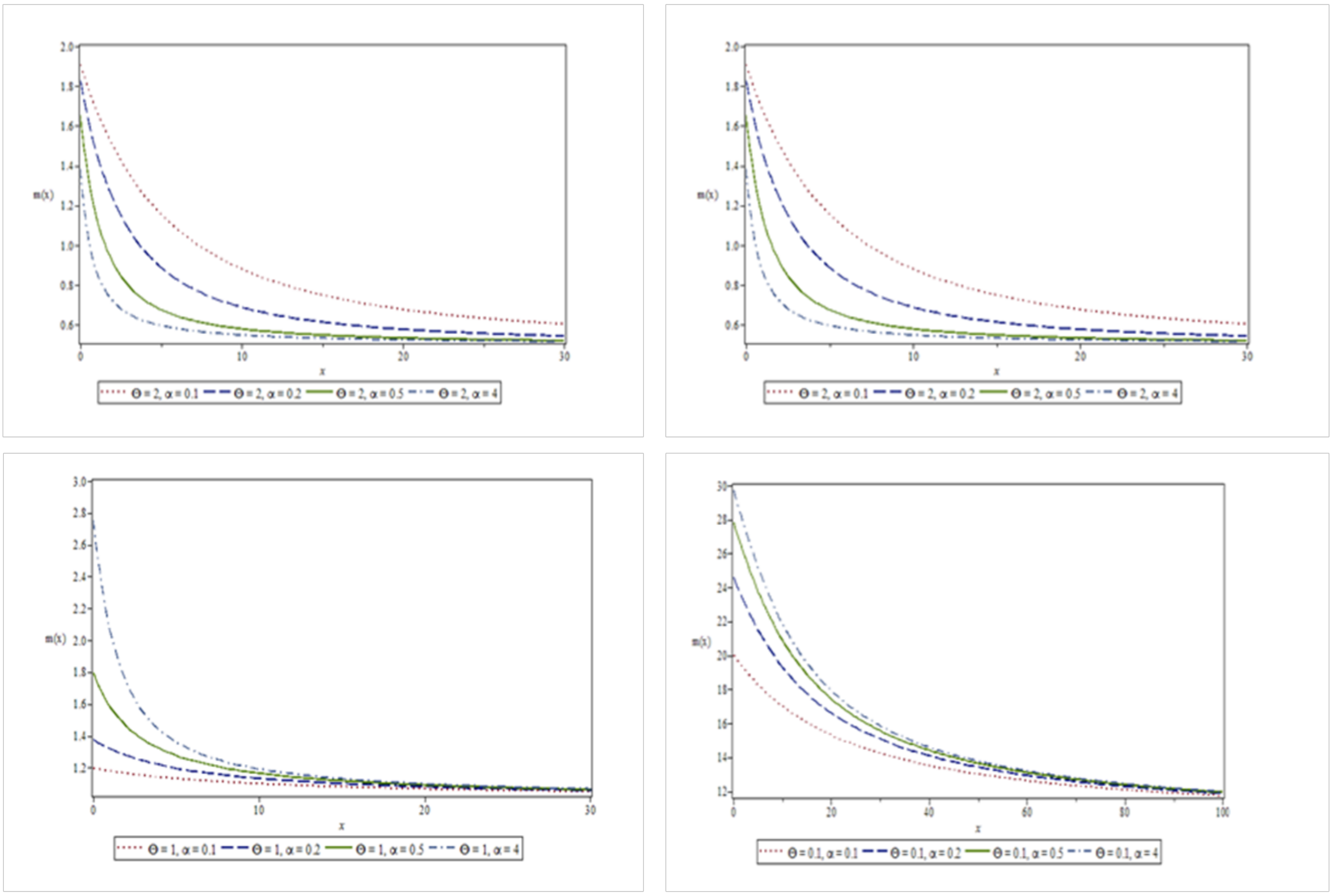

Figure 8 Behaviors of $m(x)$ of GAD for varying values of parameters θ and α .

Stochastic orderings

Stochastic ordering of positive continuous random variables is very much useful for judging their comparative behavior. A random variable X is said to be smaller than a random variable Y in the

stochastic order $(X \le_{st} Y)$ if $F_X(x) \ge F_Y(x)$ for all x

hazard rate order $(X \le_{hr} Y)$ if $h_X(x) \ge h_Y(x)$ for all x

mean residual life order $(X \le_{mrl} Y)$ if $m_X(x) \le m_Y(x)$ for all x

likelihood ratio order $(X \le_{lr} Y)$ if $\dfrac{f_X(x)}{f_Y(x)}$ decreases in x .

The following interrelationship among various stochastic orderings due to Shaker & Shanthikumar[8] are well known for establishing stochastic ordering of distributions

$$X \le_{lr} Y \Rightarrow X \le_{hr} Y \Rightarrow X \le_{mrl} Y$$
$$\Downarrow$$
$$X \le_{st} Y$$

Theorem: Let $X \sim \text{GAD}(\theta_1, \alpha_1)$ and $Y \sim \text{GAD}(\theta_2, \alpha_2)$. If $\alpha_1 = \alpha_2$ and $\theta_1 > \theta_2$ (or $\theta_1 = \theta_2$ and $\alpha_1 < \alpha_2$), then $X \le_{hr} Y$ and hence $X \le_{hr} Y$, $X \le_{mrl} Y$ and $X \le_{st} Y$.

Proof: We have

$$\frac{f_X(x;\theta_1,\alpha_1)}{f_Y(x;\theta_2,\alpha_2)} = \frac{\theta_1^3\left(\theta_2^2 + 2\alpha_2\theta_2 + 2\alpha_2^2\right)}{\theta_2^3\left(\theta_1^2 + 2\alpha_1\theta_1 + 2\alpha_1^2\right)}\left(\frac{1+\alpha_1 x}{1+\alpha_2 x}\right)^2 e^{-(\theta_1-\theta_2)x} \;;x>0$$

Now

$$\ln\frac{f_X(x;\theta_1,\alpha_1)}{f_Y(x;\theta_2,\alpha_2)} = \ln\left(\frac{\theta_1^3\left(\theta_2^2 + 2\alpha_2\theta_2 + 2\alpha_2^2\right)}{\theta_2^3\left(\theta_1^2 + 2\alpha_1\theta_1 + 2\alpha_1^2\right)}\right) + 2\ln\left(\frac{1+\alpha_1 x}{1+\alpha_2 x}\right) - (\theta_1-\theta_2)x$$

This gives

$$\frac{d}{dx}\left\{\ln\frac{f_X(x;\theta_1,\alpha_1)}{f_Y(x;\theta_2,\alpha_2)}\right\} = \frac{(\alpha_1-\alpha_2)}{(1+\alpha_1 x)(1+\alpha_2 x)} - (\theta_1-\theta_2).$$

Thus if $\alpha_1 = \alpha_2$ and $\theta_1 > \theta_2$ or $\theta_1 = \theta_2$ and $\alpha_1 < \alpha_2$, $\frac{d}{dx}\ln\frac{f_X(x;\theta_1,\alpha_1)}{f_Y(x;\theta_2,\alpha_2)} < 0$.

This means that $X \le_{lr} Y$ and hence $X \le_{mrl} Y$, $X \le_{mrl} Y$ and $X \le_{st} Y$. Thus, GAD is ordered with respect to the strongest 'likelihood ratio ordering'

Deviations from the Mean and the Median

The mean deviation about the mean, $\delta_1(X)$ and the mean deviation about the median, $\delta_2(X)$ are defined as

$$\delta_1(X) = \int_0^\infty |x-\mu| f(x)\,dx \quad \text{and} \quad \delta_2(X) = \int_0^\infty |x-M| f(x)\,dx,$$

respectively, where $\mu = E(X)$ and $M = \text{Median}(X)$. These

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expressions can be further simplified as

$$\delta_1(X)=\int_0^{\mu}(\mu-x)f(x)dx+\int_{\mu}^{\infty}(x-\mu)f(x)dx=2\mu F(\mu)-2\int_0^{\mu}x\,f(x)dx \quad (3.4.1)$$

and

$$\delta_2(X)=\int_0^{M}(M-x)f(x)dx+\int_{M}^{\infty}(x-M)f(x)dx=\mu-2\int_0^{M}x\,f(x)dx \quad (3.4.2)$$

Using pdf (2.1) and the mean of GAD, we get

$$\int_0^{\mu}x\,f(x)dx=\mu-\frac{\left\{\theta^3(\mu+2\alpha\mu^2+\alpha^2\mu^3)+\theta^2\left(1+4\alpha\mu+3\alpha^2\mu^2\right)+\theta\left(4\alpha+6\alpha^2\mu\right)+6\alpha^2\right\}e^{-\theta\mu}}{\theta\left(\theta^2+2\theta\alpha+2\alpha^2\right)} \quad (3.4.3)$$

$$\int_0^{M}x\,f(x)dx=\mu-\frac{\left\{\theta^3\left(M+2\alpha M^2+\alpha^2M^3\right)+\theta^2\left(1+4\alpha M+3\alpha^2M^2\right)+\theta\left(4\alpha+6\alpha^2M\right)+6\alpha^2\right\}e^{-\theta M}}{\theta\left(\theta^2+2\theta\alpha+2\alpha^2\right)} \quad (3.4.4)$$

Using expressions from (3.4.1), (3.4.2), (3.4.3), and (3.4.4), $\delta_1(X)$ and $\delta_2(X)$ of GAD are

$$\delta_1(X)=\frac{2\left\{\theta^2\left(1+2\alpha\mu+\alpha^2\mu^2\right)+4\alpha\theta(1+\mu)+6\alpha^2\right\}e^{-\theta\mu}}{\theta\left(\theta^2+2\theta\alpha+2\alpha^2\right)} \quad (3.4.5)$$

$$\delta_2(X)=\frac{2\left\{\theta^3\left(M+2\alpha M^2+\alpha^2M^3\right)+\theta^2\left(1+4\alpha M+3\alpha^2M^2\right)+\theta\left(4\alpha+6\alpha^2M\right)+6\alpha^2\right\}e^{-\theta M}}{\theta\left(\theta^2+2\theta\alpha+2\alpha^2\right)}-\mu \quad (3.4.6)$$

Bonferroni and lorenz curves and indices

The Bonferroni and Lorenz curves are proposed by Bonferroni[9] which are used in economics to study income and poverty and other fields of knowledge including reliability, demography, insurance and medicine, some among others. The Bonferroni and Lorenz curves are defined as

$$B(p)=\frac{1}{p\mu}\int_0^{q}x\,f(x)dx=\frac{1}{p\mu}\left[\int_0^{\infty}x\,f(x)dx-\int_q^{\infty}x\,f(x)dx\right]=\frac{1}{p\mu}\left[\mu-\int_q^{\infty}x\,f(x)dx\right] \quad (3.5.1)$$

and $$L(p)=\frac{1}{\mu}\int_0^{q}x\,f(x)dx=\frac{1}{\mu}\left[\int_0^{\infty}x\,f(x)dx-\int_q^{\infty}x\,f(x)dx\right]=\frac{1}{\mu}\left[\mu-\int_q^{\infty}x\,f(x)dx\right] \quad (3.5.2)$$

The Bonferroni and Gini indices are further simplified as

$$B=1-\int_0^1 B(p)dp \quad (3.5.3)$$

and $$G=1-2\int_0^1 L(p)dp \quad (3.5.4)$$

Using pdf of GAD (2.1), we get

$$\int_q^{\infty}x\,f(x)dx=\frac{\left\{\theta^3\left(q+2\alpha q^2+\alpha^2q^3\right)+\theta^2\left(1+4\alpha q+3\alpha^2q^2\right)+\theta\left(4\alpha+6\alpha^2q\right)+6\alpha^2\right\}e^{-\theta q}}{\theta\left(\theta^2+2\theta\alpha+2\alpha^2\right)} \quad (3.5.5)$$

Now using equation (3.5.5) in (3.5.1) and (3.5.2), we get

$$B(p)=\frac{1}{p}\left(1-\frac{\left\{\theta^3(q+2\alpha q^2+\alpha^2q^3)+\theta^2\left(1+4\alpha q+3\alpha^2q^2\right)+\theta\left(4\alpha+6\alpha^2q\right)+6\alpha^2\right\}e^{-\theta q}}{\left(\theta^2+4\theta\alpha+6\alpha^2\right)}\right) \quad (3.5.6)$$

and

$$L(p)=1-\frac{\left\{\theta^3\left(q+2\alpha q^2+\alpha^2q^3\right)+\theta^2\left(1+4\alpha q+3\alpha^2q^2\right)+\theta\left(4\alpha+6\alpha^2q\right)+6\alpha^2\right\}e^{-\theta q}}{\left(\theta^2+4\theta\alpha+6\alpha^2\right)} \quad (3.5.7)$$

Now using equations (3.5.6) and (3.5.7) in (3.5.3) and (3.5.4), we have

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$$B = 1 - \frac{\left\{\theta^3\left(q + 2\alpha q^2 + \alpha^2 q^3\right) + \theta^2\left(1 + 4\alpha q + 3\alpha^2 q^2\right) + \theta\left(4\alpha + 6\alpha^2 q\right) + 6\alpha^2\right\} e^{-\theta q}}{\left(\theta^2 + 4\theta\alpha + 6\alpha^2\right)} \quad (3.5.8)$$

$$G = \frac{2\left\{\theta^3\left(q + 2\alpha q^2 + \alpha^2 q^3\right) + \theta^2\left(1 + 4\alpha q + 3\alpha^2 q^2\right) + \theta\left(4\alpha + 6\alpha^2 q\right) + 6\alpha^2\right\} e^{-\theta q}}{\left(\theta^2 + 4\theta\alpha + 6\alpha^2\right)} - 1 \quad (3.5.9)$$

Stress-strength parameter

Suppose X is the random strength and Y be the random stress of a component. $Y > X$, the component fails instantly and the component will function satisfactorily till $X > Y$. Therefore, $R = P(Y < X)$ is a measure of component reliability and in statistics it is known as stress-strength parameter. It has applications in almost all areas of knowledge including medical science, sociology, psychology, and engineering, some among others.

Let X and Y be independent strength and stress random variables having GAD (2.1) having parameters (θ_1, α_1) and (θ_2, α_2) respectively. Then the stress-strength reliability R of GAD (2.1) can be obtained as

$$R = P(Y < X) = \int_0^\infty P(Y < X \mid X = x) f_X(x)\, dx$$

$$= \int_0^\infty f(x;\theta_1,\alpha_1) F(x;\theta_2,\alpha_2)\, dx$$

$$= 1 - \frac{\theta_1^3 \left(\begin{array}{l} \theta_2^6 + 2\left(2\theta_1 + (2\alpha_2 + \alpha_1)\right)\theta_2^5 + 2\left(3\theta_1^2 + 10\alpha_2\theta_1 + \left(3\alpha_2^2 + 6\alpha_2\alpha_1 + \alpha_1^2\right)\right)\theta_2^4 + \\ + \left(\begin{array}{l} 4\theta_1^3 + (18\alpha_2 + 6\alpha_1)\theta_1^2 + \left(18\alpha_2^2 + 28\alpha_2\alpha_1 + 4\alpha_1^2\right)\theta_1 + \\ \left(24\alpha_2^2\alpha_1 + 4\alpha_2\alpha_1^2 + 12\alpha_2^2\alpha_1^2\right) \end{array} \right)\theta_2^3 + \\ \left(\theta_1^4 + (10\alpha_2 + 2\alpha_1)\theta_1^3 + \left(20\alpha_2^2 + 20\alpha_2\alpha_1 + 2\alpha_1^2\right)\theta_1^2 + \left(40\alpha_2^2\alpha_1 + 20\alpha_2\alpha_1^2\right)\theta_1 + \left(40\alpha_2^2\alpha_1^2\right)\right)\theta_2^2 + \\ 2\left(\theta_1^3 + (5\alpha_2 + 2\alpha_2\alpha_1)\theta_1^2 + \left(10\alpha_2\alpha_1 + 2\alpha_1^2\right)\theta_1 + 10\alpha_2\alpha_1^2\right)\alpha_2\theta_1\theta_2 + \\ 2\left(\theta_1^2 + 2\alpha_1\theta_1 + 2\alpha_1^2\right)\alpha_2^2\theta_1^2 \end{array} \right)}{\left(\theta_1^2 + 2\theta_1\alpha_1 + 2\alpha_1^2\right)\left(\theta_2^2 + 2\theta_2\alpha_2 + 2\alpha_2^2\right)(\theta_1 + \theta_2)^5}$$

Clearly at $(\alpha_1 = \theta_1, \alpha_2 = \theta_2)$ and $(\alpha_1 = 0, \alpha_2 = 0)$, the above expression reduces to the corresponding expression of R for Aradhana and exponential distributions.

Parameter estimation

In this section maximum likelihood estimation of parameters of GAD has been discussed. Suppose $(x_1, x_2, x_3, ..., x_n)$ be a random sample from GAD (2.1)). The natural log likelihood function is thus obtained as

$$\ln L = 3n\ln\theta + n\ln\left(\theta^2 + 2\theta\alpha + 2\alpha^2\right) + 2\sum_{i=1}^{n}\ln(1 + \alpha x_i) - n\theta\bar{x},$$

where $\bar{x}$ is the sample mean.

The maximum likelihood estimates (MLE) $(\hat{\theta}, \hat{\alpha})$ of parameters (θ, α) are the solutions of the following non-linear log likelihood equations

$$\frac{\partial \ln L}{\partial \theta} = \frac{3n}{\theta} - \frac{2n(\theta + \alpha)}{\theta^2 + 2\theta\alpha + 2\alpha^2} - n\bar{x} = 0$$

$$\frac{\partial \ln L}{\partial \alpha} = \frac{-2n(\theta + 2\alpha)}{\theta^2 + 2\theta\alpha + 2\alpha^2} + 2\sum_{i=1}^{n}\frac{x_i}{1 + \alpha x_i} = 0$$

It is difficult to solve these two natural log likelihood equations directly because they are not in closed forms. But these equations can be solved using Fisher's scoring method. For, we have

$$\frac{\partial^2 \ln L}{\partial \theta^2} = -\frac{3n}{\theta^2} + \frac{2n\theta(\theta + 2\alpha)}{\left(\theta^2 + 2\theta\alpha + 2\alpha^2\right)^2}$$

$$\frac{\partial^2 \ln L}{\partial \theta \partial \alpha} = \frac{2n\left(\theta^2 + 4\theta\alpha + 2\alpha^2\right)}{\left(\theta^2 + 2\theta\alpha + 2\alpha^2\right)^2}$$

$$\frac{\partial^2 \ln L}{\partial \alpha^2} = \frac{8n\alpha(\theta + \alpha)}{\left(\theta^2 + 2\theta\alpha + 2\alpha^2\right)^2} - 2\sum_{i=1}^{n}\frac{x_i^2}{(1 + \alpha x_i)^2}$$

The solution of following equations gives MLE's $(\hat{\theta}, \hat{\alpha})$ of (θ, α) of GAD

$$\begin{bmatrix} \frac{\partial^2 \ln L}{\partial \theta^2} & \frac{\partial^2 \ln L}{\partial \theta \partial \alpha} \\ \frac{\partial^2 \ln L}{\partial \theta \partial \alpha} & \frac{\partial^2 \ln L}{\partial \alpha^2} \end{bmatrix}_{\substack{\hat{\theta} = \theta_0 \\ \hat{\alpha} = \alpha_0}} \begin{bmatrix} \hat{\theta} - \theta_0 \\ \hat{\alpha} - \alpha_0 \end{bmatrix} = \begin{bmatrix} \frac{\partial \ln L}{\partial \theta} \\ \frac{\partial \ln L}{\partial \alpha} \end{bmatrix}_{\substack{\hat{\theta} = \theta_0 \\ \hat{\alpha} = \alpha_0}}$$

where θ_0 and α_0 are the initial values of θ and α, respectively. These equations are solved iteratively till sufficiently close values of $\hat{\theta}$ and $\hat{\alpha}$ are obtained.

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Applications

The goodness of fit of GAD using maximum likelihood estimates has been discussed with two real lifetime dataset and the fit has been compared with exponential, Lindley and Aradhana distributions and a generalization of Sujatha distribution (AGSD) proposed by Shanker et al.[7] It has been observed that GAD is more suitable to positively skewed lifetime data. In general, majority of the real lifetime datasets in medical science and engineering are positively skewed and hence GAD is suitable for positively skewed data in these fields of knowledge. Since GAD is a new distribution, still more research is required to find several applications of the distribution in various fields of knowledge. The real lifetime datasets which are positively skewed are as follows

Data set 1: This data represents the lifetime's data relating to relief times (in minutes) of 20 patients receiving an analgesic and reported by Gross & Clark.[10]

1.1	1.4	1.3	1.7	1.9	1.8	1.6	2.2	1.7	2.7	4.1	1.8
1.5	1.2	1.4	3	1.7	2.3	1.6	2				

Data set 2: This data set is the strength data of glass of the aircraft window reported by Fuller et al.[11]

18.830	20.800	21.657	23.030	23.230	24.050	24.321	25.500
25.520	25.800	26.690	26.770	26.780	27.050	27.670	29.900
31.110	33.200	33.730	33.760	33.890	34.760	35.750	35.910
36.980	37.080	37.090	39.580	44.045	45.29	45.381	

For comparing the goodness of fit of GAD, AGSD, Aradhana, Lindley and exponential distributions, values of -2ln L , AIC (Akaike Information Criterion), kolmogorove-Smirnov Statistics (K-S Statistics) and p value of these distributions for two real lifetime datasets have been computed and presented in Table 5. Since the best fit of the distribution corresponds to the lower values of -2ln L, AIC, K-S statistics, it is obvious from table 5 that GAD provides better fit than exponential, Lindley, Aradhana and AGSD.

Conclusions

A generalized Aradhana distribution (GAD) has been introduced which includes both Aradhana distribution proposed by Shanker[2] and Lindley distribution proposed by Lindley[1] as particular cases. The nature of probability density function and cumulative distribution function of GAD has been studied. The raw moments and central moments of the distribution have been obtained and behavior of coefficient of variation, coefficient of skewness, coefficient of kurtosis and index of dispersion of GAD have been studied with varying values of the parameters. Hazard rate function and mean residual life function have been studied with varying values of the parameters. The stochastic ordering, mean deviations, Bonferroni and Lorenz curves, and stress-strength reliability have also been discussed. The method of maximum likelihood has been discussed for estimating parameters. Two examples of real lifetime, one from medical science and one from engineering, have been presented to show the applications and goodness of fit of GAD over exponential, Lindley and Aradhana distributions and AGSD and it has been observed that GAD gives much better fit.

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Conflict of interest

The author declares there is no conflict of interest.

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